Interaction of plane gravitational waves

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The interaction of two colliding homogeneous plane gravitational wave packets of arbitrary polarization is investigated. The behavior of the rays of both gravitational waves is analyzed by means of the equations for the optical scalars. It is shown that the rays are focused on a spacelike hypersurface. On this same surface, the optical scalars and invariants of the curvature tensor are singular. An equation is obtained for the singular surface in terms of the initial values of the optical scalars—the expansion parameters of the rays. The asymptotic behavior of the optical scalars and the invariant components of the curvature tensor near the singular surface is investigated.

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The interaction of plane gravitational waves in the general theory of relativity leads to an interesting effect—increase in the intensity of the gravitational field to infinitely large values, reflected mathematically in the occurrence of a singularity in the solution. In 1965, Penrose,[1] investigating the focusing effect of a plane gravitational wave on light rays, concluded that in a collision of two plane gravitational waves the focusing by each wave of the rays of the other wave leads to the appearance of a physical singularity in four-space. In[2–4] this conclusion was confirmed by examples of exact solutions describing the collision of gravitational pulses with constant (the same or opposite) polarization. In this case, as Szekeres[4] showed, a singularity arises in the solution for all profiles of the original wave packets. If the gravitational waves have constant polarization, the effect of each wave on the rays of the other oncoming wave remains the same: some rays of each wave converge all the time and others diverge. This necessarily leads to focusing of the rays and, as a consequence, to a singularity in the solution.

If the colliding gravitational waves have aribtrary polarization, their effect on lightlike rays does not remain constant. *A priori,* it is therefore not obvious whether rays of interacting waves will be focused and a singularity arise in the solution. Nevertheless, we show in the present paper that in this case of interaction of gravitational waves with arbitrary polarization focusing occurs and a physical singularity in the solution as well. We obtain an equation for the singular surface in terms of the initial values of the optical scalars of the rays of the gravitational waves. This equation enables us to relate readily the properties of the singular surface to the original profiles of the gravitational waves. We also investigate the asymptotic behavior of the various quantities that describe the gravitational field near the singular surface.

Suppose that two plane homogeneous wave packets move toward each other. Each of them is a solution of Einstein's equations in vacuum of type N. The quantities that describe one of the waves depend only on the phase u (u-wave); those that describe the other depend only on the phase v (v-wave). One can always find a coordinate system $x^0=u$, $x^1=v$, x^A $(A=2,3)$ such that the metric within each of the wave packets before the collision has the form

$$ds^2=2du\,dv-F_0dx^{2^2}-2G_0dx^2\,dx^3-H_0dx^{3^2}, \tag{1}$$

where the functions F_0, G_0, H_0 (for each wave) depend on u for the u-wave and on v for the v-wave.

Because of the homogeneity of the original wave packets, the metric in the interaction region can be written in the form

$$ds^2=2a\,du\,dv-Fdx^{2^2}-2Gdx^2\,dx^3-Hdx^{3^2}, \tag{2}$$

where the functions $a>0$, F, G, H now depend on u and v. Figure 1 shows the two-dimensional sections in four-space of the isotropic hypersurfaces $u=u_1$, $u=v_1$, $u=u_2$, $v=v_2$, which are represented by the straight lines. The equations $u=u_1$, $v=v_1$ describe the propagation in space of the fronts of the colliding waves; the equations $u=u_2$, $v=v_2$, the propagation of the trailing edges.

In the space between the colliding waves, for $u<u_1$, $v<v_1$, the metric has the form

$$ds^2=2du\,dv-dx^{2^2}-dx^{3^2}. \tag{3}$$

The metric forms (1), (3) are the initial conditions for finding the metric (2) in the interaction region. However, it is convenient to investigate this problem by means of the geometry of isotropic geodesic congruences of wave packets of gravitational waves: $\tilde{k}_\alpha=u_{,\alpha}$ and $\tilde{m}_\alpha=v_{,\alpha}$. Therefore, for the solution of the problem the Newman–Penrose formalism is convenient. The behavior of the congruences $\tilde{k}^\alpha$ and $\tilde{m}^\alpha$ is described by the so-called optical scalars, which characterize the rate of expansion and deformation (shear) of small bundles of rays. If we introduce the complex vector t_α satisfying the conditions $t_\alpha\tilde{k}^\alpha=t_\alpha\tilde{m}^\alpha=t_\alpha t^\alpha=0$, $t_\alpha\bar{t}^\alpha=-1$, the expansion parameters of the congruences $\tilde{k}^\alpha$ and $\tilde{m}^\alpha$ are, respectively, equal to

