relaxation times in the liquid crystal phase. Each of these facts is necessary although the form in which they are taken into account for calculations can be made more precise. One may hope that in the framework of the concepts given here the kinetic phenomena in liquid crystals can be connected with the sizes and shape of molecules in as far as this can allow the single molecule approximation used here.

1) The values of the stress tensor of a suspension of ellipsoids given in our paper[13] are erroneously understated by the amount $2\mu\varphi\gamma_{ik}$ which must be added to the appropriate expressions. This reduces to the fact that the value of ω in the definition (4.9) (see[13]) must be increased by unity. The remaining expressions then remain valid. The numerical values of the quantities ω and V in Figs. 1 and 3 must be increased by unity. In Eqs. (7.5), in the equation that follows, and in (9.3) one should read 2.5 instead of 1.5.

2) In a similar manner we can establish the form of the rotational diffusion equation for a particle of arbitrary shape, the oreintation of which is determined by two mutually perpendicular unit vectors **e** and **c**

$$\frac{\partial w}{\partial t}+e_s\frac{\partial(\Omega_{js}w)}{\partial e_j}+c_s\frac{\partial(\Omega_{js}w)}{\partial c_j}=e_{jks}e_{lin}\left\{\frac{\partial}{\partial e_j}\left[D_{ki}\left(e_se_n\frac{\partial w}{\partial e_l}+e_sc_n\frac{\partial w}{\partial c_l}\right)\right]+\frac{\partial}{\partial c_j}\left[D_{ki}\left(c_sc_n\frac{\partial w}{\partial c_l}+c_se_n\frac{\partial w}{\partial e_l}\right)\right]\right\}.$$

The antisymmetric angular velocity tensor Ω_{js} and the symmetric diffusion tensor D_{ki} are here functions of the vectors **e** and **c** and of the tensor S_{jk}.

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On magnetic forces at the surface of a superconductor

N. I. Shikina

Solid State Physics Institute, USSR Academy of Sciences

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We construct a theory of the ponderomotive forces at the surface of a superconductor of arbitrary dimensions, which is valid in the framework of the applicability of the London equations or the Ginzburg-Landau equations. The formalism obtained is used to describe a number of observable effects: the effect of magnetic forces on the dispersion law of the bending oscillations of a plate of arbitrary thickness in a parallel magnetic field, the appearance of an electrical quadrupole moment in a superconducting sphere in a uniform magnetic field, and so on. The presence of a potential difference between the equator and the poles of such a sphere in a magnetic field is experimentally confirmed. We propose, in connection with the problem of the calculation of the magnetic forces at the surface of a thin plate of a type-I superconductor, a consistent perturbation theory for the solution of the Ginzburg-Landau equations under the stated conditions.

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One of the fundamental properties of superconductors is their capacity to expel from their volume an external magnetic field with a field strength less than the critical one (the Meissner effect). This fact leads to many consequences which can be experimentally verified. In particular, the Meissner effect is accompanied by the appearance of well defined magnetic forces at the surface of the superconductor. We are dealing with a pressure from the magnetic field on the surface of the superconductor, which for bulk superconductors and weak magnetic fields is given by the well known phenomenological relation ([1], p. 216)

$$P=-H_0^2/8\pi, \tag{1}$$

where $H_0(\mathbf{r})$ is the local value of the magnetic field at the surface of the sample.

Another effect, which is less well known, but is organically connected with the action of the pressure (1), turns out to be the appearance at the surface of the superconductor of a well defined charge density. The point is that the pressure (1) is in fact applied to the electron gas, which is the carrier of the superconducting properties of the metal. For that reason the initial magnetic forces lead only to a deformation of the density of the electron component of the superconductor. This deformation destroys the electro-neutrality of the metal, and this is equivalent to the appearance of surface charges which compensate the action of the magnetic forces on the electrons. As a result, the pressure (1) starts to act upon the lattice of the metal via the field of the charges. Electrical fields induced by the magnetic field, which in magneto-mechanical effects are a necessary intermediary between the electron and the lattice effects of the problem, can be observed also independently. Such a possibility follows, for example, from Bock and Klein's experiments[2] in which the appearance of a potential difference between the poles and the equator of a bulk superconducting sphere in a uniform magnetic field was detected and was proportional to the square of the magnetic field.

Magneto-mechanical effects must occur for superconductors of any dimensions. However, when the dimensions of the sample are reduced the phenomenological relations which determine the amplitude of the magnetic pressures ultimately cease to be valid. Under such conditions it is necessary to use for the calculation of the magnetic forces equations that retain their meaning for finite-size superconductors.

The aim of the present paper is the construction of a theory of magneto-mechanical effects at the surface of a superconductor which is valid for superconductors of any size. In the first part of the paper we use the London definition of the magnetic forces and calculate on this basis some concrete magneto-mechanical effects. Later we discuss the problem of the magnetic forces at the surface of a superconductor in the framework of the Ginzburg–Landau theory.

