Effect of trapped electrons on Alfvén waves in a tokamak

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The interaction between Alfvén waves and trapped electrons in a plasma located in a toroidal magnetic field is investigated theoretically. It is shown that the interaction may result in wave buildup at a finite ratio of the plasma and magnetic-field pressures. The growth rate depends on the compressibility of the trapped electrons and on effects due to the finite value of the longitudinal electric field of the perturbations. The range of phenomena investigated may be of interest from the viewpoint of thermonuclear tokamak reactors and also of the physics of the magnetosphere plasma.

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1. INTRODUCTION

Interest has increased of late in the problem of non-potential instabilities of a tokamak plasma. This is due to the ever improving understanding of the circumstance that to solve successfully the thermonuclear problem on the basis of tokamak reactors it is necessary to deal with a finite-pressure plasma, also called plasma with finite β (β is the ratio of the plasma pressure to the magnetic-field pressure).[1] That the non-potential instabilities assume a more important role when the parameter β is increased was pointed out, in particular, by one of us.[2] The concrete form of the manifestation of the non-potential instabilities in a tokamak plasma with finite β is the buildup of Alfvén waves. In the approximation in which the magnetic field lines are straight, the buildup of Alfvén waves in a collisionless plasma was first considered by one of us and Rudakov,[3] and in a collision-dominated plasma by one of us and Pogutse.[4,5] In both cases, the instability is due to the finite longitudinal electric field of the perturbation, $E_\parallel \neq 0$.

An important role under tokamak conditions is played by the trapping of the particles as a result of the corrugated configuration of the magnetic field. The particles are classified in this case as trapped and untrapped. The buildup of Alfvén waves by untrapped electrons was first investigated by one of us.[6] It was observed in addition[7] that under tokamak conditions Alfvén waves can be built up also by untrapped ions. An instability of this type is due to the compressibility of the ions in the toroidal magnetic field and the toroidal satellites of $E_\parallel$.

Until recently, the buildup of Alfvén waves by trapped electrons remained uninvestigated. The filling of this gap in the theory of non-potential instabilities of a tokamak plasma was initiated by one of us[8] and by Tang *et al.*[9] In[8] there was considered the buildup of Alfvén waves due to the compressibility of the trapped electrons. It was assumed that $E_\parallel = 0$. Tang *et al.*[9] to the contrary, neglected the compressibility of the trapped electrons, but took into account the fact that $E \neq 0$. In both papers,[8,9] the toroidal satellite of $E_\parallel$, which was considered in the case of ion buildup,[7] was likewise neglected. In contrast to[8], where the role of the toroidality of the magnetic field of the tokamak was investigated qualitatively, the analysis by Tang *et al.*, is based on a model, so that the relations obtained in[9] are in effect estimates.

In connection with the foregoing, it is of interest to develop a theory that takes into account in a unified manner the effects considered separately earlier,[8,9] as noted by Lominadze *et al.*,[10] and also the effect of the toroidal $E_\parallel$ satellite.[7] This is in fact the subject of the present paper.

In Sec. 2 we present the initial equations. In Sec. 3 we obtain a dispersion equation that takes into account all the effects listed above. In Sec. 4 it is shown that in the case of a plasma containing only hot electrons (without an admixture of cold electrons) the most important role is played by the effects connected with the longitudinal electronic field of the fundamental harmonic. In Sec. 5 we consider a plasma containing an admixture of cold electrons besides the hot electrons. It is shown that the cold electrons exert a stabilizing effect, as a result of which the longitudinal electric field (both of the fundamental harmonic and of the satellite) turn out to be small, and the compressibility effect comes to the forefront. The results are discussed in Sec. 6.

2. INITIAL EQUATIONS

We shall characterize the perturbations of the magnetic and electric fields by the quantities ψ and ξ:

$$\tilde{B}^a = (\mathbf{B}_0\nabla)\zeta, \quad E_\parallel = -\mathbf{e}_0\nabla\psi, \tag{2.1}$$

where $\mathbf{B}_0$ is the vector of the unperturbed magnetic field, $\mathbf{e}_0 = \mathbf{B}_0/B_0$, $\tilde{B}^a$ is the contravariant component of the perturbations of the magnetic field $\tilde{\mathbf{B}}$, and $E_\parallel = \mathbf{e}_0\mathbf{E}$ is the longitudinal component of the electric field. The coordinate and time dependences of the perturbed quantities are chosen in the form

$$\zeta = \hat{\zeta}\exp\{-i\omega t + im\theta - in\varphi + ik_a a\},$$
$$\psi = \sum_{j=-1,0,1} \hat{\psi}_j \exp\{-i\omega t + im\theta - in\varphi + ik_a a\} e^{ij\theta}. \tag{2.2}$$

The coordinates a, θ, and φ are respectively the distance of the running magnetic surface from the magnetic axis and the angular variables with period 2π along the major and minor azimuths of the tokamak. It is assumed that the perturbation is localized near a value $a = a_0$ such that

$$m-nq(a_0)=s\approx 1,\quad (m,n)\gg 1. \tag{2.3}$$

Here $q=aB_s/RB_\theta$, B_s and B_θ are respectively the toroidal and poloidal magnetic fields averaged over the magnetic surface, and R is the major radius of the tokamak. The reasons for neglecting the harmonics of ζ are given in[8].