$$\tilde{\rho}=-{}^1/_2\tilde{k}^\alpha{}_{,\alpha}=\tilde{k}_{\alpha;\beta}t^\alpha\bar{t}^\beta;\ \tilde{\mu}={}^1/_2\tilde{m}^\alpha{}_{,\alpha}=-\tilde{m}_{\alpha;\beta}t^\alpha\bar{t}^\beta; \tag{4}$$

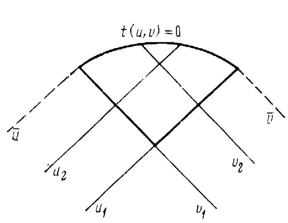

FIG. 1.

the complex shear parameters are

$$\tilde{\sigma}=\tilde{k}_{\alpha;\beta}t^{\alpha}t^{\beta}, \quad \tilde{\lambda}=-\tilde{m}_{\alpha;\beta}\bar{t}^{\alpha}\bar{t}^{\beta}. \tag{5}$$

The optical scalars together with the tetrad components of the curvature tensor satisfy the system of Newman–Penrose equations, which in this paper is constructed on the basis of the vectors $k_\alpha=\tilde{k}^\alpha$, $m_\alpha=a\tilde{m}_\alpha$, t_α. In the coordinate system (u, v, x^A), these vectors have the components

$$\begin{aligned} &k^\alpha=a^{-1}\delta_1{}^\alpha, \quad m^\alpha=\delta_0{}^\alpha, \quad t^\alpha=t^A\delta_A{}^\alpha, \\ &k_\alpha=\delta_\alpha{}^0, \quad m_\alpha=a\delta_\alpha{}^1, \quad t_\alpha=t_A\delta_\alpha{}^A. \end{aligned} \tag{6}$$

If we introduce the functions $\rho=a\tilde{\rho}$, $\mu=a\tilde{\mu}$, $\sigma=a\tilde{\sigma}$, $\lambda=a\tilde{\lambda}$, then the equations of the Newman–Penrose system that are needed to investigate the problem, and do not depend on the others, have the form

$$\partial\rho/\partial u=-2\rho\mu, \ \partial\mu/\partial v=2\rho\mu, \tag{7}$$

$$\partial\sigma/\partial u=-\mu\sigma-\rho\bar{\lambda}+2i\varkappa\sigma, \ \partial\lambda/\partial v=\rho\lambda+\mu\bar{\sigma}, \tag{8}$$

$$\partial\varkappa/\partial v=i(\lambda\sigma-\bar{\lambda}\bar{\sigma}), \tag{9}$$

$$a\Psi_2=\rho\mu-\lambda\sigma, \tag{10}$$

$$\partial\Psi_4/\partial v=\rho\Psi_4-3\lambda a\Psi_2, \tag{11}$$

$$\frac{\partial}{\partial u}(a^2\Psi_0)=(-\mu+2i\varkappa)a^2\Psi_0+3\sigma a\Psi_2, \tag{12}$$

where $\varkappa=\bar{\varkappa}=it_{\alpha;\beta}\bar{t}^\alpha k^\beta$; Ψ_0, Ψ_2, Ψ_4 are the complex tetrad components of the curvature tensor in the Newman–Penrose formalism.

The boundary conditions for the system (7)–(12) are specified on the two characteristics $v=v_1$ and $u=u_1$.

For $v=v_1$,

$$\rho=\sigma=\varkappa=0, \ a=1; \quad \mu=\mu_0(u), \quad \lambda=\lambda_0(u),$$

where the functions $\mu_0(u)$, $\lambda_0(u)$ satisfy for $u\geq u_1$ the equations

$$\frac{d\mu_0}{du}=-\mu_0{}^2-\lambda_0\bar{\lambda}_0, \quad \frac{d\lambda_0}{du}=-2\mu_0\lambda_0-\Psi_4{}^0(u) \tag{13}$$

with the initial conditions $\mu_0(u_1)=\lambda_0(u_1)=0$. In Eqs. (13), $\Psi_4^0(u)$ is the only nonzero tetrad component of the curvature tensor in the u-wave. Since we consider a wave packet, Ψ_4^0 satisfies the conditions $\Psi_4^0(u)\neq 0$ for $u_1<u<u_2$, $\Psi_4^0(u)=0$ for $u\leq u_1$, $u\geq u_2$.

