Vortex penetration into a Josephson junction of finite width

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The penetration of a magnetic field into a finite-width tunnel barrier is considered. It is shown that the magnetization curve differs from that for an infinitely wide barrier (B. D. Josephson, Revs. Mod. Phys. **36**, 216, 1964; I. O. Kulik, Zh. Eksp. Teor. Fiz. **51**, 1952, 1966 [Sov. Phys. JETP **24**, 1307 (1966)]; I. O. Kulik and I. K. Yanson, The Josephson Effect in Superconducting Tunnel Structures [in Russian], 1970). The mean field strength does not vary continuously in this case but changes by jumps as the individual vortices enter the junction. Expressions are found for the "supercooling" and "superheating" fields, which restrict the region of existence of weak superconductivity.

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It is known that a magnetic field can penetrate into a Josephson tunnel barrier in the form of individual vortex filaments, which form a one-dimensional linear structure in the junction.[1–3] In the limiting case of very wide tunnel barrier, formulas were obtained by Kulik[2,3] which describe the thermodynamic equilibrium magnetization curve of a weak superconductor. In particular, it was shown by him that the mean field in the barrier $\bar{H}$ is equal to zero up to some critical value H_{c1}, above which the vortices begin to penetrate into the barrier, and the mean field $\bar{H}$ increases continuously to the value of the external field $\bar{H}=H_e$ at $H_e \gg H_{c1}$.

In the present work, the character of the vortex penetration into a tunnel junction of finite width is investigated. It is shown that the equilibrium magnetization curve in the case of a barrier of finite width differs considerably from the corresponding curve for an infinitely wide barrier.[2,3] The mean field in the barrier $\bar{H}$ changes by jumps, as a result of the entrance of individual vortices into the junction, and the magnetization curve as a whole approaches the law $\bar{H}=H_e$. Expressions are also found for the "supercooling" and "superheating" fields, which restrict the region of existence of superconducting solutions with a specified number n of vortices in the barrier. A method of calculation is shown, which allows us to find the magnetization curve for any finite value of barrier width.

1. The initial equation which describes the distribution of the stationary magnetic field in the tunnel barrier has the form[2–4]

$$d^2\varphi(x)/dx^2=\sin\varphi(x), \tag{1}$$

where the quantity $\varphi(x)$ (the so-called phase difference at the barrier) is connected with the magnetic field in the barrier:

$$d\varphi(x)/dx=H(x). \tag{2}$$

In Eqs. (1) and (2), we have used dimensionless quantities: the coordinate is measured in units of λ_J (λ_J is the characteristic penetration depth of the field into a weak superconductor; usually, $\lambda_J \sim 0.1$ mm) the magnetic field is measured in units of $H_J - \Phi_0/2\pi\lambda_J\Lambda$ ($\Phi_0=hc/2e=2\times10^{-7}$ G-cm^{-2} is the flux quantum, $\Lambda=2\delta_L\sim10^{-5}$ cm is the length over which the field penetrates from the barrier into the bulk superconductor; the characteristic value of $H_J\sim10^{-1}$–1 G). We shall assume that the barrier has a width L and is placed in an external field H_e parallel to the plane of the junction (see Fig. 1). The boundary conditions for Eq. (1) are written down in this case in the form

$$\left.\frac{d\varphi}{dx}\right|_{x=0}=\left.\frac{d\varphi}{dx}\right|_{x=L}=H_e. \tag{3}$$

Equation (1) has a general solution in the form of an elliptic integral, giving the function $\varphi(x)$ in implicit form:

$$x=-\frac{1}{2}\int_{\varphi(0)}^{\varphi(x)}\frac{d\varphi}{\{\alpha^2+\sin^2(\varphi/2)\}^{1/2}},\quad H(x)=2\left\{\alpha^2+\sin^2\frac{\varphi}{2}\right\}^{1/2}. \tag{4}$$

