[Sov. Phys. Dokl. 11, 881 (1967); J. Appl. Mech. Tech. Phys 8, no. 5, 37 (1967)].
2V. E. Zakharov, Zh. Eksp. Teor. Fiz. 51, 688 (1966); 62, 1745 (1972) [Sov. Phys. JETP 24, 455 (1967); 35, 908 (1972)].
3V. E. Zakharov and R. Z. Sagdeev, Dokl. Akad. Nauk SSSR 192, 297 (1970) [Sov. Phys. Dokl. 15, 439 (1971)].
4A. V. Kats and V. M. Kontorovich, Prikl. Mekh. Tekh. Fiz. no 6, 97 (1974) [J. Appl. Mech. Tech. Phys. 15, No. 6, (1974)].
5A. V. Kats, V. M. Kontorovich, S. S. Moiseev, and V. E. Novikov, Pis'ma Zh. Eksp. Teor. Fiz. 21, 13 (1975); Zh. Eksp. Teor. Fiz. 71, 177 (1976) [JETP Lett. 21, 5 (1975); Sov. Phys. JETP 44, 93 (1976)].
6B. B. Kadomtsev and V. M. Kontorovich, Izv. vuzov, Radiofizika 17, 511 (1974) [Radiophys. Qu. Electron. 17, 386(1975)].
7A. V. Kats and V. M. Kontorovich, Zh. Eksp. Teor. Fiz. 64, 153 (1973); 65, 206 (1973) [Sov. Phys. JETP 37, 80 (1973); 38, 102 (1974)].
8V. I. Karas', Pis'ma Zh. Tekh. Fiz. 1, 1020 (1975) [Sov. Tech. Phys. Lett. 1, 438 (1975)]. V. I. Karas', S. S. Moiseev, and V. E. Novikov, Zh. Eksp. Teor. Fiz. 71, 1421 (1976) [Sov. Phys. JETP 44, 744 (1976)].
9A. V. Kats and V. M. Kontorovich, Preprint Inst. Radiophys. Electron. Acad. Sc. Ukr. SSR, No. 53, Khar'kov, 1975.
10A. S. Monin and A. M. Yaglom, Statisticheskya gidromekhanika (Statistical Hydromechanics) Vol. 2, Nauka, 1967 [translation published by MIT Press].
11E. F. Beckenbach and R. Bellman, NeraVenstva (Inequalities) Mir, Moscow, 1965, p. 72 [Russian translation of book published by Springer, Berlin, 1961].
12L. D. Landau and E. M. Lifshitz, Mekhanika (Mechanics) Nauka, 1965 [translation published by Pergamon Press, Oxford].
13V. N. Tsytovich, Teoriya turbulentnoĭ plazmy (Theory of a Turbulent Plasma) Atomizdat, 1971 [Translation to be published by Plenum Press].
14V. I. Botsman, A. V. Kats, and V. M. Kontorovich, Dokl. Akad. Nauk SSSR 220, 1053 (1975) [Sov. Phys. Dokl. 20, 119 (1975)].
15N. G. Basov, O. N. Krokhin, Yu. A. Mikahĭlov, V. V. Pustovalov, A. A. Rupasov, V. P. Silin, G. V. Sklizkov, V. T. Tikhonchuk, and A. S. Shikanov, Proceedings 2nd International Conference on the Theory of Plasmas, Kiev, 29 Oct.–1 Nov. 1974, Naukova Dumka, Kiev, 1976, p. 342.

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Excitation of ordinary waves in a plasma with a diffuse boundary under anomalous skin-effect conditions

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We study the absorption of an electromagnetic wave by a plasma with a diffuse boundary. We assume that the decrease in the particle density is exponential and that the magnetic field is parallel to the boundary of the plasma. We show that the collisionless absorption of waves with the electric vector directed along the magnetic field is connected with the excitation of ordinary waves in the plasma for those densities for which they can exist in a homogeneous plasma. We study the lineshapes of the electron and ion absorption resonances, especially in the effective collision frequency approximation. We obtain expressions for the limits of the existence of the ordinary cyclotron waves in the plasma and we solve the dispersion equation for a high-pressure uniform plasma.

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§1. INTRODUCTION

P. L. Kapitza[1] was the first to state the problem of the anomalous skin-effect in a plasma with a diffuse boundary in connection with a study of a high-frequency discharge in a plasma at high pressures. Liberman, Meĭerovich, and Pitaevskiĭ[2] constructed a theory of the skin-effect in a semi-infinite non-uniform plasma, and obtained an integro-differential equation for the electromagnetic field in the plasma for an arbitrary relation between the electron mean free path, the penetration depth of the field in the plasma, and the size of the transition region at the boundary. This equation has been solved for the case of an exponential decrease of the electron density outside the plasma under conditions of an extremely anomalous skin-effect[2] and for an arbitrary degree of anomalicity.[3] In[3] a plasma with a magnetic field directed parallel to the density gradient for any degree of anomalicity of the skin-effect was also studied. Dikman and Meĭerovich[4] considered the extremely anomalous skin-effect for the case where the magnetic field was strictly parallel to the boundary and obtained solutions for an exponential and for a power-law decrease in the electron density. We study in the present paper the absorption of electromagnetic waves in the case of arbitrary anomaly of the skin-effect when the electric field of the incident wave is parallel to the constant magnetic field which lies in the plane of the plasma boundary and we also analyze a mechanism for collisionless absorption which consists in the transformation of the incident wave into ordinary cyclotron waves.