For $u=u_1$,

$$\mu=\lambda=\varkappa=0, \quad a=1; \quad \rho=\rho_0(v), \quad \sigma=\sigma_0(v),$$

where the functions $\rho_0(v)$, $\sigma_0(v)$ satisfy for $v\geq v_1$ the equations

$$\frac{d\rho_0}{dv}=\rho_0{}^2+\sigma_0\bar{\sigma}_0, \quad \frac{d\sigma_0}{dv}=2\rho_0\sigma_0+\Psi_0{}^0(v) \tag{14}$$

with the initial conditions $\rho_0(v_1)=\sigma_0(v_1)=0$. In Eqs. (14), $\Psi_0^0(v)$ is the only nonzero component of the curvature tensor in the v-wave. The component Ψ_0^0 satisfies the conditions $\Psi_0^0(v)\neq 0$ for $v_1<v<v_2$; $\Psi_0^0(v)=0$ for $v\leq v_1$, $v\geq v_2$.

The functions Ψ_4^0 and Ψ_0^0 characterize the intensity and polarization of the u- and v-wave, respectively. In the case of constant polarization,[2-4] the functions Ψ_4^0 and Ψ_0^0 are real. Specification of the functions Ψ_4^0 and Ψ_0^0 completely determines all properties of the original gravitation waves, and therefore these functions will be used as the basic characteristics of the original waves. When we speak of the profile of the u-wave (v-wave) we shall mean namely the function Ψ_4^0 (respectively Ψ_0^0).

We make one other comment concerning the boundary conditions for the optical scalars. Consider, for example, the region $v<v_1$, $u\geq u_1$. The parameters $\mu=\mu_0(u)$, $\lambda=\lambda_0(u)$ in this region refer to isotropic rays, which one can identify, for example, with the world lines of test photons passing through the u-wave. Nonvanishing of the parameters μ_0 and λ_0 reflects focusing by the gravitational wave of rays passing through it. The contraction and deformation of the beam of photons passing through the u-wave are described by Eqs. (13). The focusing of the rays means that the functions μ_0 and λ_0 and the metric (1) have a singularity on a certain isotropic hypersurface $u=\bar{u}$.[5] Depending on the intensity and polarization of the u-wave, the caustic surface $u=\bar{u}$ may be either within the gravitational wave packet or behind its trailing edge. In Fig. 1, the dashed line shows one of the possible positions of the caustic surface $u=\bar{u}$. In the region $u<u_1$, the behavior of isotropic rays in the field of the v-wave is described by Eqs. (14). In Fig. 1, in the region $u<u_1$, $v>v_2$, we show one of the possible positions of the caustic surface $v=\bar{v}$, on which the light rays passing through the v-wave are focused and the functions ρ_0 and σ_0 become infinite. Because of the presence of the caustics, the boundary conditions for the optical scalars are specified as follows: on the characteristic $v=v_1$ in the interval $u_1\leq u<\bar{u}$ and on the characteristic $u=u_1$ in the interval $v_1\leq v<\bar{v}$.

We now turn to an exposition of the results of investigation of the system of equations (7)–(12). From Eqs. (7), it can be seen that one can introduce a function $w(u, v)$ such that

$$\partial w/\partial u=-\mu, \quad \partial w/\partial v=\rho. \tag{15}$$

The function w satisfies the equation

$$\frac{\partial^2 w}{\partial u\,\partial v}=2\frac{\partial w}{\partial u}\frac{\partial w}{\partial v} \tag{16}$$

and the boundary conditions

$$w(u, v_1)=-\int_{u_1}^{u}\mu_0(\xi)\,d\xi, \quad w(u_1, v)=\int_{v_1}^{v}\rho_0(\xi)\,d\xi, \tag{17}$$