It will be found convenient to introduce the quantities Φ, $A_\parallel$, and $\boldsymbol{\xi}$:

$$\mathbf{E}=-\nabla\Phi-\frac{1}{c}\frac{\partial A_\parallel}{\partial t}\mathbf{e}_0,$$

$$\tilde{\mathbf{B}}=-[\mathbf{e}_0\times\nabla A_\parallel],\quad \xi=ic[\mathbf{E}\times\mathbf{e}_0]/\omega B_0, \tag{2.4}$$

which, as can be easily shown, are connected with ζ and ψ by the following relations:

$$\Phi=\psi+\frac{\omega aB_s}{mc}\zeta,\quad \xi^a=\xi=\zeta+\frac{mc}{\omega aB_s}\psi,$$

$$A_\parallel=-\frac{ia}{m}\mathbf{B}_0\nabla\zeta. \tag{2.5}$$

As seen from (2.4), the quantities Φ, $A_\parallel$, and $\boldsymbol{\xi}$ denote respectively the electric potential, the longitudinal component of the magnetic potential, and the plasma displacement vector.

To obtain the dispersion equation, we start from the following scheme. We write down the expression for the perturbed current:

$$\mathbf{j}=\mathbf{j}_\perp+\frac{c}{4\pi}\mathbf{e}_0(\mathbf{e}_0\,\mathrm{rot}\,\tilde{\mathbf{B}}). \tag{2.6}$$

Taking the divergences of both halves of (2.6) and using the equations $\mathrm{div}\,\mathbf{j}=0$ and $-i\omega\tilde{\mathbf{B}}=-c$ curl $\tilde{\mathbf{E}}$, and also the relations (2.1) and (2.4), we obtain the equation

$$\mathrm{div}\,\mathbf{j}_\perp=-i\frac{c^2}{4\pi}k_\parallel^2k_\perp^2\frac{aB_s}{mc}\zeta, \tag{2.7}$$

$$k_\perp^2=k_b^2+k_a^2,\quad k_b=m/a,\quad k_\parallel=(m-nq)/qR.$$

We assume that the perturbations are strongly elongated along the field, i.e., $k_\parallel/k_\perp\ll 1$, and therefore replace n by m/q where possible.

The expression for $\mathrm{div}\mathbf{j}_\perp$ is obtained from the drift kinetic equations for the electrons and ions[8,11]:

$$\frac{d_jf_j}{dt}+\mathbf{V}_E\nabla F_j+v_\parallel\frac{\tilde{\mathbf{B}}}{B_0}\nabla F_j+\frac{M_j(\varepsilon_\perp+v_\parallel^2)}{2T_j}F_j\,\mathrm{div}\,\mathbf{V}_E$$

$$-\frac{e_jE_\parallel v_\parallel}{T_j}F_j+i\frac{M_jc^2\nabla^2\Phi}{e_jB_s^2}F_j(\omega-\omega_{pj}^*)=C_j. \tag{2.8}$$

Here f_j is the perturbation of the distribution function of the particles of sought j, F_j is the equilibrium distribution function, which is assumed to be Maxwellian with temperature $T_j(a)$ and density $N_0(a)$, while e_j and M_j are the charge and the mass of the particle

$$\omega_{pj}^*=\frac{mcT_j}{e_jB_sa}\frac{d\ln p_{0j}}{da},\quad p_{0j}=N_0T_j,$$

$$\frac{d_j}{dt}=\frac{\partial}{\partial t}+(v_\parallel\mathbf{e}_0+\mathbf{V}_{Dj})\nabla,$$

$$\mathbf{V}_{Dj}=-\frac{\varepsilon_\perp+v_\parallel^2}{\omega_{Bj}}[\nabla\ln B_0\times\mathbf{e}_0],\quad \omega_{Bj}=\frac{e_jB_s}{M_jc},\quad \varepsilon_\perp=\frac{v_\perp^2}{2},$$

$v_\perp$ and $v_\parallel$ are the transverse and longitudinal velocities, and C_j is the collision term.

Integrating Eqs. (2.8) with respect to the velocities and summing over the sorts of particles, we obtain

$$\frac{\partial}{\partial t}\sum_j e_jn_j+\mathbf{e}_0\nabla j_\parallel+\frac{iM_ic^2}{B_s^2}\nabla^2\Phi N_0(\omega-\omega_{pi}^*)$$

$$+\sum_j e_j\int\frac{\varepsilon_\perp+v_\parallel^2}{\omega_{Bj}}[\mathbf{e}_0\times\nabla\ln B_0]\nabla f_j\,dv=0. \tag{2.9}$$

In the derivation of (2.9) we have neglected in the electron equation (2.8) the last term, which is proportional to the electron mass M_e. Using the quasineutrality condition

$$\sum_j e_jn_j=0 \tag{2.10}$$

and the equation

$$\mathrm{div}\,\mathbf{j}=\mathbf{e}_0\nabla j_\parallel+\mathrm{div}\,\mathbf{j}_\perp=0,$$

we obtain

$$\mathrm{div}\,\mathbf{j}_\perp=i\frac{M_ic^2}{B_s^2}\nabla_\perp\Phi N_0(\omega-\omega_{pi}^*)+\frac{c}{B_s}\nabla(\tilde{p}_e+\tilde{p}_i)[\mathbf{e}_0\times\nabla\ln B_0] \tag{2.11}$$

$$\tilde{p}_j=M_j\int\frac{\varepsilon_\perp+v_\parallel^2}{2}f_j\,d\mathbf{v}. \tag{2.12}$$