LONDON DEFINITION OF THE MAGNETIC FORCES

We first of all consider the possible consequences of the London definition of the magnetic pressures on the surface of a superconductor. Supplementing the Maxwell equations with the connection between the superconducting current $\mathbf{j}_s$ and the magnetic field $\mathbf{H}$

$$(4\pi\delta^2/c)\,\mathrm{rot}\,\mathbf{j}_s+\mathbf{H}=0,\quad \delta^2=mc^2/4\pi n_s e^2 \tag{2}$$

(δ is the penetration depth for the magnetic field into the bulk superconductor, m and e are the electron mass and charge, c is the velocity of light in vacuo, and n_s the density of superconducting electrons) and postulating the equation[1)]

$$\frac{4\pi\delta^2}{c^2}\frac{\partial \mathbf{j}_s}{\partial t}=\mathbf{E} \tag{2a}$$

($\mathbf{E}$ is the electric field) London[3] obtained a closed set of equations describing the electrodynamic properties of local superconductors.

As one of the consequences of these electrodynamics one can find an explicit expression for the momentum flux tensor of the field and of the superconducting electrons T^s_{ik} and hence also the volume force density f acting per unit volume of the superconductor:

$$\begin{gathered} f_i=\frac{\partial T_{ik}{}^s}{\partial x_k},\qquad T_{ik}{}^s=T_{ik}{}^M+S_{ik},\\ T_{ik}{}^M=\frac{1}{8\pi}(H^2\delta_{ik}-2H_iH_k),\qquad S_{ik}=\frac{m}{n_se^2}\left(j_ij_k-\frac{1}{2}\delta_{ik}j_s{}^2\right). \end{gathered} \tag{2b}$$

Here T^M_{ik} and S_{ik} are the Maxwell and the so-called London stress tensors. It turns out that under stationary conditions $\partial T^s_{ik}/\partial x_k=0$, i.e., there are no volume forces in such an electrodynamical scheme. To make things clear we can state that in this case the Lorentz volume force which acts on a moving charged fluid is exactly cancelled by the Bernoulli pressure. As to the boundaries of the metal with vacuum, by definition the tensor T^M_{ik} is continuous on it, but S_{ik} has a discontinuity. Because of this discontinuity there are surface forces at the superconductor-vacuum boundary,

$$P_i=S_{ik}(0)\,n_k=-(m/2n_se^2)\,j_s^2(0), \tag{2c}$$

where $\mathbf{n}$ is the normal to the metal surface.

Yet another way to obtain the tensor T^s_{ik} of (2b) is of interest for what follows. It is well known from field theory that if the equation of motion (in this case the London Eq. (2)) is obtained through the variation of a functional of the form

$$F=F_0+\int\Lambda\left(q,\frac{\partial q}{\partial x_i}\right)d\mathbf{r},\qquad \Lambda=\frac{1}{8\pi}(\mathbf{H}^2+\delta^2|\mathrm{rot}\,\mathbf{H}|^2), \tag{2d}$$

the momentum flux tensor T_{ik} corresponding to this "equation of motion" is given by the expression (see[4])

$$T_{ik}=\delta_{ik}\Lambda-\frac{\partial q}{\partial x_i}\frac{\partial\Lambda}{\partial(\partial q/\partial x_k)},\qquad \frac{\partial T_{ik}}{\partial x_k}=0. \tag{2e}$$

One checks easily that for the actual functional (2d) the tensor (2e) has the form (2b).

For a more consistent study of the problem of the distribution of the forces in the volume of a superconductor it is sufficient to use London's Eq. (2), supplementing the theory by the equation of motion of a charged fluid in an electromagnetic field

$$\frac{\partial \mathbf{j}_s}{\partial t}+\frac{1}{n_se}(\mathbf{j}_s\nabla)\,\mathbf{j}_s=\frac{e^2n_s}{m}\mathbf{E}+\frac{e}{cm}[\mathbf{j}_s\times\mathbf{H}]\,. \tag{3}$$

Using (2) Eq. (3) takes the form

$$\frac{4\pi\delta^2}{c^2}\frac{\partial \mathbf{j}_s}{\partial t}-\mathbf{E}=-\frac{m}{2n_se^2}\nabla \mathbf{j}_s^2, \tag{3a}$$

i.e., it differs from (2a) by the term with $\nabla \mathbf{j}_s^2$.