We shall assume that the size of the transition zone at the boundary of the plasma is small compared to the characteristic dimensions of the plasma, but large compared to the penetration depth of an electromagnetic wave into a uniform plasma with the same electron and ion density as are reached in the bulk of the plasma. Under those conditions we can assume the particle density to be a function of a single coordinate x. When there is no electromagnetic wave the subsystems of ions and electrons are assumed separately to be equilibrium ones with temperatures T_i and T_e. The particle densities can be expressed in terms of the Boltzmann distribution formula

$$n_\alpha(x)=n_{0\alpha}\exp(-U_\alpha(x)/kT_\alpha),$$

where the $U_\alpha(x)$ are the effective potential energies of the electrons and ions (here and henceforth the subscript $\alpha=e, i$ indicates the kind of particle). It is possible to solve the problem completely if the effective potential energies change linearly outside the plasma, i.e., if the particle densities decrease exponentially:

$$n_e(x)=n_i(x)=n_0e^{x/a},\quad x\to-\infty. \tag{1.1}$$

In §3 we study not only the line shape of the cyclotron resonance but also the regions where ordinary cyclotron waves exist in the plasma, and also the solution of the dispersion equation for ordinary waves in a high-pressure plasma.

§2. SOLUTION OF THE EQUATION FOR AN ELECTROMAGNETIC WAVE POLARIZED ALONG H

The set of equations for an electromagnetic wave $E_i(x)e^{i\omega t}$ which propagates in a non-uniform plasma along the x-axis along the density gradient in the plasma consists of the Maxwell equations which connect the field components E_i with the current components j_i, and the kinetic equations for the electrons and ions in the plasma. Under anomalous skin-effect conditions the collisions in the plasma do not play an important role so that we can solve the kinetic equations by the method of integrating along the particle trajectories in the approximation of constant effective numbers of collisions ν_e and ν_i. Using this solution to express the current in terms of the variable electric field and substituting it into the Maxwell equations we get a system of integro-differential equations for the field $E_i(x)$. In the case where the incident electromagnetic wave is damped in the region in which the particle density is described by Eq. (1.1) the forces acting upon the particles do not depend on the coordinates (due to the linear x-dependence of the effective potentials $U_\alpha(x)$) and the kernel of the set obtained therefore has the form

$$K_{ij}(x, x')=e^{x/a}\tilde{K}_{ij}(x-x'),$$

where the matrix $\tilde{K}_{ij}$ depends only on the difference of the arguments; $i,j=x,y,z$.

Applying the two-sided Laplace transform

$$F_j(k)=\int_{-\infty}^{\infty}E_j(\xi)e^{-k\xi}\,d\xi,\qquad E_j(\xi)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F_j(k)e^{k\xi}\,dk,$$

where $\xi=x/a$ is the dimensionless coordinate, we get a set of functional equations for $F_j(k)$ (this procedure has been performed in detail, e.g., in[2,3,9]). The equations for the components of the external field which is directed along the constant magnetic field H parallel to the y-axis can be separated from the other two equations which connect the components E_x and E_z. We can write the equation for the Laplace transform $F(k)$ of the electromagnetic field $E_y(x)$ (we drop the index y in what follows) in the form[9]

$$(k+1)^2F(k+1)=P(k)F(k), \tag{2.1}$$

where

$$P(k)=\sum_\alpha\exp(L_\alpha-z_\alpha)\sum_{n=-\infty}^{\infty}\frac{I_n(z_\alpha)}{\gamma-n\sigma_\alpha}. \tag{2.2}$$

Here $I_n(z)$ is a Bessel function of an imaginary argument,

$$z_\alpha=-k(k+1)/2\sigma_\alpha^2,\quad \sigma_\alpha=t_\alpha\Omega_\alpha,\quad \gamma_\alpha=t_\alpha(\omega-i\nu_\alpha),\quad L_\alpha=\ln(a/\delta_{0\alpha})^3,$$

$t_\alpha=a/\overline{v}_\alpha$ is the mean time for particles to fly a distance of the order of the dimensions of the inhomogeneity in the density, $\overline{v}_\alpha=(2kT_\alpha/m_\alpha)^{1/2}$ is the thermal velocity of the particles, $\Omega_\alpha=|e|H/m_\alpha c$ their Larmor rotation frequency, and $\delta_{0\alpha}=(c^2\overline{v}_\alpha m_\alpha/4\pi e^2n_0\omega)^{1/3}$ is the penetration depth of an electromagnetic wave into a homogeneous plasma with particle density n_0 under anomalous skin-effect conditions.

Equation (2.1) was solved in its general form in[3]. We obtained for the reflection coefficient of the electromagnetic wave from the plasma the following expression:

$$r=1+i\pi\frac{\omega a}{c}\int_{-\infty}^{\infty}\frac{dw}{\mathrm{ch}^2\pi w}\ln\frac{P(iw-1/2)}{(iw+1/2)^2}. \tag{2.3}$$

The plasma impedance can be expressed in terms of the reflection coefficient through the formula

$$Z=2\pi(1-r)/c. \tag{2.4}$$

As k occurs in (2.2) in the combination $k(k+1)$ the function $P(iw-\frac{1}{2})$ depends only on the argument $w^2+\frac{1}{4}$. We write $Q(w^2+\frac{1}{4})=P(iw-\frac{1}{2})$. In that case

$$Q(\varkappa^2)=\sum_\alpha\exp\left(L_\alpha-\frac{\varkappa^2}{2\sigma_\alpha^2}\right)\sum_{n=-\infty}^{\infty}\frac{I_n(\varkappa^2/2\sigma_\alpha^2)}{\gamma-n\sigma_\alpha}. \tag{2.5}$$

We can reduce the expression for the plasma impedance to the form

$$Z=-2iR_0\int_0^{\infty}\frac{dw}{\mathrm{ch}^2\pi w}\ln\frac{Q(w^2+1/4)}{w^2+1/4}, \tag{2.6}$$

where $R_0=2\pi^2\omega a/c^2$ is the resistance of a plasma without a magnetic field under the conditions of an extremely anomalous skin-effect.[2]

It is important to note that for a uniform plasma with a particle density n_g the dispersion equation for the case of an ordinary wave propagating at right angles to the magnetic field takes the form[5,6]

$$-\varkappa^2=\frac{n_g}{n_0}Q(\varkappa^2), \tag{2.7}$$

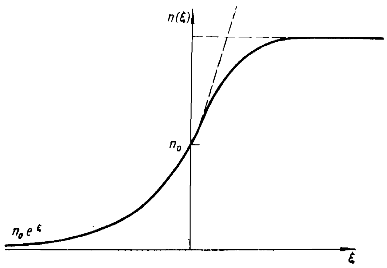

FIG. 1. Particle density n as function of the coordinate $\xi = x/a$.

where $Q(\varkappa^2)$ is the function (2.5), $\varkappa$ the dimensionless wavevector which is connected with the usual dimensional wavevector through the equation $\varkappa = \mathbf{k}a$.

