Types of vortex solutions in superfluid He^3

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Vortex solutions in the A and B phases of superfluid He^3 are classified on the basis of the Ginzburg-Landau equations. It is shown that any vortex state can be characterized by three quantum numbers. Some of the results are directly applicable to the theory of nematic liquid crystals. Volume solutions are set up which may exist for arbitrary values of the coefficient K_{22}, and it is shown that either radial or circular disclinations may exist, depending on the relation between the Frank coefficients.

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1. INTRODUCTION

One of the specific properties of superconducting quantum liquids is the possibility of the existence in them of macroscopic quantum objects—vortex filaments. In contrast with superfluid He^4 and type II superconductors, in which the order parameter is a complex scalar function, the order parameter in He^3 has the more complicated structure[1]

$$\Delta_{\alpha\beta}=\sum_{p,i}A_{pi}\hat{k}^p(\sigma\sigma_y)_{\alpha\beta}{}^i. \tag{1}$$

The order parameter is written in the momentum representation, σ are the Pauli matrices, $\hat{\mathbf{k}}$ the unit vector in the $\mathbf{p}$ direction. Different choices of A_{pi} in (1) correspond to different states of the system. The complexity of the order parameter A_{pi} indicates that we can expect in superfluid He^3 the appearance of vortex solutions of a new type, not found in He^4 and in the superconductors. Since the structure of the vortices depends significantly on the form of the coefficients A_{pi}, the experimental study of vortices should give information on the state of the system exactly the same as experiments on NMR and measurement of kinetic properties do. In addition, it turns out that there is a possibility of direct observation of states in He^3, as recently suggested in connection with the investigation of classical solutions in field theory.[2,3]

The explicit form of the solutions can be found from the Ginzburg–Landau expansion. Here, since the Fermi liquid corrections near T_c enter only as a common factor,[3,4] the weak coupling approximation can be used. The problem was considered in this form by Gor'kov and Melik-Barkhudarov for the case of anisotropic superconductors.[6] The general formulas for the free energy and current in terms of A_{pi} were deduced in[4,7].

In the second section, we study the vortex solutions in A phase; vortex interaction is considered and it is shown that an effect should exist in the A phase that is analogous to the Einstein–de Haas effect. Although it would seem that only the usual vortex states can be achieved in the B phase, the study of the more general problem is of interest, since the results are directly applicable to the theory of nematic liquid crystals. The class of volume solutions of the Oseen—Frank functions is considered which have an energy below the class of self-similar solutions studied earlier.[8] It is shown that the volume solutions can exist only in the case of arbitrary values of the coefficient K_{22}; here, in the case $K_{11}>K_{22}$, circular disclinations are more favorable, and in the case of $K_{22}<K_{11}$, radial ones. For vortex solutions in the B phase, there is a definite correlation between the change in the direction of the director $\mathbf{n}$ along the radius of disclination and the value of the circulation. These results are given in the third section.

2. TYPES OF VORTICES IN THE A PHASE

In the A phase, the order parameter has the form[1]:

$$A_{pi}=\Delta_0 A_p\hat{V}_i, \tag{2}$$

$$\mathbf{A}=\boldsymbol{\Delta}+i\boldsymbol{\Delta}',\quad (\boldsymbol{\Delta},\boldsymbol{\Delta}')=0,\quad [\boldsymbol{\Delta}\times\boldsymbol{\Delta}']=\mathbf{l}, \tag{3}$$

where $\boldsymbol{\Delta}$, $\boldsymbol{\Delta}'$, $\mathbf{l}$, $\hat{\mathbf{V}}$ are unit real vectors. The orbital momentum of the pair is directed along $\mathbf{l}$. Any state $A_{pi}(\mathbf{r})$ that is inhomogeneous in space can be specified as the deformation of the axes $\boldsymbol{\Delta}$, $\boldsymbol{\Delta}'$, $\hat{\mathbf{V}}$. Here the conditions (3) should be satisfied locally at each point. The free energy and the current, calculated by the methods of Gor'kov and Melik–Barkhudarov[6] with account of the noncommutability of the rotations around the $\mathbf{l}$ axis and the deformation of $\mathbf{l}$,[9] are identical with the expressions of Cross,[5] obtained by a different method:

$$\Delta F_s=\alpha\int\Big\{(2\delta_{ij}-l_il_j)v_{si}v_{sj}+(2\delta_{ij}-l_il_j)\frac{\partial\hat{V}_p}{\partial x_i}\frac{\partial\hat{V}_p}{\partial x_j}+\frac{1}{2}(\operatorname{div}\mathbf{l})^2$$
$$+\frac{1}{2}(\mathbf{l}\operatorname{rot}\mathbf{l})^2+\frac{3}{2}[\mathbf{l}\times\operatorname{rot}\mathbf{l}]^2-2(\mathbf{lv}_s)(\mathbf{l}\operatorname{rot}\mathbf{l})+(\mathbf{v}_s\operatorname{rot}\mathbf{l})\Big\}d^3\mathbf{r}, \tag{4}$$