In the given non-linear variant of the theory, which was also discussed by London (see[3], p. 56) and confirmed by further microscopic calculations,[5] there acts upon the density of superconducting electrons a volume force which is equal to the Bernoulli pressure, well known in hydrodynamics:

$$f_i = -{}^1/_2 n_s m v_s^2, \quad \mathbf{j}_s = e n_s \mathbf{v}_s. \tag{3b}$$

As a result, to satisfy the condition $\partial \mathbf{j}_s/\partial t = 0$ one must introduce in the problem a longitudinal volume electric field $\mathbf{E}$, i.e., destroy the local electro-neutrality of the superconductor. However, this field $\mathbf{E}$, determined from the condition $e\mathbf{E}/m - \frac{1}{2}\nabla \mathbf{v}_s^2 = 0$ does not affect the magnetic field distribution inside the superconductor as $mv_s/p_F \ll 1$ (v_s is the superfluid velocity and p_F the Fermi momentum).

The appearance of the volume forces (3b) in this variant of the theory is, as one can easily verify, accompanied by the vanishing of the surface force such as (2a) due to the discontinuity of the momentum flux tensor. Indeed, the tensor $\tilde{T}_{ik}$ corresponding to the equation of motion (3),

$$\tilde{T}_{ik}^s = T_{ik}^M + T_{ik}^E + \tilde{S}_{ik}, \quad \tilde{S}_{ik} = m j_i j_k / n_s e^2 \tag{3c}$$

(T_{ik}^M and T_{ik}^E are the Maxwell stress tensors for the magnetic and electric fields) is continuous at the metal-vacuum boundary, i.e.,

$$\tilde{T}_{ik}^s(+0) n_k - \tilde{T}_{ik}^s(-0) n_k = 0.$$

The integral action of the forces f_i of (3b) is equivalent to the presence at the surface of the superconductor of an effective magnetic pressure P_H which is equal to

$$P_H = -\int \frac{1}{2} n_s m \frac{\partial v_s^2}{\partial z} dz, \tag{3d}$$

where the integration is over a coordinate into the superconducting phase up to the point where ∇v_s^2 vanishes. If at the upper limit of the integral in (3d) the quantity v_s also vanishes, the definitions (3d) and (2c) coincide. In the cases of practical interest such a coincidence always occurs.

The more consistent system of definitions (2), (3a) to (3d), without touching upon the results referring to the magnetic field distribution over the volume of the superconducting phase and without changing the average value of the magnetic field pressure at the surface of the superconductor, therefore changes the volume distribution of the magnetic forces qualitatively due to the Meissner effect as compared to the variant (2a) to (2c). There exists a sufficient number of interesting problems for which the distribution of the magnetic forces over the volume of the superconductor is not of interest as a matter of principle. In what follows we discuss just such problems.

For a bulk superconductor the results (2c) and (3d) written in terms of $\mathbf{H}$ are the same as (1). If, however, the size of the superconductor is small (comparable to the penetration depth δ of the given superconductor) there arise, of course, deviations from the phenomenological definitions. It is convenient to trace these deviations using concrete examples.

1. We consider a superconducting plate of thickness $2d$ in a magnetic field H_0 parallel to its surface. The plate executes bending oscillations with a low frequency ω and amplitude ξ, while the wavevector $\mathbf{q}$ of the oscillations is parallel to $\mathbf{H}_0$, $\mathbf{H}_0 \parallel \mathbf{q}$. Under such conditions the magnetic forces affect the dispersion law for the bending oscillations of the plate. This effect increases particularly fast in the limit $d \gg \delta$, $q\delta \ll 1$ when the presence of the superconducting properties of the plate can be taken into account via the boundary condition for the magnetic field, which corresponds to the field not penetrating into the superconductor

$$\mathbf{Hn} = 0, \tag{4}$$

where $\mathbf{n}$ is the normal to the surface of the curved plate.

In actual fact, the bending oscillations of the plate lead, apart from the deformation of the magnetic field lines of force, which is effectively taken into account through the boundary condition (4), to the appearance near the surface of the superconductor to electric fields with a strength ([1], p. 228)

$$\mathbf{E} = [\mathbf{V} \times \mathbf{H}_0]/c, \tag{4a}$$

where $\mathbf{V}$ is the velocity of the plate motion. Below we discuss the role of this electric field.

When the shape of the surface is periodically perturbed,

$$\xi(x, t) = \xi_0 e^{i(qx - \omega t)}$$

the initial magnetic field strength H_0 ceases to satisfy the boundary condition (4) as there appears a component $H_\perp$ normal to the surface of the plate:

$$H_\perp = -H_0 \partial\xi/\partial x. \tag{5}$$

To eliminate the magnetic flux through the surface of the superconductor it is necessary to introduce an auxiliary magnetic field $\mathbf{h}(x, t)$ with boundary value $h_\perp$ with the opposite sign of (5):

$$\mathbf{h} = \nabla\varphi, \quad \Delta\varphi = 0,$$
$$\left.\frac{\partial\varphi}{\partial z}\right|_s = H_0 \frac{\partial\xi}{\partial x}, \quad \varphi|_\infty \to 0. \tag{6}$$

As a result the total magnetic field along the surface of the bent superconductor turns out to be equal to

$$H_\parallel = H_0[1 + q\xi(x, t)]. \tag{7}$$

It is somewhat larger than the average field H_0 above the convex relief of the surface and somewhat smaller above the concave one.