where the functions $\mu_0(u)$ and $\rho_0(v)$ are solutions of Eqs. (13) and (14). Equation (16) can be solved by means of the substitution $\psi=e^{-2w}$, and, using the boundary conditions (17), $w=-\frac{1}{2}\ln t$, we finally obtain

$$t=\exp\left(2\int_{u_1}^{u}\mu_0(\xi)\,d\xi\right)+\exp\left(-2\int_{v_1}^{v}\rho_0(\xi)\,d\xi\right)-1, \tag{18}$$

$$\mu=\frac{\mu_0(u)}{t}\exp\left(2\int_{u_1}^{u}\mu_0(\xi)\,d\xi\right), \tag{19}$$

$$\rho=\frac{\rho_0(v)}{t}\exp\left(-2\int_{v_1}^{v}\rho_0(\xi)\,d\xi\right). \tag{20}$$

As can be seen from (19) and (20), the functions μ and ρ become infinite on the hypersurface $t(u,v)=0$. This hypersurface is spacelike. Indeed,

$$g^{\alpha\beta}t_{,\alpha}t_{,\beta}=2a\frac{\partial t}{\partial u}\frac{\partial t}{\partial v}=-8a\mu_0\rho_0\exp\left(2\int_{u_1}^{u}\mu_0(\xi)\,d\xi-2\int_{v_1}^{v}\rho_0(\xi)\,d\xi\right),\qquad(21)$$

and since, as follows from Eqs. (13) and (14), $\mu_0<0$, $\rho_0>0$, the expression (21) is positive.

For $u\leqslant u_1$ or $v\leqslant v_1$, the functions ρ and μ are equal, respectively, to $\rho_0(v)$ and $\mu_0(u)$, which, as we have said above, become infinite for $v=\bar{v}$ and $u=\bar{u}$. The equation $t=0$ can also be considered for $u<u_1$ or $v<v_1$. Then it determines the same isotropic hypersurfaces $v=\bar{v}$ or $u=\bar{u}$. Thus, the equation $t(u,v)=0$ determines a spacelike surface that joins on continuously at the boundaries $u=u_1$, $v=v_1$ to the isotropic hypersurfaces $v=\bar{v}$, $u=\bar{u}$ (see Fig. 1).

The parameters ρ and μ have a simple geometrical meaning. Let $\Delta S=(FH-G^2)^{1/2}\Delta x^2\Delta x^3$ be the cross sectional area of a small bundle of rays of either the u- or v-wave. Then ρ and μ are related in a simple manner to the relative change of this area along the coordinates v and u:

$$\rho=-\frac{1}{2}\frac{1}{\Delta S}\frac{\partial\Delta S}{\partial v},\quad \mu=\frac{1}{2}\frac{1}{\Delta S}\frac{\partial\Delta S}{\partial u}.\qquad(22)$$

The infinite values of ρ and μ at $t=0$ means that $\Delta S=0$ ($\Delta S=\text{const}\cdot t$) on this surface, i.e., the rays of the original waves are focused on $t=0$. As will be seen from what follows, the focusing leads to an infinite value of the invariants of the curvature tensor, and therefore there is a physical singularity of spacetime on the hypersurface $t=0$. Thus, the interaction region is determined by the inequalities

$$u\geqslant u_1,\quad v\geqslant v_1,\quad t(u,v)>0.\qquad(23)$$

The form of the functions μ_0 and ρ_0 in the expression (18) for t is entirely determined by the form of the functions Ψ_4^0 and Ψ_0^0, and therefore the geometry and the size of the section of the singular surface $t=0$ are also determined by the original profiles of the wave packets.