Here $\varphi(0)$ and α are arbitrary constants. From (4) and the conditions (3), we get the relation

$$H_e=2\left\{\alpha^2+\sin^2\frac{\varphi(0)}{2}\right\}^{1/2}=2\left\{\alpha^2+\sin^2\frac{\varphi(L)}{2}\right\}^{1/2} \tag{5}$$

or

$$\alpha^2=\frac{H_e^2}{4}-\sin^2\frac{\varphi(0)}{2},\quad \varphi(L)=\pm\varphi(0)+2\pi n, \tag{6}$$

where n is an integer.[1)] We use the notation $\varphi(0)=-\varphi_e$, $0\leqslant\varphi_e\leqslant\pi$; then we must write $\varphi(L)=+\varphi_e+2\pi n$, since the currents $j=\sin\varphi$ at the points $x=0$ and $x=L$ are directed oppositely (see Fig. 1), $j(0)=-j(L)$. With account of what has been said, the solution of Eq. (1) satisfying the conditions (3) is written down in the form

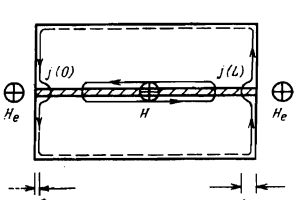

FIG. 1. Diagram of a tunnel barrier and the distribution of field and current in the case of the presence of a single vortex in the barrier.

$$x=\frac{1}{2}\int_{-\varphi_e}^{\varphi(x)}\left\{\frac{H_e^2}{4}-\sin^2\frac{\varphi_e}{2}+\sin^2\frac{\varphi}{2}\right\}^{-1/2}d\varphi,\quad \varphi(L)=\varphi_e+2\pi n. \qquad (7)$$

Here we should have labeled $\varphi(x)$ by an index n in order to denote the various solutions; however, for brevity, we shall omit this index.

The solution (7) can be represented in explicit form in terms of Jacobi elliptic functions (cf. Ref. 5), from which it follows, in particular that the functions $\varphi(x)$ and $H(x)$ are periodic. The extrema of the field correspond to the points where $d^2\varphi/dx^2=\sin\varphi=0$, i.e., $\varphi_{ext}=0, \pi, 2\pi, \ldots, 2\pi n$. The values of the field at the extremal points are

$$H_{min}=\left\{H_e^2-4\sin^2\frac{\varphi_e}{2}\right\}^{1/2},\quad H_{max}=\left\{H_e^2+4\cos^2\frac{\varphi_e}{2}\right\}. \qquad (8)$$

The schematic forms of the field and current distributions in the barrier for $n=0, 1, 2$ are shown in Fig. 2.

Using (7) and (2), it is not difficult to find the mean field in the barrier, which is described by the solution numbered n:

$$\bar{H}_n=\frac{1}{L}\int_0^L H\,dx=\frac{1}{L}\int_{-\varphi_e}^{\varphi_e+2\pi n}d\varphi=\frac{2\pi n+2\varphi_e}{L}. \qquad (9)$$

It then follows, in particular, that the number n gives the number of vortices penetrating into the barrier, with each of which there is associated a magnetic field equal to 2π (in ordinary units, this field would correspond to the flux quantum $\Phi_0=2\pi H_J\lambda_J\Lambda$). It is obvious that the quantity $2\varphi_e$ is equal to the field flux across the barrier. It is then easy to establish the physical meaning of the function $\varphi(x)$ in (1) and (2):

$$\varphi(x)=\int_0^x H\,dx-\varphi_e, \qquad (10)$$

i.e., $\varphi(x)$ is the field flux across the cut $[0,x]$ of the barrier after subtraction of the flux φ_e across the boundary (see Fig. 3a). The latter quantity is equal to the field flux on the portion from the point $x=0$ to the point $x_{ext}(j_{ext}=0)$, where the region of vortices begins (the unshaded portion in Fig. 3, to which corresponds a flux equal to 2π).

2. Substituting the value $x=L$ in (7), we obtain the relation

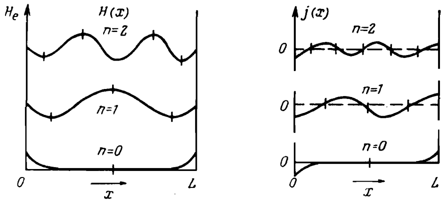

FIG. 2. Schematic distribution of field and current in a tunnel barrier of width L in the case of presence in it of $n=0$, 1, 2 vortices.