We noted above that there exists yet another perturbation of electromagnetic origin, which is connected with the occurrence of the electric fields (4a). Estimates show that the magnetic field accompanying the generation of the electric field (4a) has an amplitude $h_1 \sim \omega\xi H_0/c$. This field is less than the extra fields (6) and (7) in as far as the ratio of the sound speed in the solid to the light speed in vacuo is small. The magnetic pressure, normal to the surface of the plate, is, up to terms linear in ξ, equal to

$$P_H = T_{ik}{}^M n_k|_{\pm d} \approx -(H_0^2/8\pi)[1 \pm 2q\xi(x, t)]. \tag{8}$$

We can split the action of the pressures (8) on the plate as a whole into two parts. The main, symmetric, part of the pressures that are directed opposite to each other, exerts a compressive action on the plate:

$$P_{\uparrow\downarrow} = {}^1/_2[P_H(d) + P_H(-d)] = -H_0^2/8\pi \tag{8a}$$

and by virtue of its relative smallness does not appreciably affect the properties of the plate. The other part of the pressures (8),

$$P_{\uparrow\uparrow} = P_H(+d) - P_H(-d) = H_0^2 q\xi(x, t)/2\pi, \tag{8b}$$

bends the plate and hence affects the dispersion law for the bending oscillations of the plate.

Substituting (8b) into the equation of motion of a plate or membrane we find the corresponding dispersion laws:

$$\rho d\omega^2 = \frac{Ed^3}{12(1-\sigma^2)} q^4 + \frac{H_0^2}{2\pi} q, \tag{9}$$

ρ is the density of the material of the plate, and E, σ are the Young modulus and the Poisson coefficient;

$$\rho d\omega^2 = Tq^2 + H_0^2 q/2\pi, \tag{9a}$$

T is the tension in the membrane. Substituting into (9) $E \sim 10^{11}$ g cm^{-2} s^{-2}, $d \sim 10^{-2}$ cm, $\sigma \sim 10^{-1}$, $q \sim 1$ cm^{-1} we find that the magnetic contribution $H_0^2 q/2\pi$ in the dispersion law (9) is of the same order as the elastic one for fields $H_0 \sim 10^2$ Oe. Such field strengths are for many superconductors appreciably less than the critical field H_c.

2. One understands easily that when the thickness of the film decreases to values $d \lesssim \delta$ when it becomes incorrect to neglect the penetration of the magnetic field into the volume of the film, the effect of the magnetic field on the oscillations of the membrane, covered by the superconducting film, decreases. For a quantitative description of the function $P_H(d)$ it is necessary to "join" the magnetic field distribution inside the superconductor, determined by the London equation $\nabla^2\mathbf{h} = \delta^{-2}\mathbf{h}$, div$\mathbf{h}$ $= 0$, with the vacuum solution (6). As a result

$$h_\parallel = Hq\xi \Big/ \left(1 + \frac{q}{\gamma}\operatorname{cth}\gamma d\right), \quad \gamma^2 = \delta^{-2} + q^2. \tag{10}$$

However, it is not possible to carry out the calculation of the pressures in this case in terms of $h_\parallel$ since the quantity $\mathbf{h}$ is continuous at the metal-vacuum boundary. Writing down the appropriate superconducting currents in the film and using the London definition of the pressure (2c), (3d)[2)] we find for $P_H(d)$ an expression which generalizes the phenomenological result (8):

$$P_H(\pm d) = -\frac{H_0^2}{8\pi}\left(\operatorname{th}^2\gamma d \pm 2q\xi \frac{\gamma \operatorname{sh}\gamma d}{\gamma \operatorname{sh}\gamma d + q \operatorname{ch}\gamma d}\right). \tag{11}$$

The changes in the dispersion laws (9) and (9a) then reduce to the substitution

$$\frac{H_0^2}{2\pi} \to \frac{H_0^2}{2\pi}\frac{\gamma \operatorname{sh}\gamma d}{\gamma \operatorname{sh}\gamma d + q \operatorname{ch}\gamma d}$$

If, however, we are dealing with a composite film, consisting of an elastic substrate with parameters $2d_0$, ρ_0, T_0, and a superconducting film of thickness $2d_s$ and density ρ_s (we neglect the tension of this film), using (11) we then get for the dispersion law of such a membrane

$$(\rho_0 d_0 + \rho_s d_s)\omega^2 = T_0 q^2 + \frac{H_0^2}{2\pi} q \frac{\gamma \operatorname{sh}\gamma d}{\gamma \operatorname{sh}\gamma d + q \operatorname{ch}\gamma d}. \tag{12}$$