$$\mathbf{j}_s=\beta\{\hat{\rho}\mathbf{v}_s+\hat{C}\operatorname{rot}\mathbf{l}\}, \tag{5}$$

$$\hat{\rho}=(2\delta_{ij}-l_il_j),\quad \hat{C}=\tfrac{1}{2}(\delta_{ij}-2l_il_j),$$

$$v_{si}=\Delta_p\frac{\partial\Delta_p'}{\partial x_i}-\Delta_p'\frac{\partial\Delta_p}{\partial x_i}, \tag{6}$$

$$\alpha\sim(p_0^3/m^*)(1-T/T_c),\ \beta\sim m\alpha,\quad \alpha\sim10^{-9}\ \tau\ \text{erg-cm}^{-3}.$$

It follows from the definition (6) that[9]

$$\frac{\partial v_{sj}}{\partial x_i}-\frac{\partial v_{si}}{\partial x_j}=\frac{1}{2}\mathbf{l}\Big[\frac{\partial\mathbf{l}}{\partial x_i}\frac{\partial\mathbf{l}}{\partial x_j}\Big]. \tag{7}$$

Under steady conditions,

$$\operatorname{div}\mathbf{j}_s=0. \tag{8}$$

Account of the dipole-dipole interaction leads to the appearance of the term[10]

$$E_D=-g_D\int(\hat{\mathbf{V}}\mathbf{l})^2d^3\mathbf{r}, \qquad (9)$$
$$g_D\sim\gamma^2T_c^2(m^*p_0)^2[1-T/T_c],$$

γ is the nuclear magnetic moment, $g_D\sim10^{-3}\,\tau$ erg-cm^{-3}.

If there is an external magnetic field, then one should add another term

$$E_H=\chi\int(\hat{\mathbf{V}}\mathbf{H})^2d^3\mathbf{r}, \qquad (10)$$

$\chi\sim\chi_n\tau$, where χ_n is the susceptibility in the normal state, $\chi_n\sim5\times10^{-7}$ erg-cm-Ga^{-2}.

We are interested in solutions which depend only on x, y and not on z, and the energy per unit length of which increases no more rapidly than logarithmically with distance from the vortex axis. Therefore, we have not written out the terms in (4) which contain the gradients of the modulus of the order parameter, since it is only necessary to know the asymptotic form for the investigation of the vortex solutions. For solutions which are independent of z, the additional condition

$$j_z=0 \qquad (11)$$

must be satisfied.

As is seen from (4), the finding of a complete set of exact solutions is a complex mathematical problem; however, various types of vortex solutions can be determined from simple topological considerations. A state that is spatially inhomogeneous can be described with the aid of local transformations

$$\Delta'_{\alpha\beta}\to R_A\hat{R}_V\Delta_{\alpha\beta}.$$

The transformation $\hat{R}_A$, which satisfies the conditions (3), represents several three-dimensional rotations, determined by the n axis and the angle of rotation θ. With account of the fact that the wave function should not change upon rotation through an angle θ that is a multiple of 2π, and the ends of the unit vectors n and $\hat{\mathbf{V}}$ lie on the surface of a unit sphere, we obtain the result that the functional (4) defines some mapping of the plane (x, y) on the surface of a torus

$$T=S^1\times S^2\times S^2. \qquad (12)$$

where S^n is the surface of the unit sphere in n-dimensional space. A more rigorous proof of (12) follows from the double-connectedness of the set of the group of rotations $S^1\times S^2$.[11] By virtue of the topological properties of the torus, the solution, which is singular on one of the surfaces S, will be singular also on T. On the other hand, since there are no nontrivial mappings of the plane on the surface of the sphere S^n with $n\geq3$,[12] all the nontrivial solutions are accounted for. Physically, this means that account of the interaction can no longer change the type of the vortex state; therefore it suffices to investigate the functional (4), keeping two of the functions θ, n or $\hat{\mathbf{V}}$ constant in pairs. If the centers of the solutions that are singular in the different characteristics coincide, then a vortex will be obtained which possesses several quantum numbers simultaneously. Upon interaction of the vortices, exchange of the quantum numbers is possible, or else decay to a vortex with a different number. The total quantum number of the entire system will be conserved. Close to the walls of the vessel, at which the vortices can be created or annihilated, the conservation laws cease to be exact. We shall return at the end of this section, to a discussion of the validity of such a simple physical picture.