Forgetting about a small damping caused by the collisions of the particles with one another we see that the main contribution to the real part of the impedance (2.6) which expresses the absorption of the electromagnetic wave comes only from the regions where Q is negative:

$$R = \operatorname{Re} Z = R_0 \pi \int_0^\infty \frac{dw}{\operatorname{ch}^2 \pi w} (1 - \operatorname{sign} Q(w^2 + {}^1/_4)). \qquad (2.8)$$

Just in those regions there exist solutions of Eq. (2.7), i.e., for any w for which $Q(w^2 + \frac{1}{4}) < 0$ one can find such

$$n_g(w^2 + {}^1/_4) = -n_0 \frac{w^2 + {}^1/_4}{Q(w^2 + {}^1/_4)}, \qquad (2.9)$$

that in a homogeneous plasma with particle density $n_g(w^2 + \frac{1}{4})$ there will exist eigenwaves with a wavevector $\varkappa = (w^2 + \frac{1}{4})^{1/2}$. In the opposite case, $Q > 0$, eigenwaves can not exist for any density n_g. This means that the incident electromagnetic wave excites ordinary waves in the plasma.

It is clear from (2.6) that the intensity of the excitation of waves with wavevectors with magnitudes lying in an interval $d\varkappa$ around $\varkappa$ is given by the quantity

$$\varkappa(\varkappa^2 - {}^1/_4)^{-1/2} \operatorname{ch}^{-2} \pi (\varkappa^2 - {}^1/_4)^{1/2} d\varkappa, \quad \varkappa > {}^1/_2.$$

Hence it is clear that an incident wave with a magnitude of the wavevector outside the plasma $\omega a/c \ll 1$ excites in the plasma ordinary waves with magnitudes of the wavevector $\varkappa > \frac{1}{2}$ and the intensity of the excitation decreases rapdily with increasing $\varkappa$. It is characteristic that the intensity of the excitation is determined solely by the wavevector and the possibility that waves can exist (the region $Q < 0$). The further evolution of the excited wave depends on the actual profile of the particle density inside the plasma, but we do not study that problem in the present paper.

The absorption mechanism discussed here indicates that waves with different wavevectors are excited at different places in the plasma and just in that region where the particle density is the same as $n_g(\varkappa^2)$, i.e., close to $\xi(\varkappa^2)$ (see (1.1) and (2.9)):

$$\xi(\varkappa^2) = -\ln(|Q(\varkappa^2)|/\varkappa^2). \qquad (2.10)$$

This becomes obvious if the Larmor radius of the particles is small; however, even for a large Larmor radius and, hence, for an appreciable non-localization of the skin-effect, this statement is valid; this is corroborated by the behavior of the impedance or the reflection coefficient. The imaginary part of these quantities is connected with the distance from the point of reflection of the wave. If the whole wave is reflected from some point, when we change the distance to that point by $\Delta\xi$ the reflection coefficient is multiplied by the quantity

$$\exp(2i\omega a\Delta\xi/c) \approx 1 + i2\omega a c^{-1}\Delta\xi,$$

and, correspondingly, the imaginary part of the reflection coefficient is increased by $(2\omega a/c)\Delta\xi$ and the impedance by $-(2R_0/\pi)\Delta\xi$ (if one uses Eqs. (2.3), (2.4)). For our case we can write the imaginary part of the impedance in the form

$$X = \operatorname{Im} Z = 2R_0 \int_0^\infty \frac{dw}{\operatorname{ch}^2 \pi w} \xi\left(w^2 + \frac{1}{4}\right), \qquad (2.11)$$

whence we conclude that the incident wave propagates into a plasma up to $\xi \approx \pi X/2R_0$. It is necessary to note that the integral (2.11) receives contributions not only from the regions in which the excitation of ordinary waves is possible, but also from the regions $Q > 0$. Analyzing (2.11) we see that in the latter case Eq. (2.10) also determines the point of reflection because of the introduction of the modulus sign.

It is now relevant to make a few remarks about the applicability of the whole of the theory proposed here. In actual fact the particle density can not increase exponentially in the whole range of change of ξ. The actual form of the density is shown in Fig. 1. It is possible to apply the theory given here if the wave is not damped until the density starts to differ strongly from $n_0 e^\xi$. If the particle density inside the plasma equals $\tilde{n}$ this condition can be written in the form

$$n_0 \exp(\pi X/2R_0) \ll \tilde{n}. \qquad (2.12)$$

The results obtained are qualitatively valid also in the case where the inequality (2.12) is not a strong one. In that case we must take into account only those waves which can exist in the plasma for densities $n_g < \tilde{n}$, i.e., integrate not over the region $Q < 0$, but over the region $Q/\varkappa^2 < -n_0/\tilde{n}$.

§3. REGIONS OF EXISTENCE OF ORDINARY WAVES. CYCLOTRON RESONANCE LINE SHAPES

Since there are no ordinary waves in a cold plasma without allowance for the thermal motion of the electrons and ions when one neglects the displacement current, all of the absorption of an incident wave in a plasma is connected with the electron and ion cyclotron resonances. We can use the quantities $\varepsilon = m_e/m_i \ll 1$ and $\beta = T_e/T_i$ to express the ion parameters in terms of the electron ones:

$$\Omega_i = \varepsilon\Omega_e, \quad \exp L_i = (\varepsilon\beta)^{1/2} \exp L_e,$$
$$t_i = (\beta/\varepsilon)^{1/2} t_e, \quad \sigma_i = (\varepsilon\beta)^{1/2} \sigma_e.$$