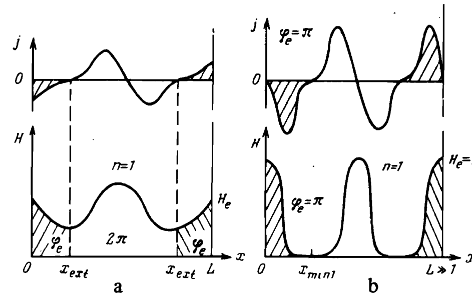

FIG. 3. Determination of the quantity φ_e. In the figure a—the case $\varphi_e<\pi$; the region where the magnetic field flux is equal to φ_e is shaded. A flux equal to 2π corresponds to the vortex; in the figure b, —the case $\varphi_e=\pi$; at $L\gg1$, $H_{min}=0$ and the solid curve on the portion $0\le x<x_{min\,1}$ is described by the Ferrell—Prange solution (see note 3).

$$L=\frac{1}{2}\int_{-\varphi_e}^{\varphi_e+2\pi n}\left\{\frac{H_e^2}{4}-\sin^2\frac{\varphi_e}{2}+\sin^2\frac{\varphi}{2}\right\}^{-1/2}d\varphi,\quad \alpha^2=\frac{H_e^2}{4}-\sin^2\frac{\varphi_e}{2}\geqslant 0, \qquad (11)$$

which determines the value of φ_e as a function of n, L, H_e. Since $0\leqslant\varphi_e\leqslant\pi$, it is seen from (11) that there exists some minimum field for given L and n, below which the requirement (11) cannot be satisfied and, consequently, there is no solution number n. It is evident that this minimum field H_{un} corresponds to the value $\varphi_e=0$ and is found from the relation

$$L=\frac{1}{2}\int_0^{2\pi n}\left\{\frac{H_{un}^2}{4}+\sin^2\frac{\varphi}{2}\right\}^{-1/2}d\varphi. \qquad (12)$$

At $H_e<H_{un}$, the solution changes from φ_n to φ_{n-1}.

We find in similar fashion the maximum field H_{sn}, above which there are no solutions with number n (this field corresponds to the value $\varphi_e=\pi$):

$$L=\frac{1}{2}\int_0^{2\pi(n+1)}\left\{\frac{H_{sn}^2}{4}-\cos^2\frac{\varphi}{2}\right\}^{-1/2}d\varphi. \qquad (13)$$

At $H_e>H_{sn}$, the solution should change from φ_n to φ_{n+1}.[2)]

We now study (12) and (13) in more detail. The relation (12), which determines the field H_{un}, can be written in the form

$$L/2n=k_{un}K(k_{un}),\quad k_{un}=\{1+H_{un}^2/4\}^{-1/2}, \qquad (14)$$

while the relation (13) for the field H_{sn} takes the form

$$L/2(n+1)=k_{sn}K(k_{sn}),\quad k_{sn}=2/H_{sn}, \qquad (15)$$

where $K(k)$ is a complete elliptic integral of the first kind:

$$K(k)=\int_0^{\pi/2}\frac{dx}{(1-k^2\sin^2 x)^{1/2}}. \qquad (16)$$

We now give the asymptotic expansion of the function $kK(k)$ at k close to zero and unity:

$$kK(k)=\frac{\pi}{2}\left(k+\frac{k^3}{4}+\ldots\right),\quad k\ll 1,$$
$$kK(k)=\ln\frac{4}{k'}-\frac{1}{4}\left(\ln\frac{4}{k'}+1\right)k'^2+\ldots,$$
$$k'=\sqrt{1-k^2}\ll 1. \tag{17}$$

The graph of the function $kK(k)$ is shown in Fig. 4. The asymptotes of (17) are given by dashed lines in the figure. We note the following circumstance, which will be useful later: at a value of $k=\sqrt{3}/2\approx 0.86$, the function $kK(k)\approx 1.86$.

The roots of Eq. (14) are determined by the intersections of the curve $kK(k)$ with the horizontal straight lines $L/2n$ (see Fig. 4). At small $L/2n$, we find, by using (14) and (17),

$$H_{un}=2[(\pi n/L)^2-1]^{1/2},\quad L/2n\ll 1. \tag{18}$$

At large $L/2n$, we obtain

$$H_{un}=8e^{-L/2n},\quad L/2n\gg 1. \tag{19}$$

At $L/2n\approx 1.86$, we have $k=\sqrt{3}/2$ (see Fig. 4); therefore, $H_{un}^2=\frac{4}{3}$, i.e.,

$$H_{un}=2/\sqrt{3}\ \text{at}\ L/2n\approx 1.86. \tag{20}$$

The graph of the curve $H_{un}(p)$, where $p=L/2n$, is shown schematically in Fig. 5a.