Substituting (2.11) in (2.7) and taking m-th harmonic of the resultant equation, we get

$$\omega(\omega-\omega_{pi}^*)\hat{\xi}_0-c_A^2k_\parallel^2\hat{\zeta}=\frac{k_b}{k_\perp}\frac{i}{R\rho}(\hat{p}_{+1}e^{-i\alpha}-\hat{p}_{-1}e^{i\alpha}), \tag{2.13}$$

$c_A^2=B_s^2/4\pi M_iN_0$, $\rho=M_iN_0$, $\tan\alpha=k_b/k_a$, $\hat{p}_{\pm1}$ and $\hat{\xi}_0$ are the corresponding amplitudes of the harmonics of $\tilde{p}$ and ξ (cf. (2.2)).

Equations (2.13) and (2.10) make up the initial system of equations.

3. DERIVATION OF THE DISPERSION EQUATION

To calculate the perturbed quantities $\tilde{n}_j$ and $\tilde{p}_j$ it is necessary to solve Eq. (2.8).

a) *Ions*. The trapping of the ions is inessential, since the ion velocities are small in comparison with the Alfvén velocity, so that the ions cannot traverse the distance between the reflection points during one period of the field oscillation. Assuming that $\omega\gg k_\parallel v_\parallel$, $k_\perp V_D$, and $C_i=0$, we get

$$f_i=-F_i'\xi+\left(1-\frac{\hat{\omega}_i^*}{\omega}\right)\frac{M_i(\varepsilon_\perp+v_\parallel^2)}{RT_i}F_i\left(\xi\cos\theta-i\frac{a}{m}\frac{\partial\xi}{\partial a}\sin\theta\right)$$

$$+\frac{M_ic^2}{eB_s^2}F_i\nabla^2\Phi\left(1-\frac{\omega_{pi}^*}{\omega}\right), \tag{3.1}$$

$$\hat{\omega}_i^*=\frac{mcT_i}{eaB_s}\frac{d\ln F_i}{da}.$$

The prime denotes differentiation with respect to a.

We obtain therefore for the harmonic of the perturbed ion density

$$\left(\frac{\hat{n}_i}{N_0}\right)_0=-\frac{N_0'}{N_0}\hat{\xi}_0-\frac{k_\perp^2}{k_b}\frac{\omega}{\omega_{Bi}}\left(1-\frac{\omega_{pi}^*}{\omega}\right)\hat{\xi}_0+\frac{1}{i}\left(1-\frac{\omega_{pi}^*}{\omega}\right)\frac{k_\perp}{k_bR}(\hat{\xi}_{-1}e^{i\alpha}+\hat{\xi}_{-1}e^{-i\alpha}), \tag{3.2}$$

$$\left(\frac{n_i}{N_0}\right)_{\pm1}=-\frac{N_0'}{N_0}\hat{\xi}_{\pm1}-\frac{k_\perp^2}{k_b}\frac{\omega}{\omega_{Bi}}\left(1-\frac{\omega_{pi}^*}{\omega}\right)\hat{\xi}_{\pm1}\pm\frac{1}{i}\left(1-\frac{\omega_{pi}^*}{\omega}\right)\frac{k_\perp}{k_bR}\hat{\xi}_0e^{\mp i\alpha}.$$

The contribution made to the perturbed pressure of the ions will be taken into account only by using the first term of (3.1). It can be shown that the remaining terms lead in the case $k_\parallel\approx 1/qR$ only to a correction to the real part of the frequency. We have

$$(\hat{p}_i)_{\pm1}=-p_0'\hat{\xi}_{\pm1}=-aeB_sN_0\omega_{pi}^*\hat{\xi}_{\pm1}/mc. \tag{3.3}$$

b) *Electrons*. It is more convenient to rewrite Eq. (2.8) for the hot component of the electrons in the form

$$\frac{d_ef_e}{dt}=-F_e'\frac{d_e\xi}{dt}+\frac{e}{T_e}F_e\frac{d_e\psi}{dt}+i\frac{eF_e}{T_e}(\omega-\hat{\omega}_e^*)\psi-i(\omega-\hat{\omega}_e^*)M_eF_e(v_\parallel^2+\varepsilon_\perp)\frac{k_\perp}{k_bRT_e}\xi\sin(\theta+\alpha)+C_e(f_e). \tag{3.4}$$

We break up f_e into two parts

$$f_e=f^{(1)}+f^{(2)}, \tag{3.5}$$

where

$$f^{(1)}=-F_e'\xi+eF_e\psi/T_e, \tag{3.6}$$

and $f^{(2)}$ satisfies the equation

$$\left(\frac{\partial}{\partial\theta}-inq\right)f^{(2)}=\frac{qR}{v_\parallel}\left\{i\omega f^{(2)}+C(f^{(2)})+i(\omega-\omega_e^*)\frac{eF_e}{T_e}\psi-i(\omega-\omega_e^*)\frac{E}{T_e}\frac{F_e}{R}\frac{k_\perp}{k_b}\xi\sin(\theta+\alpha)\right\},\quad \hat{\omega}_e^*=-\frac{mcT_e}{eaB_s}\frac{d\ln F_e}{da}. \tag{3.7}$$

Here E is the electron energy.