We shall show how this program can be realized in the variables $\mathbf{v}_s$, l and V. Formula (7) determines the velocity v_s with accuracy to a function $\tilde{\mathbf{v}}_s$, which is the gradient of the phase:

$$\tilde{\mathbf{v}}_s=\nabla\varphi. \qquad (13)$$

The solutions that are vortical in $\mathbf{v}_s$ are connected both with the solutions vortical in l and with the solutions $\tilde{\mathbf{v}}_s$ vortical in (13). These solutions can be uniquely distinguished from one another. On the left side of (7) is the Jacobian of the mapping of the plane (x, y) onto the surface of a unit sphere, which is specified by the vector l,[3,13] therefore, as a result of integration of (7) over x and y, we obtain the value of the circulation

$$\Gamma\equiv\oint\mathbf{v}_s\,d\mathbf{r},$$

on the left side. This integration is carried out over an infinitely removed contour, and $2\pi q$ appears on the right side, where q is the degree of mapping of the plane (x, y) on the sphere. The difference between Γ and $2\pi q$ is connected with the vortex solutions (13). If there are antivortices in the system, in addition to the vortices, then one must compare Γ and $2\pi q$ for each separate vortex. Thus, for the determination of the different types of vortex solutions, it suffices to investigate the functional (4) at the following three points: 1) l and $\hat{\mathbf{V}}$ constant and $\mathbf{v}_s$ equal to the gradient of the phase, 2) the vector $\hat{\mathbf{V}}$ is constant and the circulation $\Gamma=2\pi q$, 3) l and φ are constant.

Thus, let l and $\hat{\mathbf{V}}$ be constants. Then we can reduce (4) to diagonal form by a linear transformation of the coordinates (x, y). In the new variables (x', y'), the functional (4) is multiplied by the Jacobian $[2(1+l_z^2)]^{1/2}$; therefore, the state with l lying in the plane of the vortex will possess the least energy. The vortex solutions in the A phase were also investigated in Ref. 14.

As the only stable solution, they proposed in that reference the state with l parallel to the vortex. Actually, such a solution has a maximum energy and is unstable. We also note that the deformations of the $\hat{\mathbf{V}}$ axis were not considered in Ref. 14. Choosing the vector l lying along the x axis, we find that the streamlines are ellipses

$$v_{s_x}^2+2v_{s_y}^2=\text{const}.$$

which are extended along the x axis. Each vortex state, as in ordinary He II, can be characterized by the circulation index n; however, the solution will be stable only with $n=1$. The energy of interaction of the two vortices is proportional to $\ln(R/|\mathbf{r}_1-\mathbf{r}_2|)$ (R is of the order of the dimensions of the vessel). Generally speaking, this does not include the its particular solutions that can be obtained (in the variables x', y') from the equation

$$\Delta'\varphi=d_i\tilde{d}_j\ldots\frac{\partial\ldots}{\partial x_i'\,\partial x_j'\ldots}\delta_2(\mathbf{r}'), \tag{14}$$

where d, $\tilde{\mathbf{d}}$, ... are constant vectors. The general solution of (14) corresponds to states in which there is a certain number of vortices and antivortices. It is possible that such states will be sufficiently long-lived.[15] It is not difficult to construct such a series of three-dimensional solutions with $\delta_3(\mathbf{r})$ on the right side of (14) having finite energy and satisfying the condition of continuity. Such states could be formed in flow over sharply pronounced surface irregularities. In contrast to the vortices in He II, the total energy (4) is not identical, as is seen from (5), with the kinetic energy. The difference in the total and kinetic energies is connected with the internal Josephson effect and with the anisotropies. It follows from (4) that for a bulk sample the vector l is always directed along $\pm\mathbf{v}_s$. By periodic variation of $\mathbf{v}_s$ we can obtain an effect that is exactly equivalent to the Einstein–de Haas effect, since a certain elementary orbital momentum is associated with l. Since the $\hat{\mathbf{V}}$ axis will always be collinear to l because of the dipole forces, flipping of l can always be brought about with the help of the magnetic field (10). Similar experiments would be the most direct proof of the fact that there is a nonzero mean density of the orbital momentum in the A phase, and would allow the direct measurement of ρ_s.

We now proceed to the consideration of spin vortices. After reduction to diagonal form, we obtain the functional studied in the work of Belavin and Polyakov.[3] We present a simple derivation of the results of their work. The energy (4), on transforming from the unit vector $\hat{\mathbf{V}}$ to the spherical coordinates θ and φ, can be written exactly in the form

$$E_s=\alpha/2\int\left\{\left(\frac{\partial\theta}{\partial x}+\sin\theta\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\theta}{\partial y}-\sin\theta\frac{\partial\varphi}{\partial x}\right)^2 +2\sin\theta\left(\frac{\partial\varphi}{\partial x}\frac{\partial\theta}{\partial y}-\frac{\partial\varphi}{\partial y}\frac{\partial\theta}{\partial x}\right)\right\}dx\,dy. \tag{15}$$