Equations (8) and (9) do not permit one to obtain an explicit solution for the functions σ, λ, $\varkappa$ except for the case considered by Szekeres,[4] when $\varkappa=0$ and σ and λ are real functions. However, using the expressions for ρ and μ, one can analyze the behavior of the shear parameters near the singular surface and find their asymptotic behavior as $t\to 0$, which, in general, is the behavior in which one is mainly interested. In the Appendix we show that the functions σ, λ, $\varkappa$ have the behavior

$$\sigma\sim\alpha(x)t^{-1}+o(t^{-1}),\quad \lambda\sim\bar{\alpha}(x)f(x)t^{-1}+o(t^{-1}),\quad \varkappa\sim o(t^{-1}),\qquad(24)$$

where x is the coordinate orthogonal to t:

$$x(u,v)=\exp\left(2\int_{u_1}^{u}\mu_0(\xi)\,d\xi\right)-\exp\left(-2\int_{v_1}^{v}\rho_0(\xi)\,d\xi\right),\qquad(25)$$

$$f(x)=\lim_{t\to 0}\left[\mu_0(u)\rho_0^{-1}(v)\exp\left(2\int_{u_1}^{u}\mu_0(\xi)\,d\xi+2\int_{v_1}^{v}\rho_0(\xi)\,d\xi\right)\right].\qquad(26)$$

From Eqs. (7) and (8) one can obtain two further integrals:

$$(\rho^2-|\sigma|^2)\rho^{-1}-\frac{\partial\ln a}{\partial v}=(\rho_0^2-|\sigma_0|^2)\rho_0^{-1},\qquad(27)$$

$$(\mu^2-|\lambda|^2)\mu^{-1}+\frac{\partial\ln a}{\partial u}=(\mu_0^2-|\lambda_0|^2)\mu_0^{-1},\qquad(28)$$

which with allowance for (19), (20), and (24) enable one to find the asymptotic behavior of the functions $a(u,v)$. Bearing in mind that

$$\frac{\partial}{\partial u}=\mu_0(u(t,x))\,(t+x+1)\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right),$$

$$\frac{\partial}{\partial v}=-\rho_0(v(t,x))\,(t-x+1)\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial x}\right),$$

and taking into account the expressions for σ and λ, we obtain from Eqs. (27) and (28)

$$\frac{\partial\ln a}{\partial t}\simeq\gamma(x)t^{-1}+o(t^{-1}),\qquad(29)$$

$$\gamma(x)=2|\alpha(x)|^2(1-x)^{-2}\rho_0^{-2}(x,t=0)-\frac{1}{2}.\qquad(30)$$

From (29),

$$a\sim t^{\gamma(x)}.$$

Finally, we give the asymptotic expressions for the tetrad components of the curvature tensor. For this it is convenient to operate with the functions $\tilde{\Psi}_4=\Psi_4$, $\tilde{\Psi}_2=a\Psi_2$, $\tilde{\Psi}_0=a^2\Psi_0$. It follows from (10) with allowance for the above results that

$$\operatorname{Re}\tilde{\Psi}_2=\rho\mu-{}^1/_2(\sigma\lambda+\bar{\sigma}\bar{\lambda})$$
$$\sim-{}^1/_2\rho_0(x,\ t=0)\,\mu_0(x,\ t=0)\,(1-x^2)\gamma(x)t^{-2}+o(t^{-2}),\qquad(31)$$

$$\operatorname{Im}\tilde{\Psi}_2=\frac{i}{2}(\sigma\lambda-\bar{\sigma}\bar{\lambda})\simeq o(t^{-2}).\qquad(32)$$

Integration of Eqs. (11) and (12) gives

$$\tilde{\Psi}_4\sim\mu_0(x,\ t=0)\,(1+x)\bar{\alpha}(x)f(x)\gamma(x)t^{-2}+o(t^{-2}),\qquad(33)$$

$$\tilde{\Psi}_0\sim\rho_0(x,\ t=0)\,(1-x)\alpha(x)\gamma(x)t^{-2}+o(t^{-2}).\qquad(34)$$

The invariants $I_1=R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$ and $I_2=\overset{*}{R}_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$ can be expressed in terms of Ψ_0, Ψ_2, Ψ_4 by

$$I_1-iI_2=16(\Psi_0\Psi_4+3\Psi_2)=16a^{-2}(\tilde{\Psi}_0\tilde{\Psi}_4+3\tilde{\Psi}_2^2).\qquad(35)$$

Taking into account (31)–(34), we obtain

$$I_1\sim t^{-2\gamma-4},\quad I_2\sim o(t^{-2\gamma-4}).\qquad(36)$$

The proof given in the Appendix does not give the explicit form of the function $\alpha(x)$, but, as follows from the expression (30) for $\gamma(x)$, $2\gamma+4\geqslant 3$ (the equality sign holds for the case not excluded when $\alpha(x)=0$, and $\gamma=-\frac{1}{2}$). This means that the invariant I_1 increases as $t\to 0$ not slower than t^{-3}, and the invariant I_2 increases more slowly. Thus, the interaction of plane gravitational waves of arbitrary polarization also leads to the occurrence of a physical singularity.