We seek the solution of (3.7) for the trapped electrons by a method similar to that described earlier.[8] Assuming that during the period of the oscillations of the field and during the time of the collisions the electrons pass along the force line many times, we shall regard $qR/v_\parallel$ as a small parameter. We can seek the solution of (3.7) in the form of an expansion in the small parameter: $f^{(2)}=f_0^{(2)}+f_1^{(2)}$. In the zeroth order we obtain $f_0^{(2)}=He^{inq\theta}$, where H does not depend on θ. From the condition that the right-hand side of (3.7) be orthogonal to the zeroth-approximation solution[8] we obtain

$$\oint\frac{d\theta}{v_\parallel}\left\{i\omega H+C(H)+i\frac{e}{T_e}F_e(\omega-\omega_e^*)\psi e^{-inq\theta}-i(\omega-\omega_e^*)\frac{E}{T_e}\frac{F_e}{R}\frac{k_\perp}{k_b}\xi\sin(\theta+\alpha)e^{-inq\theta}\right\}=0. \tag{3.8}$$

We break up H into two terms:

$$H=H_1+H_2, \tag{3.9}$$

which are connected with the perturbations of ξ and ψ:

$$\oint\frac{d\theta}{v_\parallel}\left\{i\omega H_1+C(H_1)-i(\omega-\hat{\omega}_e^*)\frac{E}{T_e}\frac{F_e}{R}\frac{k_\perp}{k_b}\sin(\theta+\alpha)e^{-inq\theta}\right\}=0,\quad \oint\frac{d\theta}{v_\parallel}\left\{i\omega H_2+C(H_2)+i\frac{e}{T_e}(\omega-\hat{\omega}_e^*)\psi e^{-inq\theta}\right\}=0. \tag{3.10}$$

Next, we introduce h and g_j defined by the relations

$$H_1=\frac{k_\perp}{k_bR}F_eh\hat{\xi},\qquad H_2=\sum_{j=-1,0,1}F_e\frac{e}{T_e}g_j\hat{\psi}_j. \tag{3.11}$$

From (3.10) we find that h and g_j satisfy the equations

$$\omega h-i\bar{C}(h)=(\omega-\hat{\omega}_e^*)ES(\lambda)/T_eL(\lambda),\quad \omega g_j-i\bar{C}(g_j)=-(\omega-\hat{\omega}_e^*)M_j(\lambda)/L(\lambda), \tag{3.12}$$

where, in analogy with[8,12,13], we put

$$L(\lambda)=\oint\frac{d\theta}{\sigma(1-\lambda B)^{1/2}},\quad M_j(\lambda)=\oint\frac{d\theta e^{is\theta}}{\sigma(1-\lambda B)^{1/2}}e^{ij\theta},$$

$$S(\lambda)=\oint\frac{d\theta}{\sigma(1-\lambda B)^{1/2}}\sin(\theta+\alpha)e^{is\theta},\quad \lambda=\frac{\mu}{E},$$

$$\bar{C}(X)=2\nu\left(\frac{T_e}{E}\right)^{3/2}\frac{1}{L(\lambda)}\frac{\partial}{\partial\lambda}\left[D(\lambda)\frac{\partial X}{\partial\lambda}\right],$$

$$D(\lambda)=\lambda\oint\frac{d\theta}{\sigma B}(1-\lambda B)^{1/2},$$

$$\nu=\bar{\nu}[1+G(\sqrt{E/T_e})],\quad \bar{\nu}=\omega_{pe}^2e^2M_e^{1/2}\Lambda/(2T_e)^{3/2},$$

$$G(z)=\frac{1}{z\sqrt{\pi}}e^{-z^2}+\left(1-\frac{1}{2z^2}\right)\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt,$$

$\omega_{pe}^2=4\pi N_0e^2/M_e$, Λ is the Coulomb logarithm, and $\sigma=\pm1$ indicates the direction of the longitudinal motion of the particle.

From (3.9) we get

$$f_{\text{trap}}^{(2)}=e^{inq\theta}\left\{\frac{k_\perp}{k_bR}F_eh\hat{\xi}+\frac{F_ee}{T_e}\sum_j g_j\hat{\psi}_j\right\}\exp\{-i\omega t+ik_a a-in\varphi\}. \tag{3.13}$$

For the untrapped particles $f^{(2)}$ is equal to zero. Therefore the boundary conditions for (3.12) are[8,13]

$$h\left(\lambda=\frac{1}{B_{max}}\right)=0,\quad \frac{\partial h}{\partial\lambda}\left(\lambda=\frac{1}{B_{min}}\right)<\infty. \tag{3.14}$$

From (3.5), (3.6), (3.9), and (3.11) we obtain expressions for the harmonics of the perturbed electron density

$$\left(\frac{n_e}{N_0}\right)_0=-\frac{N_0'}{N_0}\hat{\xi}+\frac{e}{T_e}\hat{\psi}_0+\left(\frac{n_e}{N_0}\right)_0',\quad \left(\frac{\hat{n}_e}{N_0}\right)_{\pm1}=\frac{e}{T_e}\hat{\psi}_{\pm1}+\left(\frac{n_e}{N_0}\right)_{\pm1}', \tag{3.15}$$

where

$$\left(\frac{n_e}{N_0}\right)_0'=\frac{1}{8\pi}\left\langle\int d\lambda B_sM_0^*\left(\frac{k_\perp}{k_bR}h\hat{\xi}+\frac{e}{T_e}\sum_j g_j\hat{\psi}_j\right)\right\rangle,\quad \left(\frac{n_e}{N_0}\right)_{\pm1}'=\frac{1}{8\pi}\left\langle\int d\lambda B_sM_{\pm1}^*\left(\frac{k_\perp}{k_bR}h\hat{\xi}+\frac{e}{T_e}\sum_j g_j\hat{\psi}_j\right)\right\rangle. \tag{3.16}$$