We can rewrite Eq. (2.5) for Q in the form

$$Q(\varkappa^2,\omega)=t_e^{-1}e^{L_e}\Big(\sum_{n=-\infty}^{\infty}\frac{e^{-z_e}I_n(z_e)}{\omega-i\nu_e-n\Omega_e}+\varepsilon\sum_{n=-\infty}^{\infty}\frac{e^{-z_i}I_n(z_i)}{\omega-i\nu_i-n\Omega_i}\Big), \quad (3.1)$$

where $z_\alpha=\varkappa^2/2\sigma_\alpha^2$. When considering the regions of the existence of high-frequency ordinary waves $\omega \gtrsim \Omega_e \gg \Omega_i$ we can neglect the contribution of the ions (the second sum in Eq. (3.1)) and we have:

$$Q(\varkappa^2,x)=\sigma_e^{-1}e^{L_e}\sum_{n=-\infty}^{\infty}\frac{e^{-z_e}I_n(z_e)}{x-n}, \quad (3.2)$$

where $x=(\omega-i\nu_e)/\Omega_e$.

We study the behavior of the sum occurring in Eq. (3.2) using the method used, for instance, in[7]. This sum has simple poles at integer points. The analytical function

$$\psi(z,x)=\sin\pi x\sum_n\frac{e^{-z}I_n(z)}{x-n} \quad (3.3)$$

is an entire function of the variable x. We transform $\psi(z,x)$ as follows:

$$e^z\psi(z,x)=\sum_n(-1)^nI_n(z)\frac{\sin\pi(x-n)}{x-n}$$

$$=\sum_n(-1)^nI_n(z)\int_0^\pi\cos(x-n)\varphi\,d\varphi=\int_0^\pi d\varphi\cos x\varphi\sum_n(-1)^nI_n(z)\cos n\varphi,$$

or, using the well known formula for a series of Bessel functions,

$$\psi(z,x)=\int_0 d\varphi\cos x\varphi\exp[-z(\cos\varphi+1)]. \quad (3.4)$$

Using (3.3) we have

$$Q=\sigma_e^{-1}e^{L_e}\psi(z_e,x)/\sin\pi x,$$

whence it is clear that the poles of Q are connected solely with the factor $1/\sin\pi x$ which is independent of z_e. There occurs therefore always a change in sign of Q for $\omega=n\Omega_e$. The other boundary of the region of existence of ordinary waves is connected with the zeroes of $\psi(z_e, x)$. We consider the asymptotic behavior of $\psi(z_e,x)$ in z_e.

When $z\gg 1$ the main contribution to the integral (3.4) comes from the region of φ close to $\varphi=\pi$. In that case

$$\psi(z,x)=(\pi/2z)^{1/2}\exp(-x^2/2z)\{\cos\pi x+\sin\pi x[\Phi(ix/\sqrt{2z})-\Phi(-ix/\sqrt{2z})]/2i\}, \quad (3.5)$$

$\Phi(x)$ is the error function. Expression (3.5) is valid for any x. In the case $x\ll z^{1/2}$ we can simplify (3.5) after which we can write Q in the form

$$Q=\frac{e^{L_e}}{\sigma_e}\sqrt{\frac{\pi}{2z_e}}\Big(\operatorname{ctg}\pi x+\frac{2x}{\sqrt{2\pi z_e}}\Big), \quad (3.6)$$

$z_e\gg 1$ and $x\ll z_e^{1/2}$.

In the case $a\Omega_e/\bar{v}_e\to 0$ (3.6) goes over into the equation corresponding to Eq. (2.16) of[4]. It is simpler to consider the other limiting case $z_e\ll 1$ by starting from Eq. (3.2) retaining in it only the main term and the resonance term:

$$Q=\frac{e^{L_e}}{\sigma_e}\Big(\frac{1}{x}+\frac{Z_e^n}{n!2^n(x-n)}\Big), \quad z_e\ll 1, \quad x\approx n. \quad (3.7)$$

Putting the right-hand sides of Eqs. (3.5) to (3.7) to zero we find the lower bound of the negative values of Q for $\nu_e=0$:

$$x=(n-1/2)[1+(2/\pi^3z_e)^{1/2}], \quad z_e\gg 1, \quad n^2\ll z_e; \quad (3.8a)$$

$$x=n[1-(2\pi z_e)^{-1/2}\exp(-n^2/2z_e)], \quad z_e\gg 1, \quad n^2\gg z_e; \quad (3.8b)$$

$$x=n(1-z_e^n/2^nn!), \quad z_e\ll 1. \quad (3.8c)$$

The form of the regions of existence of ordinary waves for different values of n are shown in Fig. 2a.

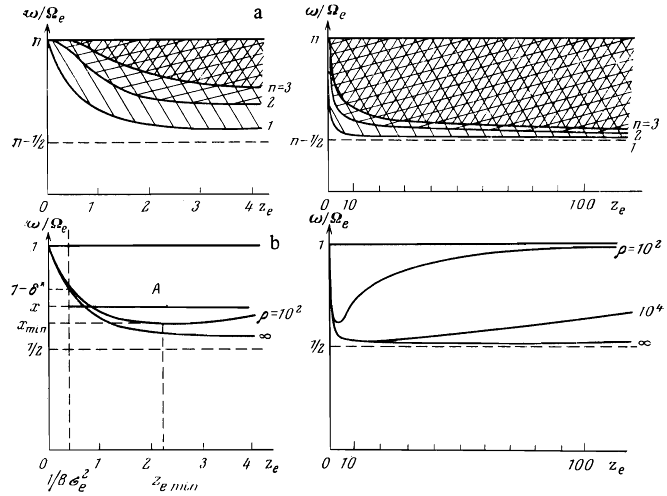

FIG. 2. The shape of the regions of existence of ordinary waves in the ω/Ω_e, z_e—plane ($z_e=\varkappa^2\bar{v}_e^2/2a^2\Omega_e^2$): a) for different numbers n (indicated by the numbers at the boundaries of the regions) of the resonance; the regions where the waves exist are hatched; b) for $n=1$. A is the line of integration in the integral (2.6). Inside the region of existence of the waves we have drawn the dispersion curves for a uniform plasma for different values of $\rho=8\pi\tilde{n}_ekT_e/H^2$, indicated by the numbers at the curves.