It follows from Eqs. (24) that $\varkappa$ is small compared with σ and λ as $t\to 0$. Therefore, near the singular surface the behavior of the gravitational field in the case of collision of waves with arbitrary polarization is the same as in the case of collision of waves with the same or opposite polarization.

This picture of the interaction of two gravitational waves can be well explained by invoking two well-known facts: 1) scattering of radiation on curvature of space-time; 2) focusing by a gravitational wave of type N of radiation passing through it. The passage of gravitational waves through one another is accompanied by continuous reflection of both waves. Because of this, the original u-wave is transformed into a complicated mixture of an infinite number of direct and reflected waves propagating in the similarly modified v-wave. The increase in the intensity of the gravitational field is due to the fact that the waves that move in one direction and are either the remainder of the original wave or are generated by reflections of both waves are focused by the waves approaching one another. It is a characteristic of the nonlinear theory that even weak original gravitational waves amplify each other to infinitely large values in a collision.

The singularity is essentially due to the focusing of the rays. But, for example, a plane gravitational wave of type N does not always focus a diverging beam of light rays so that a caustic appears. It is therefore to be expected that the interaction of gravitational waves with nonplanar wave surfaces will not necessarily lead to a singularity in the solution.

APPENDIX

To find the asymptotic behavior of the functions σ, λ, $\varkappa$, we investigate the behavior of the functions

$$\varphi_1=\frac{t|\sigma|}{\rho_0(v)}\exp\left(2\int_{v_1}^{v}\rho_0(\xi)\,d\xi\right),\quad \varphi_2=\frac{t|\lambda|}{|\mu_0(u)|}\exp\left(-2\int_{u_1}^{u}\mu_0(\xi)\,d\xi\right). \tag{A.1}$$

We replace the coordinates u and v by

$$\zeta(u)=\exp\left(2\int_{u_1}^{u}\mu_0(\xi)\,d\xi\right)-\frac{1}{2},\quad \eta(v)=\exp\left(-2\int_{v_1}\rho_0(\xi)\,d\xi\right)-\frac{1}{2}, \tag{A.2}$$

and ζ and η vary from $\frac{1}{2}$ to $-\frac{1}{2}$ when u varies from u_1 to $\bar{u}$ and v from v_1 to $\bar{v}$, respectively. The coordinates t and x defined by Eqs. (18) and (25) are related to ζ and η by

$$t=\zeta+\eta,\quad x=\zeta-\eta.$$

The interaction region is shown in the coordinates ζ and η in the form of the triangle AOB (Fig. 2). From Eqs. (8), one can also obtain equations for φ_1, φ_2:

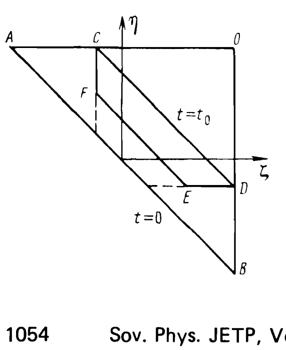

FIG. 2.

$$\frac{\partial\varphi_1}{\partial\zeta}=\frac{1}{2}(\varphi_1+\varphi_2\cos\alpha),\quad \frac{\partial\varphi_2}{\partial\eta}=\frac{1}{2}(\varphi_1\cos\alpha+\varphi_2), \tag{A.3}$$

$$\alpha=\arg\sigma+\arg\lambda.$$

We investigate Eqs. (A.3) for $t_0\geq t>0$, regarding the values of the functions φ_1 and φ_2 on $t=t_0$ as initial conditions. From Eqs. (A.3) we obtain these integral relations for φ_1 and φ_2:

$$\varphi_1(\zeta,\eta)=\left(\frac{t}{t_0}\right)^{1/2}\varphi_1(t_0-\eta,\eta)-\frac{t^{1/2}}{2}\int_{\zeta}^{t_0-\eta}\frac{\varphi_2(\zeta',\eta)\cos\alpha(\zeta',\eta)}{(\zeta'+\eta)^{3/2}}\,d\zeta', \tag{A.4}$$

$$\varphi_2(\zeta,\eta)=\left(\frac{t}{t_0}\right)^{1/2}\varphi_2(\zeta,t_0-\zeta)-\frac{t^{1/2}}{2}\int_{\eta}^{t_0-\zeta}\frac{\varphi_1(\zeta,\eta')\cos\alpha(\zeta,\eta')}{(\zeta+\eta')^{3/2}}\,d\eta'. \tag{A.5}$$