The form of the curve $H_{sn}(L)$ is found in similar fashion from (15):

$$H_{sn}=2\pi(n+1)/L,\quad L/2(n+1)\ll 1,$$
$$H_{sn}=2(1+8e^{-L/(n+1)}),\quad L/2(n+1)\gg 1, \tag{21}$$
$$H_{sn}\approx 4/\sqrt{3}\ \text{at}\ L/2(n+1)\approx 1.86.$$

The value of the field $H_{s0}=2$ at $L\gg 1$, which restricts the region of existence of the solution of the Meissner type ($n=0$), was found earlier by Kulik.[2,3] The graph of the curve $H_{sn}(p)$, where $p=L/2(n+1)$ is given in Fig. 5b.

Figure 6 shows schematically the behavior of the mean field $\overline{H}_n(H_e)$ (9) in a barrier of finite width with n vortices (solid lines); the points on the ends of these lines indicate the location of the fields H_{un} (the "supercooling" field) and H_{sn} (the "superheating" field). (We use here the terminology used for ordinary superconductors[6] to designate the boundaries of the regions of existence of

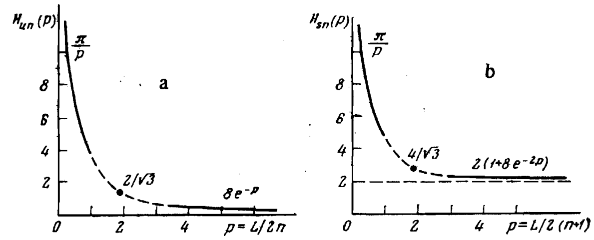

FIG. 5. a) Minimum field H_{un}, below which there are no solutions corresponding to n vortices in a barrier of width L; b) the maximum field H_{sn}, above which there are no solutions corresponding to n vortices in a barrier of width L.

superconductivity of films in a magnetic field.) The solution with n vortices exists in the range of fields $H_{un}\leq H_e\leq H_{sn}$. The field H_{un} corresponds to the value $\varphi_e=0$, the field H_{sn} to the value[3)] $\varphi_e=\pi$. The intermediate points on the curves correspond to the values $0\leq\varphi_e\leq\pi$.

It is seen from Fig. 6 that at a specified external field H_e solutions can exist with different n, but not larger than several units. The quantity $\overline{H}_{un}=\overline{H}(H_e=H_{un})=2\pi n/L$ corresponds to the maximum value of n, consistent with the given H_e. The large dots, which bound on the left of the region of existence of superconductivity, lie on the curve

$$\overline{H}_u(H_e)=4\pi\Big/\int_0^{2\pi}\left\{\frac{H_e^2}{4}+\sin^2\frac{\varphi}{2}\right\}^{-1/2}d\varphi=\frac{\pi}{k_uK(k_u)},$$
$$k_u=\left\{1+\frac{H_e^2}{4}\right\}^{-1/2}. \tag{22}$$

Using the expansions (17), we find

$$\overline{H}_u(H_e)=\pi\Big/\ln\frac{4\sqrt{4+H_e^2}}{H_e},\quad H_e\ll 1,$$
$$\overline{H}_u(H_e)=\sqrt{4+H_e^2},\ H_e\gg 1, \tag{22'}$$
$$\overline{H}_u\approx\pi/1{,}86\ \text{at}\ H_e=2/\sqrt{3}.$$

At a given field $H_e>H_{s0}$ there exists some minimum n, to which corresponds the value $\overline{H}_{sn}=\overline{H}(H_{sn})=2\pi(n+1)/L$ on Fig. 6 (we note that $\overline{H}(H_{u,n+1})=\overline{H}(H_{sn})$). The large dots, which restrict the region of exist of superconductivity to the right, lie on the curve

$$\overline{H}_s(H_e)=4\pi\Big/\int_0^{2\pi}\left\{\frac{H_e^2}{4}-\cos^2\frac{\varphi}{2}\right\}^{-1/2}d\varphi=\frac{\pi}{k_sK(k_s)},\quad k_s=\frac{2}{H_e},\quad H_e\geq 2. \tag{23}$$

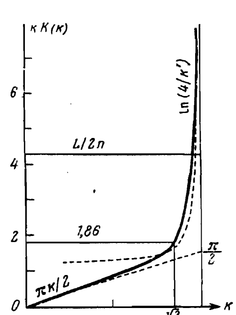

FIG. 4. Graph of the function $kK(k)$. The roots of Eqs. (14), (15) are determined by the intersection of the lines $L/2n$, or $L/(2n+1)$ with the heavy curve. The asymptotes of the function $kK(k)$ are given by the narrow lines.