The Jacobian of the transformation $(x, y)\to(\varphi, \theta)$ in units of 4π is equal to the order of the mapping and for a given homotopic class is a constant quantity; therefore, the minimum is reached if the first two terms are equal to zero, which is equivalent to the requirement of analyticity of the function $\cot(\theta/2)\exp(i\varphi)$. Its explicit form is determined from the requirement of single valuedness of $\hat{\mathbf{V}}$. For the wave function (1), (2), only the axis of spin projection has a meaning, but not the direction of the axis. This is also the case when all the physical quantities (energy, spin current, etc.) are invariant relative to the substitution $\hat{\mathbf{V}}\to-\hat{\mathbf{V}}$. (An exception is the phase transition A_1-A_2, near which, however, the expansion (4) ceases to be valid.) Therefore the $\hat{\mathbf{V}}$ axis is determined to within its sign. As a result, the mapping has the form

$$w_1\equiv\operatorname{ctg}\frac{\theta}{2}\exp(i\varphi)=\Pi\frac{(z-z_i^{(0)})^{m_i}}{(z-z_j^{(0)})^{n_j}}, \tag{16}$$

where $z=(x+iy)/\lambda$; λ is an arbitrary scale factor; $z_i^{(0)}\neq z_j^{(0)}$; m_i and n_j are arbitrary half integers. The solution (16) describes a system of spin disclinations located at the points $z_i^{(0)}$ and having the Frank index $2m_i$, and antidisclinations (disclinations with negative index) at the points $z_j^{(0)}$ with indices $2n_j$. The plane of the disclination is not connected with the location of the (x, y) plane. The order of the mapping is equal to

$$q=\max\left\{\sum_i m_i,\ \sum_j n_j\right\}. \tag{17}$$

It follows from the formulas (15) and (17) that the energy of the system of spin disclinations does not depend on $z_i^{(0)}$ or $z_j^{(0)}$ (this is also confirmed by direct calculations in a number of cases), which leads to the paradoxical conclusion that the disclinations do not interact. This is connected with the degenerate properties of the model (the presence of exact conformal symmetry) and the absence of a superposition principle. In fact, the requirement of uniqueness of the vector $\hat{\mathbf{V}}$ leads to the result that even for large distances between the centers of the disclinations it is impossible to assign an arbitrary distribution to $\hat{\mathbf{V}}$, and this should correspond to a nonequilibrium situation. Any departure from the framework of the model (15) leads to interaction. In the given case, this is the account of the dipole energy (9) and the account of the terms of fourth order in the expansion in gradients in (4). We first consider an isolated disclination at the coordinate origin. For disclinations with $q=\frac{1}{2}$, $E_D\propto R$, $q=1$, $E_D\propto\ln(R/\lambda)$, the dipole energy diverges with depature from the axis. It can be made finite if we take into account the deformation of l (and for this, the vortex should not necessarily have two quantum numbers). At $q>1$,

$$E_{D_1}=\lambda^2 g_D\frac{4\pi^2}{q^2}\frac{1}{\sin(\pi/q)}. \tag{18}$$

For two disclinations, found at distances $2a$ from the x axis,

$$w_1=\left(z-\frac{a}{\lambda}\right)\left(z+\frac{a}{\lambda}\right).$$

at $a\ll\lambda$,

$$E_D=E_{D2}+\lambda^2 g_D\frac{\pi^2}{4}\left(\frac{a}{\lambda}\right)^4. \tag{19}$$

Attraction at small distances can be understood from Eq. (18), since a decrease in E_D takes place with increase in q.

At $a\gg\lambda$

$$E_D=\frac{\pi g_D\lambda^4}{a^2}\ln\frac{a}{\lambda}. \tag{20}$$

Repulsion at large distances takes place because of the effective decrease in the scale λ at $a \gg \lambda$, $\lambda_{eff} \sim \lambda^2/a$. Because of the absence of a principle of superposition for m disclinations, the approximation of pair interaction is inapplicable. At $a \gg \lambda$, for not too large m, the relation

$$E_D \propto (a/\lambda)^{-2(m-1)} \ln (a/\lambda)$$

is valid.

Account of the terms of fourth order in (4) at $a \gg \lambda$ leads to attraction of the vortices

$$E \sim \alpha \xi^2 a^2/\lambda^4 \sim \alpha \xi^2/\lambda_{eff}^2 .$$

For $q \sim 1$, these terms give the correction $\Delta E \propto q^2$, and for $q \gg 1$, $\Delta E \propto q^3$, which leads to repulsion of the vortices at small distances. Concurrent account of terms of fourth order and the dipole energy give an estimate $\lambda_0 \sim (R_D \xi)^{1/2}$, $R_D^2 \equiv \alpha/g_D$ for the optimum scale of the vortices.

For the model of a planar Heisenberg magnet, the fourth order terms give a single mechanism of violation of the conformal invariance. In this case, we would have had $\lambda_0 \gtrsim b(J/T)^{1/2}$, b is the lattice constant, J the constant of interaction, T the temperature. Qualitatively, this corresponds to the assumption advanced by Belavin and Polyakov, [3] on the loss of long-range order in vortex formations.

By virtue of the invariance $\varphi \to -\varphi$, conformal mappings with the functions

$$w_2 \equiv \operatorname{ctg} (\theta/2) \exp (-i\varphi) \qquad (21)$$

are also achievable. The joint use of w_1 and w_2 allows us to expand the class of single-valued solutions.