To determine the resistivity of the plasma we must evaluate the integral (2.8). Introducing $g=R/R_0$ we get

$$g=\sum_j\pm4[1+\exp(2\pi\sqrt{2\sigma_e^2z_e^{(j)}-1/4})]^{-1}, \quad (3.9)$$

where $z_e^{(j)}$ are the points where the line of integration A ($\omega=\text{const}$, $z_e>1/8\sigma_e^2$) intersects the boundaries of the region of existence of the waves (see Fig. 2b). The plus sign in (3.9) is used in the case when for increasing z_e we go from a region where there are no waves to a region where they exist, and the opposite, minus sign in the opposite case. If $z_e=1/8\sigma_e^2$ occurs already in the region where the waves exist, the sum (3.9) includes a term equal to 2 corresponding to that point.

It is clear from Eq. (3.9) and Fig. 2 that $R=0$ for $n-1<x<n-\frac{1}{2}$, $R=2R_0$ for $n-\delta^*<x<n$, where δ^* is the point where the lower limit (3.8) of the region of existence of the waves intersects the line $z_e=1/8\sigma_e^2$. In the case $\Omega_e\ll\bar{v}_e/a$, $n\ll\bar{v}_e/a\Omega_e$:

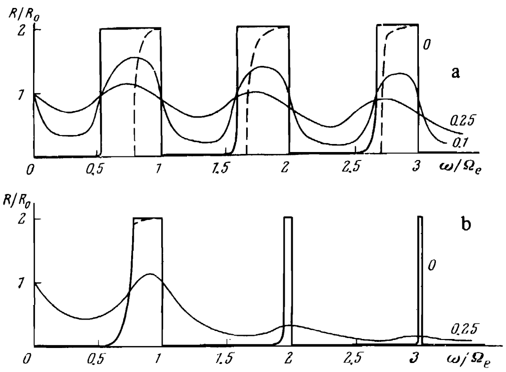

FIG. 3. Line shapes of absorption resonance: a: for $a\Omega_e/\bar{v}_e = 0.1$; b: $a\Omega_e/\bar{v}_e = 0.5$. The numbers at the curves indicate the values of the parameter ν_e/Ω_e. The line shapes for $\rho = 10^2$ are qualitatively indicated by a dotted line.

$$\delta_1^* = {}^1/_2 - 4\pi^{-3/2}(n - {}^1/_2)a\Omega_e/\bar{v}_e; \tag{3.10a}$$

if, however, in that case the number of the resonance is very large, $n \gg \bar{v}_e/a\Omega_e$, with logarithmic accuracy

$$\ln \delta_2^* = -4n^2a^2\Omega_e^2/\bar{v}_e^2. \tag{3.10b}$$

In the case of strong magnetic fields $\Omega_e \gg \bar{v}_e/a$:

$$\delta_3^* = (4a\Omega_e/\bar{v}_e)^{-2n}/(n-1)! \tag{3.10c}$$

In the interval between $n - \frac{1}{2}$ and $n - \delta^*$ R increases and has at $x = n - \delta^*$ a square-root singularity and is exponentially small at $x = n - \frac{1}{2}$ (Fig. 3). The actual form of $g = R/R_0$ in some limiting cases is given by the equations

$$g = 2 - 2\,\mathrm{th}\left\{\frac{\pi}{2}\left[\left(\frac{({}^1/_2 - \delta_1^*)\Omega_e}{\omega - (n - {}^1/_2)\Omega_e}\right)^2 - 1\right]^{1/2}\right\}, \tag{3.11a}$$

$$\Omega_e \ll \bar{v}_e/a, \quad n \ll \bar{v}_e/a\Omega_e, \quad n - {}^1/_2 < \omega/\Omega_e \leq n - \delta_1^*;$$

$$g = 2 - 2\,\mathrm{th}\left\{\frac{\pi}{2}\left[\frac{\ln \delta_2^*}{\ln(n - \omega/\Omega_e)} - 1\right]^{1/2}\right\}, \tag{3.11b}$$

$$\Omega_e \ll \bar{v}_e/a, \quad n \gg \bar{v}_e/a\Omega_e, \quad 0 < n - \delta_2^* - \omega/\Omega_e \ll 1;$$

$$g = 2 - 2\,\mathrm{th}\left\{\frac{\pi}{2}\left[\left(\frac{n\Omega_e - \omega}{\delta_3^*\Omega_e}\right)^{1/n} - 1\right]^{1/2}\right\}, \tag{3.11c}$$

$$\Omega_e \gg \bar{v}_e/a, \quad 0 < n - \delta_3^* - \omega/\Omega_e \ll 1;$$

$$g = 4\exp\left\{-\frac{4}{\sqrt{\pi}\,\bar{v}_e}\frac{(n - {}^1/_2)a\Omega_e^2}{(\omega - (n - {}^1/_2)\Omega_e)}\right\}, \tag{3.11d}$$

$$\Omega_e \text{ -- arbitrary}, \quad 0 < \omega/\Omega_e - n + {}^1/_2 \ll {}^1/_2 - \delta^*.$$

The absorption of an electromagnetic wave in the regions of x close to half-odd-integral values is connected with large particle densities. As inside the plasma the density reaches the limiting value $\tilde{n}$, it is useful to estimate for which densities our solution is correct and how it changes when the particle density changes. The dispersion curves for a uniform plasma have been studied earlier analytically only for the case of a low-density plasma or in the limiting cases when the dispersion curve lies around the resonance frequency (see, e.g., [5,6]). If the dispersion curve is removed from resonance, it has been studied numerically or qualitatively (e.g., [8]). In the case of a high-pressure plasma $\rho = 8\pi\tilde{n}_e kT_e/H^2 \gg 1$ we can obtain the equation for the dispersion curve (Fig. 2b) for $z_e \gg 1$ by substituting into Eq. (2.7) the asymptotic form (3.6):

$$-2z_e = \rho x\left(\frac{\pi}{2z_e}\right)^{1/2}\left(\mathrm{ctg}\,\pi x + \frac{2x}{\sqrt{2\pi z_e}}\right).$$