The asterisk denotes complex conjugation, while the integration with respect to λ is carried out from $1/B_{\max}$ to $1/B_{\min}$

$$\langle\ldots\rangle=\frac{2}{\sqrt{\pi}}\int_0^\infty\left(\frac{E}{T_e}\right)^{1/2}\exp\left(-\frac{E}{T_e}\right)(\ldots)d\frac{E}{T_e}.$$

For the harmonics of the pressure of the electrons we obtain with the aid of (2.12) and (2.13)

$$(\hat{p}_e)_{\pm 1}=N_0 e\psi_{\pm 1}+(\hat{p}_e)_{\pm 1}{}', \tag{3.17}$$

$$(\hat{p}_e)_{\pm 1}{}'=\frac{N_0 T_e}{16\pi}\left\langle \frac{E}{T_e}\int d\lambda\, B_s M_{\pm 1}{}^{\cdot}\left(\frac{k_\perp}{k_b R}h\hat{\xi}+\frac{e}{T_e}\sum_j g_j\hat{\psi}_j\right)\right\rangle. \tag{3.18}$$

We note that a factor $\frac{1}{4}$ was left out of the right-hand side of formula (4.1) of[8] for $\hat{p}$. Substituting (3.15) and (3.2) in the quasineutrality equation (2.10), we get

$$\frac{e}{T_e}\left(1-\frac{\omega_e^*}{\omega}\right)(\hat{\xi}_0-\hat{\zeta})+\frac{mc}{\omega a B_s}\left(\frac{n_e}{N_0}\right)_0'$$

$$=-\frac{ck_\perp^2}{\omega_{Bi}B_s}\left(1-\frac{\omega_{pi}^*}{\omega}\right)\hat{\xi}_0+\frac{1}{i}\frac{ck_\perp}{R\omega B_s}\left(1-\frac{\omega_{pi}^*}{\omega}\right)(\hat{\xi}_{-1}e^{i\alpha}-\hat{\xi}_{+1}e^{-i\alpha}), \tag{3.19}$$

$$\frac{e}{T_e}\left(1-\frac{\omega_e^*}{\omega}\right)\hat{\xi}_{\pm 1}+\frac{mc}{\omega a B_s}\left(\frac{\hat{n}_e}{N_0}\right)_{\pm 1}'=-\frac{ck_\perp^2}{\omega_{Bi}B_s}\left(1-\frac{\omega_{pi}^*}{\omega}\right)\hat{\xi}_{\pm 1}$$

$$\pm\frac{1}{i}\frac{ck_\perp}{R\omega B_s}\left(1-\frac{\omega_{pi}^*}{\omega}\right)\hat{\xi}_0 e^{\pm i\alpha}.$$

Here

$$\omega_e^*=-(mcT_e/eaB_s)\,d\ln N_0/da.$$

We have used the relation (2.5) between ξ, ζ, and ψ.

Using (3.19) to express ζ and $\xi_{\pm 1}$ in terms of ξ_0, and using next (3.3), (3.17), and (2.13), we get

$$Q_0(\omega)\hat{\xi}_0=\frac{\omega mc}{aB_s}\frac{T_e}{e}\Gamma\left(\frac{\hat{n}_e}{N_0}\right)_0'-i\frac{T_e}{RM_i}\frac{k_b}{k_\perp}\Gamma\left\{\left(\frac{\hat{n}_e}{n_0}\right)_{+1}'e^{-i\alpha}-\left(\frac{\hat{n}_e}{N_0}\right)_{-1}'e^{i\alpha}\right\}$$

$$+\frac{ik_b}{k_\perp R\rho}\{\hat{p}_{+1}{}'e^{-i\alpha}-\hat{p}_{-1}{}'e^{i\alpha}\}, \tag{3.20}$$

where

$$Q_0(\omega)=\omega(\omega-\omega_{pi}^*)-c_A{}^2k_\parallel{}^2(1+\tau z_i\Gamma), \tag{3.21}$$

$$\tau=\frac{T_e}{T_i},\quad z_i=\frac{k_\perp{}^2T_i}{M_i\omega_{Bi}{}^2},\quad \Gamma=\frac{1-\omega_{pi}^*/\omega}{1-\omega_e^*/\omega}.$$