For vortex solutions in 1, we first consider the very simple case $\mathbf{l} \parallel \hat{\mathbf{V}}$ of vortices with two quantum numbers. The importance of this case is made clear in the following. At $\mathbf{l} \parallel \hat{\mathbf{V}}$, the term which depends only on 1 in (4) is again brought to the form (15). Following de Gennes, [1] we shall call the vortex solutions in 1 disgyrations.

We consider an isolated disgyration at the origin. We shall seek a solution in the form

$$\theta = \theta(r), \quad \varphi = m\chi + \varphi_0, \qquad (22)$$

where χ is the polar angle, m an integer. It follows from (7) that $v_{sz} = 0$. It is clear from symmetry conditions that $\mathbf{v}_s$ is parallel to the circulation. From the condition (11), we obtain the result that either $m = 1$, $\varphi_0 = 0$, or $\theta(r)$ is a function of determined form. The first solution has the lesser energy. Taking the condition $\Gamma = 2\pi$, into account, we obtain, finally,

$$v_{sx} = \frac{y}{2r^2}(1+\cos\theta), \quad v_{sy} = -\frac{x}{2r^2}(1+\cos\theta), \quad w_1 = z. \qquad (23)$$

The Bohr–Sommerfeld quantization rule is satisfied for the solution (23) only asymptotically as $r \to \infty$. It is effectively replaced by quantization in phase space θ, φ. The vortex energy at $R \gg \lambda$ is

$$E \approx 4\pi\alpha \ln \frac{R}{\lambda} + 9, \qquad (24)$$

where R is the dimension of the system. It follows from (24) that $\lambda \sim R$. As is seen from (5) and (23), near the center of the vortex, the current has no singularity. The total orbital momentum of the state (23) will be composed of the elementary orbital momenta of the pairs

$$\mathbf{L}_a \propto (\hbar\rho_s/m) \int \mathbf{l}\, d^2\mathbf{r}$$

and the orbital momentum associated with $\mathbf{j}_s$

$$\mathbf{L}_c \propto \int [\mathbf{r} \times \mathbf{j}_s]\, d^2\mathbf{r} .$$

The angular momenta $\mathbf{L}_a$ and $\mathbf{L}_c$ are of the same order of magnitude and have different signs.

Like the vortices in He II, the disgyrations (23) can be obtained by rotating the system. If the condition $\hat{\mathbf{V}} = \text{const}$ were to be satisfied, then for the creation of vortices there would be a threshold value ω_0, determined by the dipole energy, $\omega_0 \sim 1/mR_D^2 \sim 10^3$ rad-sec^{-1}. For solutions with $\mathbf{l} \parallel \hat{\mathbf{V}}$, the barrier, which is associated with the dipole energy, is absent, and vortices are formed starting with the values

$$\omega_{c_1} \sim (1/m\lambda^2) \ln R/\lambda .$$

For the vortices (14) we have

$$\omega_{c_1} \sim (1/mR^2) \ln (R/\xi).$$

The most direct method of obtaining disgyrations consists of threading a fine filament through the superfluid He^3. (It can be assumed that at infinity the orientation of 1 is specified by the magnetic field and the dipole energy.) In the case of sufficiently rapid motion of the filament, the vector 1 will be oriented along the motion. [16] The solution considered by Mermin and Ho [9] represents an example of disgyrations with half-integer order of mapping.

The classification of the solutions by means of quantum numbers was obtained through the use of the functional (4). Since solutions of the type of spin disclinations and disgyrations are singular only in the mathematical sense and do not have physical singularities (in the current distribution, in the spin density, and so on), the expansion (4) is applicable for them everywhere, including the core of the vortex. For the vortices (14), beginning with distances $r \lesssim \xi$, the order parameter goes to zero, and the expansion (4) ceases to be valid. Nevertheless, it can be assumed practically throughout the entire region that its interaction is described by the hydrodynamic approximation (4). In fact, for example, the value of the frequency ω_{c_2} at which the interaction of the vortex cores begins to be significant,

$$\omega_{c_1} \sim 1/m\xi^2(T) \sim 10^7 \tau \text{ rad-sec}^{-1}.$$

is unachievable for mechanical frequencies, with the exception of the extremely narrow region of T_c. Therefore, the physical picture, based on the introduction of quantum numbers, will be valid in practice throughout the entire region of interest to us.

3. TYPES OF VORTICES IN THE *B* PHASE

The order parameter for the B phase has the form

$$A_{pi} = \exp(i\varphi) R_{pi}. \tag{25}$$

where φ is the phase of the wave function, and R_{pi} is the matrix of three-dimensional rotation, which can be specified with the help of the axis n and the angle of rotation θ.