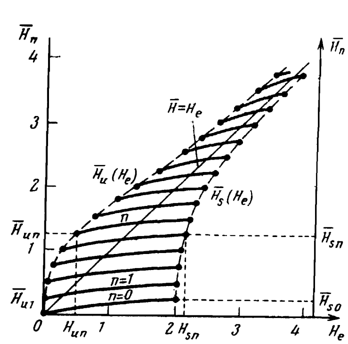

FIG. 6. The region of existence of superconducting solutions. The dashed curves H_u and H_s correspond to "supercooling" and superheating" fields. The heavy lines correspond to a definite number n vortices in the barrier.

Using the expansions (17), we find

$$\bar{H}_s(H_e)=\pi\Big/\ln\frac{4H_e}{(H_e^2-4)^{1/2}},\quad 0\leqslant H_e-2\ll 1,$$
$$\bar{H}_s(H_c)=H_c,\quad H_c\gg 1,$$
$$\bar{H}_s\approx\pi/1.86\quad\text{at}\quad H_e=4/\sqrt{3}. \tag{24}$$

The curves $\bar{H}_u(H_c)$ and $\bar{H}_s(H_v)$ are shown in Fig. 6 by dashed lines.

We now write out one more relation that follows from (14):

$$L_1(H_c)=2k_uK(k_u),\quad k_u^2=(1+H_c^2/4)^{-1}. \tag{25}$$

The quantity L_1 is the minimum size of the barrier at which one vortex still exists at the given field H_e. At $L<L_1$, there can be no vortices in the barrier, and only the Meissner state $n=0$ is achieved. Thus, in a specified field, vortices can exist in barriers whose width is not too small.

3. We now proceed to a more detailed investigation of the mean field in the barrier $\bar{H}_n(H_e)$. As is seen from (9), for this, we must know the function $\varphi_e(n, L, H_e)$, which should be found from the relation (11). At $L\ll 1$, it is necessary for existence of a solution of Eq. (11) that $\alpha^2\gg 1$, whence we obtain

$$L=\frac{1}{2\alpha}(2\pi n+2\varphi_e),\quad \varphi_e(n,L,H_e)=\frac{L}{2}H_e-\pi n,\quad L\ll 1,$$
$$H_{un}\leqslant H_c\leqslant H_{sn},\quad H_{un}\approx 2\pi n/L,\quad H_{sn}\approx 2\pi(n+1)/L. \tag{26}$$

Thus, at $H_e\sim H_{un}\varphi_e\sim 0$; at $H_e\sim H_{sn}\varphi_e\sim\pi$.

In the case $n=0$, it is not difficult to find the dependence $\varphi_e(n=0, L, H_e)$ at arbitrary L, but small H_e, when $\varphi_e\ll 1$. Here we get from (11)

$$L=\ln\frac{H_e+\varphi_e}{H_e-\varphi_e},\quad \varphi_e(n=0,L,H_e)=H_e\,\text{th}\frac{L}{2},\quad H_e\ll 1, \tag{27}$$

and from (8) we can find the value of the field at the center of the barrier:

$$H\left(x=\frac{L}{2}\right)=H_e\left(1-\text{th}^2\frac{L}{2}\right)^{1/2},\quad n=0. \tag{28}$$

At $L\gg 1$, we have $H(x=L/2)=2H_ee^{-L/2}$—the exponential decay of the field inside the barrier that is characteristic of the Meissner solution ($n=0$). At $L\ll 1$, we have $H_{n=0}(x)=H_e$.