The dispersion curve has a minimum

$$x_{min} = n - {}^1/_2 + 4\pi^{-3/2}({}^2/_3)^{3/4}(n - {}^1/_2)^{1/2}\rho^{-1/4}, \tag{3.12}$$

which is reached for $z_{e\,\min} = (n - \frac{1}{2})(\rho/6)^{1/2}$; $z_e \sim z_{e\,\min}$

$$x = (n - {}^1/_2)\{1 + \pi^{-3/2}[\rho^{-1}(n - {}^1/_2)^{-2}(2z_e)^{3/2} + (z_e/2)^{-1/2}].$$

For $z_e \gg \rho^{2/3}$

$$x = n[1 - \rho\pi^{-1/2}(2z_e)^{-3/2}], \tag{3.13}$$

while for $z_e \ll 1$ it is the same as Eq. (3.8c) and in the case $\rho \gg 1$ it is to a first approximation independent of ρ. The asymptotic expressions (3.13) and (3.8c) are the same as Eqs. (5.7.1.5) and (5.7.1.7) of the book[5].

It is clear from (3.12) that for large ρ the limiting dispersion curve, corresponding to the density $\tilde{n}$, can approach $x = n - \frac{1}{2}$ rather closely, without, however, going through that value.[8] Under realistic conditions for particle densities in the plasma of $\tilde{n} \sim 10^{15}$ cm^{-3} and $T_e \sim 10^6$ K[1] we have $\rho \sim 3 \times 10^6/H^2$(Oe). In the region between $n - 1$ and $x_{\min}$ waves cannot be excited and the cyclotron resonance line acquires the form qualitatively shown in Fig. 3 by a dotted curve.

We give some results when one takes into account the non-vanishing collision frequency in the plasma. We consider first of all the case of an extremely anomalous skin-effect $\Omega_e \ll \bar{v}_e/a$. If $\nu_e \gg \Omega_e$, we have

$$g = 1 - 4\pi^{-1}\exp\left(-2\pi\frac{\nu_e}{\Omega_e}\right)\sin\left(2\pi\frac{\omega}{\Omega_e}\right) - 2I_1\frac{\omega a}{\bar{v}_e}, \quad \omega \ll \frac{\bar{v}_e}{a}, \tag{3.14}$$

where

$$I_1 = 2\pi^{-1/2}\int_0^\infty dx \Big/ \left(\sqrt{x^2+1}\,\mathrm{ch}^2\frac{\pi x}{2}\right) = 0.6461.$$

In the opposite case, $\nu_e \ll \Omega_e$, it is necessary to consider the absorption coefficient in different sections of the spectrum:

$$g = 1 - \mathrm{sign}\sin\left(2\pi\frac{\omega}{\Omega_e}\right) + \frac{4\nu_e}{\Omega_e}\left[\frac{1}{\sin(2\pi\omega/\Omega_e)} - \frac{1}{2}I_1\left(\frac{\pi\omega a}{\bar{v}_e\cos^2(\pi\omega/\Omega_e)} + \frac{\Omega_e a}{\bar{v}_e}\mathrm{tg}\,\pi\frac{\omega}{\Omega_e}\right)\right]$$

when $|\omega - n\Omega_e| \gg \nu_e, \quad |\omega - (n - {}^1/_2)\Omega_e| \gg \max\{\nu_e, ({}^1/_2 - \delta_1^*)\Omega_e\}$;

$$g = 1 - \frac{2}{\pi}\frac{\omega - n\Omega_e}{\nu_e} - \frac{2\pi\omega a}{\bar{v}_e}\frac{\nu_e}{\Omega_e}I_1 \tag{3.15}$$

when $|\omega - n\Omega_e| \ll \nu_e$;

$$g = 1 + \frac{2}{\pi}\frac{\omega - (n - {}^1/_2)\Omega_e}{\nu_e} - \frac{2a\Omega_e\omega}{\pi\bar{v}_e\nu_e}I_1$$

when $|\omega - (n - {}^1/_2)\Omega_e| \ll \nu_e, \quad \nu_e \gg ({}^1/_2 - \delta_1^*)\Omega_e$.

When $\Omega_e \gg \bar{v}_e/a$ we have in the case $\nu_e \gg \Omega_e\delta_3^*$ the Lorentz formula for the absorption line:

$$g = \frac{2}{\pi}\mathrm{arctg}\frac{\nu_e}{\omega} + \frac{\nu_e\Omega_e}{(\omega - n\Omega_e)^2 + \nu_e^2}\delta_3^* I_{-2n}, \tag{3.16}$$

$$I_{-2n} = \int_0^\infty \frac{dw}{\mathrm{ch}^2\,\pi w}(4w^2+1)^n = \pi^{-1}\sum_{k=0}^{n} C_k^n|(2^{2k}-2)B_{2k}|,$$

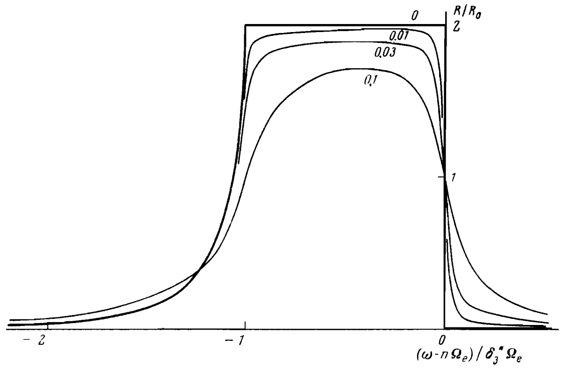

FIG. 4. The electron cyclotron resonance line shape for $a\Omega_e/\overline{v}_e \gg 1$ and weak damping. The numbers at the curves indicate the values of the parameter $\nu_e/\delta_3^*\Omega_e$.