In the inhomogeneous state, φ, n and θ depend on r:

$$F_s = \alpha \int \{5(\nabla\varphi)^2 + 4(\nabla\theta)^2 - 2(\mathbf{n}\nabla\theta)^2 + 8\nabla\theta[\mathbf{n}\times\operatorname{rot}\mathbf{n}]\sin\theta + 4\sin\theta(\mathbf{n}\nabla\theta)\operatorname{div}\mathbf{n} - 4(1-\cos\theta)(\nabla\theta\operatorname{rot}\mathbf{n}) - 4\sin\theta(1-\cos\theta)(\mathbf{n}\operatorname{rot}\mathbf{n})\operatorname{div}\mathbf{n} - 4(1-\cos\theta)(\mathbf{n}\nabla\theta)(\mathbf{n}\operatorname{rot}\mathbf{n}) + 2(1-\cos\theta)(3-\cos\theta)(\operatorname{div}\mathbf{n})^2 - 2(1-\cos\theta)(3+\cos\theta)(\mathbf{n}\operatorname{rot}\mathbf{n})^2 + 8(1-\cos\theta)[\mathbf{n}\times\operatorname{rot}\mathbf{n}]^2\} d^3\mathbf{r}, \tag{26}$$

$$E_D = g_D \int (\cos\theta + 2\cos^2\theta) d^3\mathbf{r}. \tag{27}$$

$$\mathbf{j}_s = \frac{1}{3}\nabla\varphi. \tag{28}$$

As estimates show, the action of the centrifugal forces is too small to produce a transition from the B phase to the A phase. Therefore, the vortex current states in the B phase are entirely similar to the usual ones (whereupon the center of the vortex will not consist of A phase).

The only mechanism which could lead to the formation of vortices along θ and n is the orienting action of the surface. If the effect of the surface enters only as a common factor in (25) [17] (which should be rigorously based on the microtheory) then the existence of such vortex solutions will only be theoretically possible. However, we shall consider this case, keeping in mind the application to the theory of nematic liquid crystals.

A qualitative picture of the vortex solution in θ can easily be obtained if we take into account the smallness of the dipole energy in comparison with $F_s(R_D \gg \xi)$. At distances $\xi \ll r \ll R_D$, the dipole energy can be neglected, and the usual vortex solution is obtained. If we introduce the two-dimensional vector ν in place of the angle θ

$$\nu = (\sin\theta, \cos\theta).$$

then the picture of the distribution of ν is equivalent to flow of a laminar flow around the vortex.

For the investigation of disclinations in n, we consider first a functional of the form (the Oseen–Frank functional)

$$F = \int \{K_{11}(\operatorname{div}\mathbf{n})^2 + K_{22}(\mathbf{n}\operatorname{rot}\mathbf{n})^2 + K_{33}[\mathbf{n}\times\operatorname{rot}\mathbf{n}]^2\} d\mathbf{r}. \tag{29}$$

For the case of an individual disclination, we can consider the n distribution to be radially symmetric and seek a solution in the form (22). For solutions of such a form, we have

$$\operatorname{div}\mathbf{n} = \left(\frac{m\sin\theta}{r} - \cos\theta\frac{\partial\theta}{\partial r}\right)\cos(\varphi-\chi), \tag{30}$$

$$(\mathbf{n}\operatorname{rot}\mathbf{n}) = \left(\frac{m\cos\theta\sin\theta}{r} - \frac{\partial\theta}{\partial r}\right)\sin(\varphi-\chi). \tag{31}$$

We first consider the case of integral q (17). As r changes from zero to infinity, the angle θ changes by π; therefore, terms of the type

$$\frac{\sin\theta\cos\theta}{r}\frac{\partial\theta}{\partial r}$$

will vanish upon integration over r, while the addition of some function

$$\frac{f(\theta)}{r}\frac{\partial\theta}{\partial r}$$

can always reduce the energy to the sum of a constant term and a square of some function dependent on r. As in (15), this allows us to write down immediately the energy for the given class of solutions

$$m=1,\ E_s = 4\pi K_{22}\int_{-1}^{1}(1+a\cos^2\varphi_0+bu^2\sin^2\varphi_0)^{1/2}(1+au^2\cos^2\varphi_0+b\sin^2\varphi_0)^{1/2}du, \tag{32}$$

$$m\neq 1,\ E_s = 4\pi|m|K_{22}\int_{-1}^{1}\left(1+\frac{1}{2}a+\frac{1}{2}bu^2\right)^{1/2}\left(1+\frac{1}{2}au^2+\frac{1}{2}b\right)^{1/2}du, \tag{33}$$

$$a = \frac{K_{11}-K_{33}}{K_{33}}, \qquad b = \frac{K_{22}-K_{33}}{K_{33}}.$$

It is seen from (32), (33) that it can always be arranged by choice of the constant φ_0 that the sum of the energies of individual disclinations is smaller than the energy of a single disclination with $m = k$. Here, if $K_{22} > K_{11}$, then $\varphi_0 = 0, \pi$, i.e., radial disclinations are more favorable, and for $K_{22} < K_{11}$ we have $\varphi_0 = \pm\pi/2$, i.e., circular disclinations. For $K_{11} = K_{22}$, we have neutral equilibrium in relation to these two types. In contrast with the self-similar solutions, [8] the energy of the volume disclinations depends linearly on the Frank index. We also note that even in the limit of as large a value of the coefficient K_{22} as desired, existence of volume solutions is possible. The explicit form of the function $\theta(r)$ depends on the different relations among the coefficients K_{ii}.