The right-hand side of (3.20) must be expressed in terms of $\hat{\xi}_0$. To this end we use relations (3.19), (3.16), and (3.18). Neglecting the terms that are quadratic in the trapped particles, we obtain the dispersion equation in the form

$$Q_0(\omega)=\frac{1}{8\pi}\left\langle B_s\int d\lambda\left\{\frac{c_s}{R}S^{\cdot}\left(\frac{E}{T_e}-2\Gamma\right)+\omega(\tau z_i)^{1/2}\Gamma M_0^{\cdot}\right\}\right.$$

$$\left.\times\left\{\frac{c_s}{R}h-\omega(\tau z_i)^{1/2}\Gamma g_0+\Gamma\frac{c_s}{R}G\right\}\right\rangle, \tag{3.22}$$

$$G=-i(g_{+1}e^{i\alpha}-g_{-1}e^{-i\alpha}),\quad c_s{}^2=T_e/M_i.$$

We put

$$c_s/R\equiv d,\quad \omega(\tau z_i)^{1/2}\equiv b,$$

$$hd-g_0 b+\Gamma G d\equiv\Pi,\quad (E/T_e-2\Gamma)Sd+M_0 b=K. \tag{3.23}$$

We then find from (3.12) that the function Π satisfies the equation

$$\omega\Pi-i\bar{C}(\Pi)=(\omega-\hat{\omega}_e^*)K/L. \tag{3.24}$$

Thus, the problem has reduced to finding the growth rate due to the right-hand side of the equation

$$Q_0(\omega)=\frac{1}{8\pi}\left\langle B_s\int d\lambda\, K^{\cdot}\Pi\right\rangle, \tag{3.25}$$

where the function Π satisfies Eq. (3.24) with the boundary conditions

$$\Pi\left(\lambda=\frac{1}{B_{max}}\right)=0,\quad \frac{\partial\Pi}{\partial\lambda}\left(\lambda=\frac{1}{B_{min}}\right)<\infty. \tag{3.26}$$

Equations (3.24) and (3.25) are fully identical with the system (3.18) and (4.4) of[8]. We can therefore write down immediately an expression for the instability growth rate:

$$\gamma=-\frac{1}{\partial Q_0/\partial\omega}\frac{(2\varepsilon)^{1/2}}{\pi}\left\langle I\left(1-\frac{\hat{\omega}_e^*}{\omega}\right)\left\{d\left(\frac{E}{T_e}-2\Gamma\right)\sin\alpha+b\right\}^2\right\rangle, \tag{3.27}$$

where

$$I=\frac{1}{2^{1/2}}\xi_0\left(\ln\frac{16}{\xi_0}\right)^{1/2},\quad \xi_0=\left(\frac{2\nu}{\varepsilon\omega}\right)^{1/2}\left(\frac{T_e}{E}\right)^{3/4},\quad \varepsilon=\frac{a}{R},$$

or ultimately:

$$\gamma=-\frac{1}{2\pi}\left(\frac{\bar{\nu}}{\omega}\right)^{1/2}\ln^{1/2}\left\{\frac{128\varepsilon\omega}{\bar{\nu}}\right\}\left(\frac{\partial Q_0}{\partial\omega}\right)^{-1}\left\{d^2\sin^2\alpha I_1\left[1-\frac{\omega_e^*}{\omega}\left(1+\eta_e\frac{\tilde{I}_1}{I_1}\right)\right]\right.$$

$$+2dI_2\sin\alpha(b-2\Gamma\sin\alpha)\left[1-\frac{\omega_e^*}{\omega}\left(1+\eta_e\frac{\tilde{I}_2}{I_2}\right)\right]$$

$$\left.+(2d\Gamma\sin\alpha-b)^2I_3\left[1-\frac{\omega_e^*}{\omega}\left(1+\eta_e\frac{\tilde{I}_3}{I_3}\right)\right]\right\}, \tag{3.28}$$

$$\eta_e=\frac{d\ln T_e}{d\ln N_e},$$

$$I_j=\langle z^{9/4-j}[1+G(z)^{1/2}]^{1/2}\rangle;$$

$$\tilde{I}_j=\left\langle\left(z-\frac{3}{2}\right)z^{9/4-j}[1+G(z^{1/2})]^{1/2}\right\rangle;$$

$$\langle\ldots\rangle=\frac{2}{\sqrt{\pi}}\int_0^\infty z^{1/2}e^{-z}(\ldots)\,dz. \tag{3.29}$$

Calculation of the integral yields: $I_1\approx 2.47$, $I_2\approx 1.35$, $I_3\approx 1.61$, $\tilde{I}_1\approx 3.22$, $\tilde{I}_2\approx 0.44$, $\tilde{I}_3\approx -1.07$.

We note that in expression (5.29) of[8] for the growth rate the numerical coefficient has been incorrectly calculated; in addition, the figure under the logarithm sign in this formula should be 128 rather than 16. The correct replacing (5.29) of[8] is obtained from Eq. (3.28) of our paper by retaining in the latter only the terms that do not contain b and Γ. We note also that a factor $z^{-3/4}$ is missing from the right-hand side of formulas (5.30) of[8] but they contain the unnecessary factor $z^{1/2}$ instead.