It is interesting to note that the deviations from the phenomenological values for the pressures $P_{\uparrow\downarrow}$ of (8a) and $P_{\uparrow\uparrow}$ of (8b) arise at different regions of film thickness. According to (8a) and (11) the pressure $P_{\uparrow\downarrow}$ is equal to

$$P_{\uparrow\downarrow} = -(H_0^2/8\pi)\operatorname{th}^2\gamma d, \tag{11a}$$

i.e., it starts to depend on d when

$$\operatorname{th}\gamma d < 1. \tag{13}$$

As far as the pressure $P_{\uparrow\uparrow}$ is concerned, according to (11) the deviations from (8b) must arise in the region

$$\operatorname{th}\gamma d < q\delta. \tag{13a}$$

Using the inequality $q\delta \ll 1$, which is satisfied with a large margin for the bending oscillations of a plate, the requirement (13a) for the thickness of the plate turns out to be more rigid than (13).

3. We calculate also the electric quadrupole moment of a superconducting sphere in a uniform magnetic field. The original definitions for the component D_{zz} of the quadrupole moment of the magnetic field distribution near the field are

$$D_{zz} = \int \rho(3\cos^2\theta - 1)r^2\,dV, \tag{14}$$

$$H_r = 3H_0\frac{\delta a}{r^2}\frac{\operatorname{sh}(r/\delta)}{\operatorname{sh}(a/\delta)}\left(\operatorname{cth}\frac{r}{\delta} - \frac{\delta}{r}\right)\cos\theta,$$

$$H_\theta = \frac{3H_0}{2}\frac{\delta a}{r^2}\frac{\operatorname{sh}(r/\delta)}{\operatorname{sh}(a/\delta)}\left[\operatorname{cth}\frac{r}{\delta} - \frac{\delta}{r}\left(1 + \frac{r^2}{\delta^2}\right)\right]\sin\theta, \quad H_\varphi = 0.$$

Here ρ is the charge density in the volume of the superconductor, dV an element of volume of the superconductor, a the radius of the sphere, and the definitions (14) are written in a spherical system of coordinates with the origin at the center of the sphere and the polar axis along the direction of the unperturbed magnetic field.

In the limit $a \gg \delta$ the distribution of the pressures P_H on the surface of the sphere is

$$P_H = \frac{9H_0^2}{32\pi}\sin^2\theta = \frac{3H_0^2}{16\pi}[p_0 - p_2(\cos\theta)], \tag{15}$$

$p_\lambda(x)$ are the Legendre polynomials.

The presence of the pressures P_H at the surface of the sphere leads to a shift of the electron density relative to the ion framework of the superconductor, as a result of which there appear surface charges on the surface of the superconductor. In the incompressible electron fluid approximation we are dealing with a shift of electrons from the equatorial region of the sphere to its poles. The equilibrium surface charge density distribution is determined from the condition that the work done by the magnetic pressure when producing the charges is balanced by the Coulomb energy which arises. Using the auxiliary concept of the shift $\xi(\theta)$ of the boundary of the electron density relative to the surface of the sphere we write the work done Q by the magnetic pressure when the charges are separated in the form (ds is a surface area element of the sphere)

$$Q = \int P_H(\theta)\xi(\theta)\,ds = -P_H^0 \cdot 2\pi a^3 \xi_2/5,$$
$$P_H = P_H^0[p_0 - p_2(\cos\theta)], \quad P_H^0 = \frac{3}{16}\frac{H_0^2}{\pi},$$
$$\xi(\theta) = a\xi_2 p_2(\cos\theta), \quad \int \xi(\theta)\,ds = 0, \tag{16}$$

On the other hand, the Coulomb energy of the surface charges which arise is analogous to the Coulomb energy which turns up in the calculation of the natural oscillations of an incompressible charged drop of nuclear matter (see, e.g., [6]) if we understand by the surface charge density the expression

$$\sigma(\theta) = en_0\xi(\theta) = en_0 a\xi_2 p_2(\cos\theta), \tag{17}$$

n_0 is the ion volume density.

Using (17) and the equations given in Davydov's book[6] we have for the Coulomb energy V_e in terms of n_0 and ξ the following expression:

$$V_e = {}^4/_3\pi a^5 e^2 n_0^2 \xi_2^2/5. \tag{18}$$

Comparing Q and V_e we find

$$\xi_2 = 3P_H^0/2a^2e^2n_0^2. \tag{19}$$

The surface charges on a superconducting sphere in a magnetic field are thus determined by Eq. (17) with ξ from (19).