We now obtain expressions for the commonest case $a < 0$, $b < 0$. They can be obtained from the formula

$$\left\{(1+\delta-\alpha)^{1/2}\left[\ln\frac{1+u}{1-u} + \ln\frac{[(1+\delta-\alpha)(1+\delta-\alpha u^2)]^{1/2}+1+\delta-\alpha u}{[(1+\delta-\alpha)(1+\delta-\alpha u^2)]^{1/2}-1-\delta+\alpha u}\right] + 2\alpha^{1/2}\arcsin\frac{u}{[1+(1+\delta-\alpha)^{-1}\alpha]^{1/2}}\right\}\gamma = \pm\ln\frac{r}{\lambda}, \tag{34}$$

where $u = \cos\theta$ and λ is an arbitrary scale factor.

For $K_{22} > K_{11}$ we have

$$\delta=0,\ \alpha=|a|,\ \gamma=1/2(1-|a|)^{1/2}.$$

In the case $K_{22}<K_{11}$, we must differentiate the left side of (34) with respect to δ and then set

$$\delta=0,\ \alpha=|b|,\ \gamma=(1-|b|)^{1/2}.$$

Choice of the solution in the form (22) corresponds to an assumption on a certain definite fixing of the disclination density. It is therefore necessary to test the solution for stability relative to the transformation $\mathbf{n} \to \hat{R}\mathbf{n}$, where $\hat{R}$ is the rotation operator. In the general case, the calculations are extremely cumbersome; however, for small a and b the stability can be proved; here the radial and circular disclinations are stable also relative to a change in the radial and circular character, respectively. We shall assume that, far from the disclocation centers, the vector n tends to a constant value $n_z \to 1$ (or $\theta \to 0$). In the first approximation, we can substitute $\sin\theta \approx \theta$, $\cos\theta \approx 1$. Then, under the condition of conformal invariance (15), divn and $(\mathbf{n}\cdot\mathrm{curl}\mathbf{n})$ vanish with accuracy to terms of higher order in θ.[1)] Thus, the functional (29) has the property of asymptotic conformal invariance; therefore, we can write

$$w_1=(z-z_1^{(0)})\ldots(z-z_n^{(0)}). \tag{35}$$

for the disclination lattice. Equation (35) is valid under the conditions $|z-z_k^{(0)}|\gg 1$, $|z_i^{(0)}-z_k^{(0)}|\gg 1$.

For the case of half-integer q, two types of solutions are possible. For the first type, θ changes by π and $m=q$, for the second type, θ changes by $\pi/2$ and $m=2q$. In the first case, we can use Eq. (33). Solutions of the second type can be constructed only for bounded systems in a case of a special choice of scale λ. We consider a cylindrical vessel of radius R. We can then set $\lambda=R$ in (34). In this case, the energy is equal to

$$E=\frac{1}{2}E_0-2\pi K_{33}(a\cos^2\varphi_0-b\sin^2\varphi_0),\qquad m=1. \tag{36}$$

$$E=\frac{1}{2}E_0-2\pi mK_{33}\frac{(a+b)}{2},\qquad m\neq 1. \tag{37}$$

In (36) and (37), the energy E_0 is determined by (32) and (33), respectively. If we set $\lambda\sim\xi$ in (34), then the second components in (36) and (37) will change sign.

We now consider the functional (26) in the case of the equilibrium value of θ, $\cos\theta_0=-\frac{1}{4}$. In this limit, the functional was also obtained by Brinkman *et al.*[17] We limit ourselves to the case of integral q only. If $m\neq 1$, then we again obtain a functional of the type (29). In the case $m=1$,

$$E=4\pi\alpha\int_{-1}^{1}du[16-(1\overline{3}\cos\varphi_0 u+1\overline{5}\sin\varphi_0)^2]^{1/2}[16-(1\overline{3}\cos\varphi_0+1\overline{5}\sin\varphi_0 u)^2]^{1/2}$$
$$\pm\frac{16\overline{15}}{3}\pi\alpha\sin 2\varphi_0. \tag{38}$$

The plus sign should be chosen in (38) if the angle θ changes from π to 0 when r changes from 0 to ∞, and the minus sign if θ changes from 0 to π. The value for minimizing $\varphi_0^{(\min)}$ can be found only numerically; however, it is not difficult to show that for the plus sign, the optimal value $\varphi_0^{(\min)}$ lies between

$$\frac{3\pi}{4}<\varphi_{01}^{\min}<\frac{\pi}{2},\qquad -\frac{\pi}{2}<\varphi_{02}^{\min}<-\frac{\pi}{4},$$

and for the minus sign,

$$-\frac{3\pi}{4}<\varphi_{01}^{\min}<-\frac{\pi}{2}\qquad \frac{\pi}{4}<\varphi_{02}^{\min}<\frac{\pi}{2}.$$

In both cases, the energies are identical. Thus, for disclinations in the B phase, there is correlation between the direction of change of n along the radius and the value of the circulation. Considerations on the asymptotic conform invariance are applicable, finally, to the functional (26) also.