Substituting (A.5) into (A.4), we obtain for φ_1 the integral equation

$$\varphi_1=\left(\frac{t}{t_0}\right)^{1/2}\varphi_1(t_0-\eta,\eta)-\frac{1}{2}\left(\frac{t}{t_0}\right)^{1/2}\int_{\zeta}^{t_0-\eta}\frac{\varphi_2(\zeta',t_0-\zeta')\cos\alpha(\zeta',\eta)}{(\zeta'+\eta)}$$

$$\times d\zeta'+\frac{t^{1/2}}{4}\int_{\zeta}^{t_0-\eta}\frac{\cos\alpha(\zeta',\eta)}{\zeta'+\eta}\,d\zeta'\int_{\eta}^{t_0-\zeta'}\frac{\varphi_1(\zeta',\eta')\cos\alpha(\zeta',\eta')}{(\zeta'+\eta')^{3/2}}\,d\eta'. \tag{A.6}$$

A similar equation is obtained for φ_2. From Eq. (A.6) one can determine the values of the function φ_1 at all points of the segment EF (Fig. 2) at the time t if the values of φ_1 and φ_2 are known on CD (and the values of $\cos\alpha$ in the rectangle $CDEF$). We denote by m_1 and m_2 the maxima of the functions $\varphi_1(t_0, x)$ and $\varphi_2(t_0, x)$ on CD and assume

$$m=\max\{m_1, m_2\},$$

for example, $m=m_1$. Suppose also $M=\max|\varphi_1(t,x)|$ in the rectangle $CDEF$. Then from Eq. (A.6) we obtain an estimate for φ_1:

$$|\varphi_1(t,x)|\leqslant m\left(\frac{t}{t_0}\right)^{1/2}+m\left(\frac{t}{t_0}\right)^{1/2}\int_{\zeta}^{t_0-\eta}\frac{d\zeta'}{\zeta'+\eta}+\frac{Mt^{1/2}}{4}\int_{\zeta}^{t_0-\eta}\frac{d\zeta'}{\zeta'+\eta}\int_{\eta}^{t_0-\zeta'}\frac{d\eta'}{(\zeta'+\eta')^{3/2}}$$

$$=M+(m-M)\left(\frac{t}{t_0}\right)^{1/2}\left(1+\frac{1}{2}\ln\frac{t_0}{t}\right). \tag{A.7}$$

We show that the values of the function φ_1 at any time $t<t_0$ within the rectangle $CDEF$ do not exceed the maximum of the function φ_1 on CD. Suppose otherwise, and $M>m$. Then one can find a position of the line EF such that the maximum of φ_1 in the rectangle $CDEF$ is attained at one or several points on EF. Writing down the inequality (A.7) for such a point, we obtain

$$M\leqslant M+(m-M)\left(\frac{t}{t_0}\right)^{1/2}\left(1+\frac{1}{2}\ln\frac{t_0}{t}\right),$$

or $M\leqslant m$, which contradicts the assumption.

Thus, the continuous function φ_1 is bounded right up to $t=0$. It follows from (A.5) that φ_2 is also a bounded function since the integral operator on the right-hand side of (A.5) is bounded. Since the functions

$$\rho_0(v)\exp\left(-2\int_{v_1}^{v}\rho_0(\xi)\,d\xi\right)\text{ and }\mu_0(u)\exp\left(2\int_{u_1}^{u}\mu_0(\xi)\,d\xi\right)$$

are bounded in this region, the functions $t\sigma$ and $t\lambda$ are bounded. The function $\varphi_3=2t\varkappa$ can be expressed in terms of these last by means of the integral relation