Once we have $\sigma(\theta)$ we can easily evaluate D_{zz} from (16). In the limit $a \gg \delta$

$$D_{zz}^0 = 3\pi a^3 P_H^0/5n_0 e. \tag{20}$$

In the case of an arbitrary radius of the superconducting sphere $a_0 \ll a \lesssim \delta$ (a_0 is the interatomic distance) the expression for D_{zz} looks as follows:

$$D_{zz} = D_{zz}^0\varphi(a/\delta),$$
$$\varphi(x) = x^{-2}[\operatorname{cth} x(\operatorname{cth} x - x) - \operatorname{sh}^{-2} x] = \begin{cases} 1, & x \gg 1 \\ x^2/9, & x \ll 1. \end{cases} \tag{21}$$

The appearance of the charges (17) and (19) on the surface of a sphere in a magnetic field can be put in correspondence with the observed occurrence of a potential difference between the equator and the pole of a cylindrical superconductor in a transverse magnetic field.[2] According to Bock and Klein this difference is proportional to H^2 and oscillates with twice the frequency of the change in the magnetic field. The doubling of the frequency of the potential difference is connected with the fact that during one period of the change in the magnetic field the pressure P_H reaches its maximum twice.

MAGNETIC FORCES AND THE GINZBURG-LANDAU THEORY

1. To determine the forces in the framework of the Ginzburg–Landau equations (GL equations in what follows) which arise as the result of the variation of the functional F[7]

$$F = \int \Lambda\,d\mathbf{r}, \quad \Lambda = -f^2 + \frac{1}{2}f^4 + \left|\left(\frac{i}{\varkappa}\nabla - \mathbf{A}\right)f\right|^2 + \left(\frac{\partial A}{\partial \mathbf{r}}\right)^2 \tag{22}$$

($f(\mathbf{r})$ is the magnitude of the order parameter, $\mathbf{A}(\mathbf{r})$ the vector potential, $\varkappa$ the parameter of the GL theory, and Eq. (22) is written in the usual dimensionless units), it is natural to use Eq. (2e) for T_{ik}^s

$$T_{ik}^s = \Lambda\delta_{ik} - \frac{\partial f}{\partial x_i}\frac{\partial\Lambda}{\partial(\partial f/\partial x_k)} - \frac{\partial \mathbf{A}}{\partial x_i}\frac{\partial\Lambda}{\partial(\partial\mathbf{A}/\partial x_k)}. \tag{23}$$

By definition, the tensor T_{ik}^s in (23) satisfies everywhere in the volume of the superconductor the equation $\partial T_{ik}^s/\partial x_k = 0$ for the coordinate-dependent quantities $f(r)$ and $\mathbf{A}(\mathbf{r})$, which satisfy the GL equations. The situation which occurs when we use the definitions (2a) to (2c) in the London variant of the theory is thus duplicated also in the GL theory.

The non-vanishing component T_{zz}^s (z is the coordinate directed into the superconducting phase) in the one-dimensional case turns out to equal

$$T_{zz}^s = -(1-A^2)f^2 + \frac{1}{2}f^4 - \frac{1}{\varkappa^2}\left(\frac{\partial f}{\partial z}\right)^2 - \left(\frac{\partial A}{\partial z}\right)^2 = \text{const.} \tag{24}$$

Expression (24) is exactly equal to the well known first integral of the GL equations, whence we can conclude that the existence of that integral is closely connected with satisfying the condition $\partial T_{ik}^s/\partial x_k = 0$ in the one-dimensional case.

Defining (by analogy with (2c)) the pressure at the metal-vacuum boundary as the difference between the values of the momentum flux in the superconductor and in vacuo, and using the boundary conditions $f'(0) = 0$ and $A'(0) = H$, we find the following result for P_H:

$$P_H = (1-A_0^2)f_0^2 - f_0^4/2, \quad f_0 = f(0), \quad A_0 = A(0). \tag{25}$$

Using (24), where const $= \frac{1}{2}$ for the case of a bulk superconductor, we can rewrite (25) as follows:

$$P_H = -H^2 + {}^1/_2. \tag{25'}$$

The magnetic part of this pressure in dimensionless units is the same as expression (1). As to the constant $\frac{1}{2}$, its appearance must not cause any surprise, since the magnetic pressure is defined accurate to a constant that is independent of H ([1], p. 224).

2. The results (22) to (25) need some commentary. The microscopic derivation of the GL equations[8] shows that to obtain them from the general Gor'kov equations one can neglect terms $\sim \frac{1}{2} m v_s^2$ as compared to terms $\sim p_F v_s$ (v_s is the superfluid velocity and p_F the Fermi momentum). In other words, the accuracy of the GL equations is sufficient for the determination of the distribution of the magnetic field inside the superconductor and of the order parameter, but insufficient (in its existing form) to describe effects leading to the appearance of a local longitudinal electric field inside the superconductor of the kind of the Bernoulli potential (3b). For that reason the conclusion reached above about the absence of ponderomotive forces inside the superconductor when we describe its properties by the GL equations is most likely a consequence of the approximation and is not retained when the theory is made more exact, as occurs, in fact, in the London approximation. Nonetheless the expression for the effective pressure at the surface of the superconductor, which is an integral characteristic of the as yet unknown volume distribution of the forces, should, as in the London approximation, not depend on the details of the distribution of the forces over the volume of the superconductor. One can therefore use the result (25) for the calculation of actual effects which depend only on the total magnitude of the magnetic pressure at the metal surface.