4. CONCLUSION

As the results of the research show, the class of vortex solutions in superfluid He^3 is much richer than in He II. However, we can establish definitely the presence only of solutions of the disgyration type. All the consideration has been carried out on the basis of the generally accepted model of A and B phases. Thus, the spherical solutions assumed by Dribinskiĭ and Zelevinskiĭ[18] were not considered. We have also not considered the more complicated problem of the Ginzburg–Landau expansion for the isotropic solutions with the total angular momentum $J=1, 2$.[19,20]

As was already noted in the Introduction, the study of vortex states could give information on the character of the ground state. The possibility of direct experimental observation of the states proposed for the classical solutions in field theory is also of interest.

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[1)] This also follows from the equivalence of the vanishing of divn and $(\mathbf{n}\cdot\mathrm{curl}\mathbf{n})$ and the analyticity of $\exp(i\varphi)/\sin\theta$ and $\exp(i\varphi)\cot\theta$.

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Electronic properties of nondegenerate strongly doped compensated semiconductors

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A theory of the electronic properties of strongly doped compensated (SDC) semiconductors is developed for finite temperatures, and in particular for the case when the electron gas in the entire semiconductor volume may be regarded as nondegenerate. It is shown that in the nondegenerate case and in the impurity conductivity region the characteristic size of the electron "drops" in the fluctuation potential wells increases with the temperature T initially as $\sim T^{-\frac{1}{2}}$ and subsequently as $\sim T^{-\frac{1}{3}}$. The density of states $g(\epsilon)$ in the vicinity of the initial (prior to doping) bottom of the conduction band is found in an energy range of the order of the characteristic depth ζ_0 of the largest fluctuation potential wells that still remain unscreened at $T=0$. In this energy range, $g(\epsilon)\sim\epsilon^{-\frac{1}{2}}$ for the chosen shape of the potential wells. The position of the Fermi level, the mean energy, and the specific heat of the electron gas in SDC semiconductors are calculated as functions of temperature in the range $0\leq t\leq\zeta_0$, by using the obtained form of the density of states. Some arguments are presented regarding the temperature dependence of the electric conductivity of SDC semiconductors in the indicated temperature range.

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The study of the electric properties of strongly doped compensated (SDC) semiconductors has recently been the subject of a number of both theoretical and experimental papers (see, e.g., [1–11]). Interest in these materials is due, first, to the fact that they constitute apparently one of the simplest examples of disordered solids, the electronic structure of which has not yet been sufficiently well studied. Second, SDC semiconductors are of interest from the practical point of view, for example for the development of most sensitive detectors of infrared and submillimeter radiation.[9]

In the customarily accepted quite obvious model of an SDC semiconductor, say of n-type, it is assumed that the free electrons remaining as a result of incomplete compensation, and numbering nV ($n=N_d-N_a$ is the electron density averaged over the crystal volume V, while N_d and N_a are the donor and acceptor densities averaged over the volume V) fill the deepest regions of the potential relief of the bottom of the conduction band, produced by the fluctuating potential of the impurities. The concentration of the electrons in these so-called electron drops, n_d, under conditions of strong compensation, greatly exceed the average density n and can be of the order of the density of the doping impurity (for example, 10^{15}–10^{16} cm^{-3} for n-InSb[9]). Thus, at helium temperatures the electron gas in the drops in SDC semiconductors can be regarded as degenerate. This is precisely the assumption under which Shklovskiĭ and Éfros constructed a theory of SDC semiconductors.[6]

With increasing temperature, however, the assumption that the electron gas is degenerate in the drops can cease to be satisfied. Thus, for example, in n-InSb at $n\sim10^{15}$ cm^{-3} the degeneracy temperature amounts to approximately 30 °K. Thus, at temperatures on this order or higher the electron gas in the drops must apparently be regarded as degenerate. An appreciable number of the experimental data lie in this temperature region.[8,9]

In this paper we attempt to construct, using the theory of Shklovskiĭ and Éfros[6] as a model, a theory for several electronic properties of SDC semiconductors, when the electron gas in a semiconductor, including also the drops, can be regarded as nondegenerate. One of the most important parameters of the theory of Shklovskiĭ and Éfros[6] is the characteristic dimension of the electron drop $R_q=a/(Na^3)^{1/9}$ ($a=\hbar^2\varkappa/me^2$ is the Bohr radius, $\varkappa$ is the dielectric constant of the crystal, m and e are the respective mass and charge of the electron, $N\approx N_d, N_a$). This dimension is obtained from the condition that the number of the quantum states in the attracting fluctuation of the impurity concentration be equal to the excess charges in it.[6] Let us find this parameter formally by a somewhat different method.