4. ANALYSIS OF EXPRESSION (3.27)

Let us examine in greater detail the case of a zero gradient of the electron temperature $\eta_e=0$. We obtain from (3.28)

$$\gamma=-\frac{1}{2\pi}\left(\frac{\bar{\nu}}{\omega}\ln\frac{128\varepsilon\omega}{\bar{\nu}}\right)^{1/2}\left(\frac{\partial Q}{\partial\omega}\right)^{-1}\left(1-\frac{\omega_e^*}{\omega}\right)$$

$$\times\{I_1 d^2\sin^2\alpha+2I_2 d(b-2\Gamma d\sin\alpha)+I_3(b-2\Gamma d\sin\alpha)^2\}. \tag{4.1}$$

The quantity in the curly brackets, as follows from (3.27) is positive. Therefore the instability criterion coincides with that obtained earlier[8]

$$\omega<\omega_e^*, \tag{4.2}$$

$$\omega=\omega_{pi}^*/2+(\omega_{pi}^{*2}/4+c_A{}^2/q^2R^2)^{1/2}. \tag{4.3}$$

We have assumed for the sake of argument that $\omega_{pi}^* < 0$. Using (4.2) and (4.3) we can rewrite the condition for the buildup of Alfvén waves in the form

$$\left(\frac{qR\omega_{pi}^*}{c_A}\right)^2\left(1+\frac{\tau}{1+\eta_i}\right) > \frac{1+\eta_i}{\tau}. \quad (4.4)$$

At $\eta_i = \eta_e = 0$ we obtain from (4.4) the sufficient instability condition, which coincides with[8]:

$$k_b^2\rho_i^2\beta_J > \frac{T_i}{T_e}, \quad \beta_J = \frac{4\pi N_0(T_i+T_e)}{B_0^2}. \quad (4.5)$$

The contribution to the growth rate, as can be seen from (3.27), comes from three sources. The first term in the curly brackets of (3.27) is connected with the "compressibility" of the plasma in perturbations of the Alfvén-wave type ($\mathrm{div}\,\mathbf{V}_E \neq 0$). This effect was investigated in[8]. The second term is connected with the appearance of a longitudinal electric field in the satellite harmonics of the perturbation. The third term, proportional to $b = \omega(\tau z_i)^{1/2}\Gamma$, is due to the longitudinal electric field in the fundamental harmonic of the perturbation $E_\parallel$, which arises when account is taken of the finite Larmor radius of the ions (of the finite $k_\perp^2\rho_i^2$).

The growth rate connected with allowance for the finite $k_\perp^2\rho_i^2$, was calculated by Tang *et al.*[9] The instability criterion, as seen from (3.27) does not differ in this case from (4.2).

Let us compare the role of the toroidality effects (compressibility and satellites) with the effect of the finite $E_\parallel \sim k_\perp^2\rho_i^2$. They turn out to be of the same order of magnitude at $d \approx b$, i.e., at

$$k_\perp^2\rho_i^2 \approx \beta_J\varepsilon^2 T_i/T_e. \quad (4.6)$$

For the characteristic values

$$k_\perp^2\rho_i^2 \sim T_i/T_e\beta_J \quad (4.7)$$

we find that the effects of the finite Larmor radius are more substantial at

$$\beta_J < \beta_J^* \approx 1/\varepsilon, \quad (4.8)$$

i.e., at all values of β_J that are allowed by the condition of equilibrium in the tokamak. The growth increment is then of the order

$$\gamma \approx \gamma_{E_\parallel} \approx \left(\frac{\bar{\nabla} c_A}{qR}\right)^{1/2}\frac{T_e}{T_i}(k_\perp\rho_i)^2, \quad (4.9)$$

i.e., for the characteristic $k_\perp\rho_i$ defined by (4.7) we have

$$\gamma \approx \gamma_E \approx (\bar{\nabla} c_A/qR)^{1/2}\beta_J^{-1}. \quad (4.10)$$

5. EFFECT OF STABILIZATION BY COLD ELECTRONS

We shall show that if the plasma contains an admixture of a cold electron component, the role of the longitudinal electric field decreases, so that the growth increment Alfvén waves can become much smaller than would follow from relations (4.9) and (4.10). When the cold electrons are taken into account, we obtain in place of (3.25)

$$Q(\omega) = \frac{1}{8\pi}\left\langle \int d\lambda B_s K^*\Pi \right\rangle; \quad (5.1)$$

$$K = bM_0 - 2d\hat{p} + \frac{E}{T_e}\frac{N_c}{N_0}d\hat{S}; \quad b = \Gamma_0\omega(\tau z_i)^{1/2},$$

$$d = \frac{c_e}{R}, \quad \hat{p} = \frac{1}{2i}\{\Delta_{+1}M_{+1}e^{i\alpha} - \Delta_{-1}M_{-1}e^{-i\alpha}\}; \quad (5.2)$$

$$\Gamma_0 = \left(1-\frac{\omega_{pi}^*}{\omega}\right)\Big/\left(1-\frac{\omega_{eh}^*}{\omega}-\frac{c_e^2}{\omega^2}k_\parallel^2\frac{N_c}{N_h}\right),$$

$$\Delta_{\pm1} = \left(\frac{N_h}{N_0}-\frac{\omega_{pi}^*}{\omega}\right)\Big/\left(1-\frac{\omega_{eh}^*}{\omega}-\frac{c_e^2}{\omega^2}k_{\pm1}^2\frac{N_c}{N_h}\right); \quad (5.3)$$