One should state that the mentioned inaccuracy of the GL equations is not unique. In fact, the coefficients of the expansion in powers of the parameter f^2 in the free energy (22) are well defined functions of the coordinates, and this leads to an additional source for the existence of volume forces. The cause for this dependence is the observed effect of the change in the volume of the superconductor when it goes over from the normal into the superconducting state (see, e.g., Shoenberg's book[9]). Such a change in the volume in the transition region at the ns-boundary of the intermediate state must be accomplished by the appropriate volume forces. However, numerically this effect is very small (the relative change in the volume is 10^{-7} [9]).

3. Using the formalism expounded above with the reservations listed we can make the results which follow from the London definition appreciably more exact. We consider, for instance, the problem of the symmetric part $P_{\uparrow\downarrow}$ of the pressures for a thin type-I superconducting plate of thickness $2d<1/\varkappa$, $\varkappa \ll 1$ in a parallel magnetic field. The GL equations then become one-dimensional and, hence, the expression for T_{zz} retains the form (24). However, the constant, which is now equal to the momentum flux, is not equal to $\frac{1}{2}$. We must therefore use Eq. (25), and not (25′), as the definition of the pressure at the surface of the plate. Using the explicit form of the vector potential in the plate problem[7]:

$$A(z)=H_0 \operatorname{sh} f_0 z / f_0 \operatorname{ch} f_0 d, \quad \varkappa\to 0,$$

where the dependence of f_0 on H_0 and d is given by the relation[3]

$$f_0^2(f_0^2-1)=\frac{H_0^2(1-(\operatorname{sh} 2f_0 d)/2f_0 d)}{2\operatorname{ch}^2 f_0 d}, \tag{26}$$

and using (25) we find an expression for the pressures $P_{\uparrow\downarrow}$ at the surface of a thin plate:

$$P_{\uparrow\downarrow}=f_0^2\left(1-\frac{1}{2}f_0^2\right)-H_0^2 \operatorname{th}^2(f_0 d). \tag{27}$$

Comparing (27) with (11a) one easily notes the changes arising in the determination of $P_{\uparrow\downarrow}$. Apart from the appearance of f_0 in the argument of the hyperbolic tangent: $\tanh^2(d/\delta)\to\tanh^2(f_0 d/\delta)$, which we could expect in the case of thin plates, the stretching part of the pressures which for a bulk superconductor had the form $\frac{1}{2}$, begins to depend on the magnetic field. As a result the expression (27) contains the term $f_0^2(1-\frac{1}{2}f_0^2)$ which is completely absent in the definition (11a). As $H_0\to 0$ we have $f_0^2(1-\frac{1}{2}f_0^2)\to\frac{1}{2}$.

Turning to the determination of the bending part of the pressures for plates performing oscillations in a parallel magnetic field we can arrive at the conclusion that the effective increase in the penetration depth, $\delta\to\delta/f_0$, arising under the conditions $f_0\ll 1$, noticeably widens the region of plate thicknesses for which the quantity $P_H(d)$ begins to depend on d. The corresponding inequality which replaces (13a) is

$$\operatorname{th}(f_0 d/\delta)<q\delta/f_0. \tag{28}$$

It is clear that the possibility for an effective increase of the penetration depth for the magnetic field into the volume of a superconductor is, because of the fact that $f_0\to 0$ is small, a basic distinguishing feature of the theory of magneto-mechanical effects in the framework of the GL equations as compared to the analogous consideration in the London variant of the theory.

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APPENDIX

The solution of the set of Ginzburg–Landau equations for a thin type-I superconducting plate in a parallel magnetic field

$$\begin{gathered} f''=\varkappa^2[(A^2-1)f+f^3], \quad A''=f^2A, \\ f'|_{\pm d}=0, \quad A'|_{\pm d}=H \end{gathered} \tag{A1}$$

will be looked for (see[6]) in the following form

$$A_0=H\operatorname{sh} f_0 z/f_0\operatorname{ch} f_0 d, \quad f(z)=f_0+\varphi, \quad \varphi\ll f_0. \tag{A2}$$

The quantity φ is then according to[7] defined by the equation

$$\varphi''=\varkappa^2[f_0^3-f_0+(3f_0^2-1)\varphi+A_0^2 f_0], \quad \varphi'_{\pm d}=0. \tag{A3}$$

In actual fact, however, the equation for φ must in the general case of arbitrary magnetic fields contain also a term $A_0^2\varphi$. The absence of this term in (A3) does not lead to errors in the relation between f_0 and H_0 in the zeroth approximation in $\varkappa d \ll 1$, but starts to affect the results in the following approximations. In this connection there arises the necessity of a more consistent consideration of the higher approximations in $\varkappa$ in the problem of a thin plate in a parallel magnetic field.