C_k^n are the binomial coefficients, and B_{2k} the Bernoulli numbers;

$$I_{-2}=3/2\pi,\quad I_{-4}=37/15\pi,\quad I_{-6}=1129/210\pi,\ldots$$

The curves corresponding to the two limiting cases are drawn in Fig. 3. Equations (3.14) to (3.16) are correct, if ν_e is much broader than the fine structure of the line.

If, on the other hand, $\alpha \ll 1$, where $\alpha = \nu_e/(\frac{1}{2} - \delta_1^*)\Omega_e$ for $\Omega_e \ll \overline{v}_e/a$ and $\alpha = \nu_e/\delta_3^*\Omega_e$ when $\Omega_e \gg \overline{v}_e/a$ we can trace the change in the shape of the absorption curve. In that case we can split off from $\operatorname{Im}\ln Q$ which occurs in the integral of (2.6), apart from the sign, the term

$$\operatorname{Im}\ln(\omega(z_e)-\omega+i\nu_e)=\frac{\pi}{2}+\operatorname{arctg}\frac{\omega-\omega(z_e)}{\nu_e},$$

where $\omega(z_e)$ is the equation of the curve in the ω, z_e-plane (Fig. 2) which describes the boundary of the region of existence of waves in a uniform plasma. If the frequency ω lies close to the lower boundary of the region of existence of waves, but not in the immediate vicinity of the point $\omega = (n - \delta^*)\Omega_e$, we can to a first approximation in the parameter α write the change in g in the form

$$\Delta g=2\alpha f(\zeta).$$

Here

$$f(\zeta)=\int_0^\infty\frac{dw}{\operatorname{ch}^2\pi w}\{\zeta+[(4w^2+1)^{l}-1]\operatorname{sign}l\}^{-1},$$

$$\zeta=[\omega-(n-\delta_3^*)\Omega_e]/\delta_3^*\Omega_e,\quad l=n\quad\text{when}\quad\Omega_e\gg\overline{v}_e/a; \tag{3.17}$$

$$\zeta=[\omega-(n-\delta_1^*)\Omega_e]/(1/2-\delta_1^*)\Omega_e,\quad l=-1/2\quad\text{when}\quad\Omega_e\ll\overline{v}_e/a.$$

The integral (3.17) is taken in the principal-value sense. In the point $\zeta=0$ corresponding to the square root singularity in the collisionless plasma, $\omega = (n - \delta^*)\Omega_e$, and the integral (3.17) diverges. Near that point there arises for ν_e different from zero a non-analytical ν_e-dependence of g:

$$g=2-\frac{\pi}{\sqrt{2|l|}}\frac{\alpha}{(\zeta+\sqrt{\zeta^2+\alpha^2})^{1/2}}.$$

At the point $\omega = n\Omega_e$ there also arises a non-analytical behavior, but in a more familiar form:

$$g=1-\frac{2}{\pi}\operatorname{arctg}\frac{\omega-n\Omega_e}{\nu_e}.$$

The shape of the cyclotron resonance for small damping is shown in Fig. 4.

We must say a few words about the transition to the limit as $H \to 0$ in the case $\nu_e = 0$. When there are no collisions there occurs only a decrease in the line width without a change in the amplitude with an increase in the number of the resonance. It is therefore necessary when determining the resistance of the plasma to average the plasma resistance which oscillates fast between 0 and $2R_0$ for small magnetic fields. Thus, if we consider the region $\omega a/\overline{v}_e \ll 1$, we get, integrating Eq. (3.11a) over a period,

$$R=R_0(1-2I_1\omega a/\overline{v}_e),$$

i.e., Eq. (3.5) from[3]. We can similarly take the limit $H \to 0$ also in other cases.[1)] However, when decreasing H usually there starts earlier a smearing-out or a decrease in the amplitude of the resonances due to the fact that ν_e is finite (it is clear from Eq. (3.14) that the resonances are conserved only down to fields for which $\Omega_e \gtrsim \nu_e$ after which they decrease exponentially fast while the plasma resistance tends to the same limit as in the collisionless case) or the resonances start to smear out due to the fact that when the radius of the Larmor circles increases the particles leave the region of densities which can be approximated by an exponential function. In the latter case it is necessary to take into account the energy-dependence of the frequency Ω_e.

We consider now briefly the ion cyclotron waves. It is well known[5] that ion waves lie appreciably closer to the ion cyclotron frequencies than the electron waves to the corresponding cyclotron frequencies so that the lines for the excitation of ion cyclotron waves are considerably narrower. The equation which determines the boundaries of the existence of ion waves is obtained from (3.1). Since the frequencies of the ion waves are appreciably smaller than Ω_e and since the solutions of interest to us lie in magnetic field regions for which $z_e \ll 1$, we retain only the first electron term:

$$Q=t_e^{-1}e^{L_e}\left(\frac{1}{\omega-i\nu_e}+\varepsilon\sum_n\frac{\exp(-z_i)I_n(z_i)}{\omega-n\Omega_i-i\nu_i}\right).$$

We consider only the collisionless case. Taking collisions into account is qualitatively analogous to the electron cyclotron wave case. In that case the upper boundary for Q being negative is the same as the ion cyclotron frequencies $\omega = n\Omega_i$ and the lower one is determined by the equation

$$\omega=n\Omega_i(1-\varepsilon e^{-z_i}I_n(z_i)). \tag{3.18}$$

In the case $\Omega_i a/\overline{v}_i = (\varepsilon\beta)^{1/2}\Omega_e a/\overline{v}_e \ll 1$ we have for $0 < n\Omega_i - \omega < \delta_1^{**}\Omega_i$

$$g=2\operatorname{th}\left\{\frac{\pi}{2}\left[\left(\frac{\Omega_i\delta_1^{**}}{n\Omega_i-\omega}\right)^2-1\right]^{1/2}\right\},$$

$$\delta_1^{**}=2\pi^{-1/2}n\varepsilon\Omega_i a/\overline{v}_i,$$

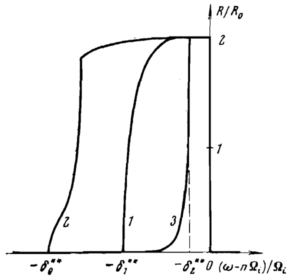

FIG. 5. Ion cyclotron resonance line shape: 1: for $8\sigma_i^2 z_{i\,\min} < 1$, 2: $8\sigma_i^2 z_{i\,\min} > 1$, 3: $8\sigma_i^2 \gg 1$, collisionless case ($\sigma_i = a\Omega_i/\bar{v}_i$).