$c_e^2 = T_e/M_e$, N_h and N_c are the densities of the hot and cold components of the electrons, $N_0 = N_h + N_c$,

$$k_\parallel = \frac{m-nq}{qR}, \quad k_{\pm1} = k_\parallel \pm \frac{1}{qR},$$

$$\omega_{eh}^* = -\frac{mcT_e}{eaB_s}\frac{d\ln N_h}{da}$$

and the function Π satisfies the equation

$$\omega\Pi - i\hat{C}(\Pi) = (\omega-\omega_h^*)N_0K/N_h. \quad (5.4)$$

From (5.2) and (5.3) we see that at $(c_e/c_A)^2N_c/N_h \gg 1$, i.e., at

$$\frac{N_cN_0}{N_h^2}\beta_e\frac{M_i}{M_e} \gg 1, \quad (5.5)$$

where $\beta_e = 4\pi N_hT_e/B_s^2$, the role of the terms with the electric field decreases:

$$\Gamma_0 \approx \frac{N_h^2}{N_0N_c}\frac{M_e}{M_i\beta} \ll 1. \quad (5.6)$$

$$\Delta_{\pm1} \approx \frac{N_h^3}{N_0^2N_c}\frac{M_e}{M_i\beta} \ll 1. \quad (5.7)$$

In the presence of a cold component of electrons, the sufficient condition for the instability (4.5) is replaced by

$$k_\perp^2\rho_i^2 > \frac{T_i}{T_e}\frac{N_h}{N_0}\frac{1}{\beta_J}. \quad (5.8)$$

The toroidality effects become comparable with the effect of the longitudinal electric field at $b \approx (N_h/N_0)d$, i.e., at

$$(k_\perp\rho_i)^2 \approx \left(\frac{M_i}{M_e}\beta\right)^2\left(\frac{N_c}{N_h}\right)^2\frac{N_0}{N_h}\frac{T_i}{T_e}\varepsilon^2\beta_J, \quad (5.9)$$

which yields at the characteristic value $(k_\perp\rho_i)^2 \approx (T_i/T_e) \times (N_h/N_0)\beta_J^{-1}$

$$\beta_J^* \approx \frac{N_h^2}{N_0N_c}\frac{1}{\varepsilon}\frac{M_e}{M_i\beta} \ll \frac{1}{\varepsilon}. \quad (5.10)$$

Expressing β in terms of β_J, we find from (5.10) that the toroidality effects are more substantial at

$$\beta_J > \beta_J^* \approx \left(\frac{N_h}{N_c}\right)^{1/2}\left(\frac{N_h}{N_0}\right)^{1/2}\left(\frac{M_e}{M_i}\right)^{1/2}q\varepsilon^{-1/2}. \quad (5.11)$$

The growth rate due to the "compressibility" effect is of the order of

$$\gamma_{\mathrm{comp}} \approx \gamma_0\varepsilon^2\beta_J, \quad \gamma_0 = (\bar{\nabla} c_A/qR)^{1/2}. \quad (5.12)$$

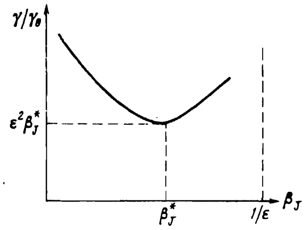

FIG. 1. Qualitative dependence of the growth increments of the Alfvén waves on the parameter β_J in a plasma with admixture of cold electrons.

The growth rate connected with the longitudinal electric field is

$$\gamma_{E_{\parallel}} \approx \gamma_0 \left(\frac{N_h^2}{N_0 N_c}\right)^2 \frac{M_e}{M_i} \beta_J^{-3} q^4 \varepsilon^{-4}. \tag{5.13}$$

The characteristic growth rate as a function of β_J is shown in Fig. 1.

Thus, in the case $M_e/M_i\,\beta \ll 1$, given a small admixture of the cold component

$$M_c/N_0 > M_e/M_i\beta, \tag{5.14}$$

causes the dominating factor in the "drift" buildup of the Alfvén waves by the trapped particles to be the "compressibility" of the plasma.

We note that in the derivation of (5.7) and in the estimate of $\Delta_{\pm 1}$ we have assumed that the difference $k_{\parallel} - 1/qR$ is not too close to zero. Otherwise $\Delta_{-1} \approx N_h/N_0$ and the effect of the longitudinal electric field in the satellite is comparable with the compressibility effect. Both toroidal effects are then the same order. At $\beta_J > \beta_J^*$, both effects make appreciable contributions to the instability growth rate.

6. DISCUSSION OF RESULTS

It was shown above that at large plasma pressure $\beta_J \approx R/a$ the buildup of Alfvén waves by trapped electrons in toroidal magnetic traps such as the tokamak is determined to an equal degree by three factors: the compressibility of the gas of the trapped electrons, the finite value of the longitudinal electric field of the fundamental azimuthal harmonic, and the longitudinal electric field of the satellite harmonics. With decreasing parameter β_J, the role of the compressibility and of the satellite electric field decreases, so that the growth increment of the perturbations is determined mainly by the effect of the longitudinal electric field of the fundamental harmonic. In the presence of a small fraction of cold electrons with density $N_c > N_0 M_e/M_i\beta$ in the plasma, the role of the effects connected with the longitudinal electric field is slight; in this case the growth rate is determined by the compressibility of the trapped electrons.

The effects considered in the present paper can take place in thermonuclear tokamak reactors, and apparently also in the earth's magnetosphere. We note in this connection that the buildup of Alfvén waves in the magnetosphere, due to the effects of the compressibility, was investigated by Pothotelov and one of us.[14,15] The analysis presented above suggests that in magnetosphere regions containing only a negligible fraction of cold particles a role can be played also by the mechanisms of the buildup of Alfvén waves, which are connected with the longitudinal electric field (the finite conductivity of the plasma along the magnetic-field force lines).

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