We write $f(z)$ and $A(z)$ as series

$$f=f_0+\varkappa^2 f_1+\varkappa^4 f_2+\ldots,\quad A=A_0(z)+\varkappa^2 A_1+\ldots, \tag{A4}$$

substitute (A4) into (A1) and collect terms of the same order of smallness

$$A_0''=f_0^2 A_0,\quad A_0'|_{\pm d}=H, \qquad f_1''=(A_0^2-1)f_0+f_0^3,\quad f_1'|_{\pm d}=0; \tag{A5}$$

$$A_1''=f_0^2 A_1+2f_0A_0f_1,\quad A_1'|_{\pm d}=0, \qquad f_2''=2A_0f_0A_1+(A_0^2+3f_0^2-1)f_1,\quad f_2'|_{\pm d}=0,\ldots \tag{A6}$$

An interesting peculiarity of the set of approximations which we have written down is the method of determining the constants which arise during the solution of the equations for f_i in each of the approximations. The zeroth boundary conditions substituted for arbitrary f_i' and not for the function f_i itself lead to the fact that the corresponding integration constants c_i of the equations for f_i can not be determined in the same approximation. One finds the values of c_i from satisfying the boundary condition for $f_{i+1}'=0$ in the next approximation.

Bearing in mind what we have said we give the solution of the set (A5):

$$\begin{gathered} A_0=H_0 \operatorname{sh} f_0z/f_0 \operatorname{ch} f_0d, \\ f_1(z)=\alpha z^2+\beta\left(\frac{\operatorname{sh}^2 f_0z}{4f_0^2}-\frac{z^2}{4}\right)+c_1, \\ \alpha=f_0(f_0^2-1)/2,\quad \beta=H_0^2/f_0 \operatorname{ch}^2 f_0d, \\ f_0^2(f_0^2-1)=\frac{H^2[1-(\operatorname{sh} 2f_0d)/2f_0d]}{2\operatorname{ch}^2 f_0d}. \end{gathered} \tag{A5a}$$

The last relation which determines f_0 in terms of H and d and which is the same as Eq. (26) of the main text is here obtained from the requirement that $f_1'(\pm d)=0$.

As to the quantity c_1 it remains as yet unknown. To find it we must solve the set (A6), using (A5a) and requiring that the condition $f_2'(\pm d)=0$ is satisfied. As a result we find

$$\begin{gathered} A_1(z)=\tilde{A}_1(z)-\frac{\partial \tilde{A}_1(z)}{\partial z}\bigg|_{\pm d}\frac{\operatorname{sh} f_0z}{f_0 \operatorname{ch} f_0d}, \\ \tilde{A}_1=\int A_0(t)\operatorname{sh} f_0(z-t)f_1(t)\,dt, \end{gathered} \tag{A7}$$

$$f_2'(d)=\int_{-d}^{d}\{2f_0A_0(z)A_1(z)+[A_0^2(z)+3f_0^2-1]f_1(z)\}dz=0. \tag{A8}$$

The integrations in (A8) are elementary, but Eq. (A8) when written out explicitly turns out to be rather cumbersome for the determination of the constant c_1. For that reason the explicit expression for c_1 has a translucent form only in the limit $f_0d<1$

$$c_1=\frac{f_0d^2}{6}\,\frac{1-0.7H_0^2d^2-{}^5/_{14}H_0^2f_0^2d^4}{1-{}^1/_3H_0^2d^2+{}^8/_{15}H_0^2f_0^2d^4}. \tag{A9}$$

In the region $f_0d\to 0$ the combination $H_0^2d^2/3-1$ in the denominator is proportional to f_0^2. As a result expression (A9) diverges as $1/f_0$ as $f_0\to 0$. This means that the solution of the GL equations obtained here has a meaning only for finite values of f_0 as long as $f_0>\varkappa^2 f_1$.

1) A consequence of the Maxwell equations and Eq. (2) is the equation

$$\frac{4\pi\delta^2}{c^2}\frac{\partial \mathbf{j}_s}{\partial t}-\mathbf{E}=\operatorname{grad}\Phi,$$

where Φ is an arbitrary scalar function. London's postulate consists in the assumption that this arbitrary function vanishes.

2) We noted above that the definitions (2c) and (3d) coincide in the present case since the superconducting current for a plate in a parallel magnetic field vanishes in the middle of the plate.

3) Equation (26) is valid in the limit $\varkappa\to 0$. In the more general case $\varkappa d<1$ the analysis[7] that allows us to determine the $\varkappa$-dependence of f_0 contains some inaccuracies. A self-consistent scheme of calculating the function $f_0(\varkappa)$ is given in the Appendix.

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