$$\varphi_3 = \frac{t}{t}\varphi_3(u, v^*) - 2it\int_{v^*}^{v} \frac{[t^2(\bar\sigma\bar\lambda - \sigma\lambda)]}{t^2(u, y)}dy, \quad (A.8)$$

where v^* is a root of the equation $t(u, v^*) = t_0$. The integral operator in (A.8) is bounded, and therefore the function $\varphi_3 = 2t\varkappa$ is bounded. Thus, taking into account the boundedness of the functions $t\sigma$ and $t\lambda$ and assuming them continuous right up to $t=0$, we obtain the asymptotic estimates

$$\sigma \sim \alpha(x)t^{-1} + o(t^{-1}), \quad \lambda \sim \beta(x)t^{-1} + o(t^{-1}), \quad (A.9)$$

where $\alpha(x)$ and $\beta(x)$ are certain continuous functions that do not in general vanish identically. Between $\alpha(x)$ and $\beta(x)$ there is the relationship

$$\beta(x) = \bar\alpha(x)f(x) \quad (A.10)$$

($f(x)$ is given by Eq. (26)), which can be obtained by assuming for simplicity

$$\frac{\partial\sigma}{\partial t} = -\alpha(x)t^{-2} + o(t^{-2}) \quad (A.11)$$

and substituting (A.9) and (A.11) into Eq. (8) for σ.

One need not assume (A.11) but instead obtain (A.10) by using the integral relationship between λ and $\bar\sigma$:

$$\lambda(u, v) = \lambda_0(u)t^{-1/2}\exp\left(\int_{u_1}^{u}\mu_0(\xi)d\xi\right)$$
$$+ |\mu_0(u)|t^{-1/2}\exp\left(2\int_{u_1}^{u}\mu_0(\xi)d\xi\right)\int_{v_1}^{v}\bar\sigma(u, \xi)t^{-1/2}(u, \xi)d\xi.$$

Finally, if (A.9) is substituted with allowance for (A.10) into Eq. (A.8), we obtain for $\varkappa$ the asymptotic behavior

$$\varkappa \sim o(t^{-1}). \quad (A.12)$$

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Calculation of the Gell-Mann–Low function in scalar theory with strong nonlinearity

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The Gell-Mann–Low function in a scalar logarithmic theory with a polynomial interaction is found in all orders of perturbation theory in the limit of strong nonlinearity of the interaction. The existence of ultraviolet-stable points in the theory is proved.

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1. INTRODUCTION

It is well known that, in quantum electrodynamics and in most of the models of quantum field theory known until recently, the physical phenomenon of screening of the interaction as a result of quantum fluctuations of the vacuum occurs and has the result that the physical charges vanish if the bare charges are sufficiently small.[1] At the same time, in many currently popular nonabelian gauge models the opposite situation obtains, viz., for sufficiently small physical charges the bare charge vanishes.[2] These results were obtained by means of perturbation theory and therefore have a limited region of applicability. To investigate the consistency of quantum electrodynamics and the other traditional models of quantum field theory, and also to study the question of the infrared catastrophe in Yang–Mills theory, it is necessary to go beyond the framework of perturbation theory. The use of the renormalization-group apparatus proposed by Gell-Mann and Low (GML)[3] is of great benefit here. Thus, e.g., the question of the consistency of quantum electrodynamics reduces to the calculation, in all orders of perturbation theory, of the so-called GML function $\psi(\alpha)$, which, by definition is equal to

$$\psi(\alpha(k^2)) = \frac{d\alpha(k^2)}{d\ln(k^2/m^2)}, \quad \alpha(k^2) \equiv \alpha_c d_c(k^2/m^2), \quad (1)$$

where $\alpha_c = e_c^2/4\pi\hbar c$. The asymptotic form of the renormalized photon Green function is related to $d_c(k^2/m^2)$ by the formula

$$D_c^{\mu\nu}(k^2)|_{k^2\gg m^2} = -\frac{g_{\mu\nu}}{k^2}d_c(k^2).$$

As is well known, there are three possibilities, leading to different physical consequences: 1) the GML function (1) has a zero: $\psi(\alpha_0) = 0$. Then, when the bare charge is chosen equal to α_0, the renormalized charge does not vanish and the asymptotic form of the Green