In fields H_c close to H_{s0}, we have $\varphi_e\approx\pi$, and we find from (11) in the case $n=0$,

$$\varphi_c=\pi-\sqrt{4-H_c^2+64e^{-L}},\quad L\gg 1,\quad 0\leqslant 2-H_c\ll 1. \tag{29}$$

Knowing $\varphi_e(H_e)$ we can find the curves $\bar{H}_n(H_v)$ from (9), which are shown schematically in Fig. 6 by solid lines. In particular, at $n=0$, the initial portions of the curve $\bar{H}_{n=0}(H_v)$ depend linearly on H_e in correspondence with (27) and at $H_e\approx H_{s0}$ a square-root dependence exists at $L\gg 1$, in accord with (29). In the case of a change in the quantity L, the curves are transformed as shown in Fig. 7.

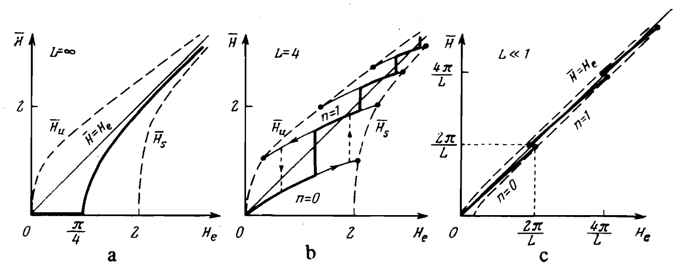

FIG. 7. The magnetization curves: a—the case of a wide barrier ($L\to\infty$), the heavy line was obtained by Kulik[2]; b—the case of a barrier of finite width ($L\sim 4$); the heavy vertical cuts correspond to jumps of the magnetization at thermodynamic equilibrium entrance of individual vortices into the junction (the location of the jumps is shown tentatively). The narrow lines illustrate the possibility of non-equilibrium (hysteretic) junctions; c—the case of point contact: the magnetization curve approaches the law $\bar{H}=H_e$.

As follows from the considerations given above, the existence of some set of solutions with different values of n is possible for a given field H_e. We now raise the question as to which of the possible solutions is realized at a given value of the field H_e. To answer this question, it is necessary to compare the thermodynamic potentials (free energy) of the states with different n; under equilibrium conditions, the system will be found in a state with the minimum free energy. We write down the Gibbs potential for the state n:

$$G_n(L, H_e)=\mathscr{E}_n(L, H_c)-\bar{H}_nH_c, \tag{30}$$

where H_n plays the role of the induction B of a weak superconductor (B is the mean macroscopic field),[4)] and $\mathscr{E}_n$ represents the energy per unit surface area of the tunnel contact (cf. Refs. 1–3):

$$\mathscr{E}_n=\frac{1}{L}\int_0^L\left(1-\cos\varphi+\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2\right)dx. \tag{31}$$

Here φ corresponds to the solution of number n, the term $1-\cos\varphi$ describes the energy connected with the distribution of currents in the barrier, and the term $\frac{1}{2}(d\varphi/dx)^2$ is the energy of the magnetic field. Using (1)–(7), it is not difficult to find

$$\mathscr{E}_n=-2\alpha^2+\frac{8}{L}n\int_0^{\pi/2}(\alpha^2+\sin^2x)^{1/2}dx+\frac{8}{L}\int_0^{\varphi_e/2}(\alpha^2+\sin^2x)^{1/2}dx,$$
$$\alpha^2=\frac{1}{4}H_e^2-\sin^2\frac{\varphi_e}{2}. \tag{32}$$

Let $L\gg 1$. At $n=0$, we find from (32)

$$\mathscr{E}_0=H_c^2/L\quad\text{at}\quad H_e\ll 1,\quad \mathscr{E}_0=8/L\quad\text{at}\quad H_e\sim 2. \tag{33}$$

Taking it into account that $\bar{H}_{n=0}=2\varphi_e/L=2H_e/L$ at $H_e\ll 1$ and $\bar{H}_{n=0}=2\pi/L$ at $H_e\sim 2$, we find from (30) and (33)

$$G_0=-\frac{H_e^2}{L}\quad\text{at}\quad H_e\ll 1,\quad G_0=\frac{8}{L}-\frac{4\pi}{L}\quad\text{at}\quad H_e\sim 2. \tag{34}$$