with a square root singularity on the boundary of the absorption spectrum; in the remaining band of frequencies $R = 0$. It is clear from (3.18) that the boundary curve has a minimum which is reached in some point $z_{i\,\min} \sim 1$ and which is equal to $n - \delta_0^{**}$, $\delta_0^{**} \sim \varepsilon$. Using this one can easily show that when $8\Omega_i^2 a^2 z_{i\,\min} < \bar{v}_i^2$ the absorption line does not change qualitatively: there is a square-root singularity at the lower boundary of the spectrum and an exponential one near 2 when $\omega \approx n\Omega_i$, $\omega < n\Omega_i$. In the opposite case $8\Omega_i^2 a^2 z_{i\,\min} > \bar{v}_i^2$ there are two square root type singularities. If $\Omega_i \gg \bar{v}_i/a$, we have for $n\Omega_i - \omega \ll \delta_0^{**}\Omega_i$

$$g = 2 - 4\exp\left(-\frac{\pi\delta_1^{**}\Omega_i}{n\Omega_i - \omega}\right) - 2\theta((n - \delta_2^{**})\Omega_i - \omega)\,\mathrm{th}\left\{\frac{\pi}{2}\left[\left(\frac{n\Omega_i - \omega}{\delta_2^{**}\Omega_i}\right)^{1/n} - 1\right]^{1/2}\right\}.$$

Here $\theta(x)$ is the Heaviside step function and

$$\delta_2^{**} = \varepsilon(4a\Omega_i/\bar{v}_i)^{-2n}/(n-1)!$$

We can near the boundary of the spectrum $\omega \approx n\Omega_i - \delta_0^{**}\Omega_i$ expand expression (3.18) in a series in $z_i - z_{i\,\min}$:

$$\omega - n\Omega_i + \delta_0^{**}\Omega_i = \varepsilon A^{-2}\Omega_i (z_i - z_{i\,\min})^2,$$

where

$$A^{-2} = e^{-z}(2I_n'(z) - I_n''(z) - I_n(z))\,|_{z = z_{i\,\min}} \sim 1.$$

Hence

$$g = 2\,\mathrm{th}\frac{\pi}{2}\left[8\sigma_i^2 z_{i\,\min} - 1 + 8\sigma_i^2 A\left(\frac{\omega - n\Omega_i + \delta_0^{**}\Omega_i}{\varepsilon\Omega_i}\right)^{1/2}\right]^{1/2} - 2\,\mathrm{th}\frac{\pi}{2}\left[8\sigma_i^2 z_{i\,\min} - 1 - 8\sigma_i^2 A\left(\frac{\omega - n\Omega_i + \delta_0^{**}\Omega_i}{\varepsilon\Omega_i}\right)^{1/2}\right]^{1/2}. \quad (3.19)$$

It is clear from (3.19) that g has a square root singularity at the point $\omega = (n - \delta_0^{**})\Omega_i$; when the second singularity approaches the same point, i.e., when $8\sigma_i^2 z_{i\,\min} = 1$, there arises a higher-order singularity, $g \sim (\omega - n\Omega_i + \delta_0^{**}\Omega_i)^{1/4}$. We show in Fig. 5 various ion resonance line shapes for $8\sigma_i^2 z_{i\,\min} < 1$ (curve 1), for $8\sigma_i^2 z_{i\,\min} \gtrsim 1$ (curve 2), and for $8\sigma_i^2 \gg 1$ (curve 3); in the last case the nature of the line is similar to the electron line, as the singularity at the boundary of the spectrum is exponentially small.

As $\delta_0^{**} \sim \varepsilon = m_e/m_i$, the maximum line width is also small and the lines are strongly smeared out by electron and ion collisions.

If we take the finite density of the plasma into account the electron resonance lines have also a similar character, as is shown in Fig. 3 by the dotted line. Taking the finite density into account for the ion resonances does not lead to qualitative differences in the absorption line shape.

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[1)]We use the opportunity to correct an error which slipped into[3]: in Eq. (3.7) the numerical factor in the coefficient in front of the exponent should be $8/\sqrt{3}$ and not $2/\sqrt{3}$.

[1]P. L. Kapitza, Zh. Eksp. Teor. Fiz. **57**, 1801 (1969) [Sov. Phys. JETP **30**, 973 (1970)].
[2]M. A. Liberman, B. E. Meĭerovich, and L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **62**, 1737 (1972) [Sov. Phys. JETP **35**, 904 (1972)].
[3]A. N. Vasil'ev and B. E. Meĭerovich, Zh. Eksp. Teor. Fiz. **67**, 1738 (1974) [Sov. Phys. JETP **40**, 865 (1975)].
[4]S. M. Dikman and B. E. Meĭerovich, Zh. Eksp. Teor. Fiz. **64**, 1653 (1973) [Sov. Phys. JETP **37**, 835 (1973)].
[5]A. I. Akhiezer, I. A. Akhiezer, R. V. Polovin, A. G. Sitenko, and K. N. Stepanov, Elektrodinamika plazmy (Plasma Electrodynamics), Nauka, 1974, § 5.7 [translation published by Pergamon Press, Oxford].
[6]A. B. Kitsenko and K. N. Stepanov, Nucl. Fusion **4**, 272 (1964).
[7]E. P. Gross, Phys. Rev. **82**, 232 (1951).
[8]T. D. Kaladze, D. G. Lominadze, and K. N. Stepanov, Zh. Tekh. Fiz. **42**, 243 (1972) [Sov. Phys. Tech. Phys. **17**, 196 (1972)].
[9]A. N. Vasil'ev, Vestnik Moskva Gos. Univ., ser. fizika, astron. **17**, 727 (1976).

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