At $n=1$, we obtain, similarly,

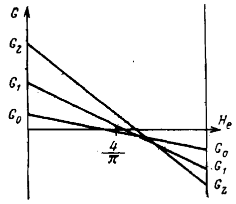

FIG. 8. The dependence on the field (schematically) of the Gibbs free energies corresponding to the different numbers $n=0, 1, 2$, vortices in the barrier. As $L\to\infty$, the point of intersection of the curves approaches $H_{c1}=4/\pi$. Under conditions of thermodynamic equilibrium the system is found in a state with minimum G.

$$G_1=\frac{8}{L}(4-\pi H_e) \quad \text{at} \quad H_e \ll 1,$$
$$G_1=\frac{4}{L}(4-\pi H_e) \quad \text{at} \quad H_e \sim 2. \tag{35}$$

The $G_n(H_e)$ curves are shown schematically in Fig. 8 in the case $L\gg 1$. As $L\to\infty$, G_0 and G_1 become small and then their intersection point tends to $H_e=4/\pi$. The value $H_{c1}=4/\pi$ is the critical field at which the first vortex appears in an infinitely wide barrier under equilibrium conditions. At $H_e>H_{c1}=4/\pi$, the function G_1 becomes smaller than G_0 and the system transforms from the state $n=0$ to the state $n=1$. The field H_{c1} was found earlier by Josephson,[1] and the entire path of the magnetization curve $\overline{H}(H_e)$ was obtained by Kulk in the case $L\to\infty$.[2,3] This curve is shown in Fig. 7a by the solid line.

We have sketched out here schematically a method of calculation of the thermodynamic potentials which allows us in principle to obtain the entire path of the magnetization curve in the case of arbitrary finite L. However, even without a detailed calculation, it is clear that at finite L, the equilibrium magnetization curve differs essentially from the curve found by Kulik (cf. Fig. 7a–c). Thus, the jumps in the magnetization at places where the system transforms in equilibrium fashion from the state n to the state $n+1$ are tentatively indicated in Fig. 7b by the heavy vertical cuts. It can be shown that these jumps reflect the "size quantization" that is characteristic for systems of finite dimensions. At $L\ll 1$, the magnetization curve approaches the curve $\overline{H}=H_e$ (Fig. 7c) and as $L\to 0$, only the state $n=0$ is realized.

Thus, we can conclude that real tunnel barriers, characterized by widths $L\sim 1$ (i.e., $L\sim\lambda_J\sim 0.1$ mm) should possess unique magnetic properties. Along with the magnetization jumps noted above, we can also observe in such barriers hysteresis phenomena connected with the possibility of existence of metastable states of the "superheating" and "supercooling" types in the magnetic field. An example of a possible hysteretic dependence is shown in Fig. 7b by the thin lines with arrows.

1) We shall assume below that $n\geqslant 0$; the case $n<0$ corresponds to vortices in which the direction of the magnetic field is opposite to that of the external field. Such a state is obviously not realized under equilibrium conditions; however, in principle, it can exist in the presence of forces that pin the vortices in the barrier.

2) The static solutions φ_n and φ_{n+1} describe essentially different field distributions (for example, in Fig. 2 at $n=0$ there is a minimum in the field at the center of the barrier, while at $n=1$ there is a maximum). For this reason, the transition from φ_n to φ_{n+1}, which is connected with the process of the entry of the vortex into the redistribution of the fields and currents, should be described by a time-dependent equation.

3) In the work of Ferrell and Prange,[4] a partial solution was obtained for φ having the form $\varphi=4\arctan e^{-x}$. This solution is obtained from (7) at $\varphi_e=\pi$ and corresponds to the external field H_e equal to the maximum "superheating" field, $H_e=H_{sn}=2$. The Ferrell–Prange solution describes the distribution of the field and the current in a wide barrier on the portion of the boundary of the barrier (where $H_e=H_{sn}=2$) up to the point $x_{\min 1}$, where the field has the first minimum (see Fig. 3b; here $H_{\min}=0$ at $\varphi_e=\pi$, in correspondence with (8); as $L\to\infty$, $x_{\min}\to\infty$ and the distributions the field and current transform into those which correspond to the Ferrell–Prange solution, cf. Fig. 12.1 of the book of Solymar[7]).

4) In ordinary units, the second term in (30) has the form $BH_e/4\pi$ and is equal to the change in the energy of the sample in the external field, while the first term corresponds to the internal energy.

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