Contribution to the theory of paramagnetic generation of acoustic phonons in crystals

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(Submitted April 29, 1976)
Zh. Eksp. Teor. Fiz. **71**, 2178–2193 (December 1976)

The role of various nonlinear processes in the suppression of parametric instability of acoustic phonons generated by a laser beam is investigated. It is shown that the problem has a range of parameters in which the principal role in the suppression of the instability is played by the corrections to the vertex of the interaction of the light with the phonons. The form of the occupation numbers and the spectrum of the generated phonons are obtained for this range of parameters.

PACS numbers: 63.20.Dj, 79.20.Ds

1. INTRODUCTION

It was shown in Ref. 1 that parametric instability (PI) occurs in a crystal upon generation of acoustic phonons of the lower, non-decay, branch with a momentum of the order of the Debye momentum q_D, by light having a spectral width $\nu_0 \gg \Delta\nu \ll \tau^{-1}$ (where ν_0 is the central frequency of the light beam and is not equal to the frequencies of the optical branches, τ is the lifetime of the equilibrium phonons with momentum q_0 determined from the condition $\nu_0 = 2\omega_{q0}$, and ω_q is the dispersion law of the generated phonons). Here we mean by PI that the occupation numbers N_q of the phonons become infinite at some finite value of the light intensity $\tilde{J}_0$. It has also been shown that here, in contrast with the case of generation by monochromatic light,[2] the damping of the generated phonon (GPh) at the PI threshold does not necessarily vanish.

The nonlinear damping due to the sticking together of two GPh into phonons of higher decay branches was regarded in Ref. 1 as the limiting mechanism. It was shown in this case that there exists a region of values $\tilde{J} > \tilde{J}_0$ in which other nonlinear mechanisms can be neglected. For these values of $\tilde{J}$, the value of N_q was found in a form turned out to be dependent on the spectral shape of the light $\varphi(\nu) \leqslant 1$ and its intensity. For the Lorentz shape of the light with width $\Delta\nu$ and center ν_0, the quantity N_q also has a Lorentz shape with center $\omega_0 = \nu_0/2$ and width $\frac{1}{2}\Delta\nu(1-\tilde{\xi})^{1/2}$, where $\tilde{\xi}$ is a dimensionless quantity which depends on the intensity, and tends to unity as $\tilde{J} \to \infty$.

However, a situation is possible in which the GPh either cannot stick into phonons of higher branches, because of nonsatisfaction of the law of conservation of energy-momentum, or this mechanism has added smallness in comparison with the others. For example, in the generation of different phonons, the effectiveness of generation is determined by the density of the two-phonon states with total momentum ≈ 0, and the effectiveness of the sticking is determined by the single-particle densities of states of the GPh and of those phonons into which the GPh can stick, but these densities can differ considerably. Moreover, it is necessary to investigate a region of intensities that exceed the limits of applicability of Ref. 1. Therefore, the consideration of the role of other nonlinear mechanisms is of interest. In the present work, for the same statement of the problem as in Ref. 1, we describe a general approach to consideration of nonlinearities, and show that the corrections of next order (following the nonlinear damping) are those to the vertex of the interaction of light with phonons. The form of the function N_q, which is obtained when these corrections are taken into account, is also determined.

2. GENERAL INVESTIGATION OF ABOVE-THRESHOLD BEHAVIOR

Assuming the nonlinearity to be weak, we limit ourselves to the following model Hamiltonian of the interaction:

$$H_{\text{int}} = \int d\mathbf{x} \left(\frac{g}{2} E\varphi^2 + \frac{b}{2} \psi\varphi^2 + \frac{a}{4!} \varphi^4 \right) + H_T. \tag{1}$$

Here E is the classical random field of the pump, φ is the GPh field, ψ is the field of the decay phonon, into which the φ phonons can stick, H_T is the interaction Hamiltonian with the thermostat. The term of the form $\int \varphi^3 d\mathbf{x}$, which is unimportant in what follows, because of the assumed non-decay character of the φ phonon (here, it reduces to an interaction of the type $\int \varphi^4 d\mathbf{x}$), is not written out, for brevity. The Hamiltonian (1) is chosen in a local form because the dependence of the form factors on the momenta will not be important in those equations which will be solved. For definiteness, we have chosen the case of generation of two identical phonons and for simplicity we assume the problem to be spherically symmetric, the light E to be scalar and the temperature $T = 0$.

For the general approach to consideration of the nonlinearity, we make use of the Keldysh diagram technique[3] (its formulation is given in the Appendix). The stationary nonequilibrium state of the phonons is described by the Dyson equations for the retarded (D_r) and the statistical (D_s) Green's functions of the phonons. In this case, the spectral characteristics of the phonons—the renormalization of the spectrum $\Delta\omega_q$ and the width γ_q—should be found simultaneously with their distribution function N_q. The Dyson equations in general form are not closed relative to D_r and D_s (Green's functions of higher orders enter here), but, by using the weakness of the interaction and the expected "narrowness" of N_q, we can isolate the principal terms in these equations. This isolation is conveniently carried out by applying the "skeleton" perturbation theory,[4] i.e., a diagram representation for the Green's functions with exact single-particle Green's functions and with "bare" vertices. When the corrections to the vertices are not small, they must be taken into account.

We represent D_r and D_s in the following form[5]:

$$D_r(q)=\frac{\omega_q^2}{\omega^2-\omega_q^2-\omega_q^2\Pi_r(q)}\approx\frac{\omega_q^2}{(\omega+i\gamma(q)/2)^2-\tilde{\omega}^2(q)} \quad (2)$$

$$D_s(q)=\Pi_s(q)\,|D_r(q)|^2$$
$$=-i\pi\omega_q(2N(q)+1)[\Delta(\omega-\tilde{\omega}(q))+\Delta(\omega+\tilde{\omega}(q))], \quad (3)$$

where $\Pi_{r,s}$ are the corresponding polarization operators,

$$\gamma(q)=-\omega_q\,\mathrm{sgn}\,\omega\,\mathrm{Im}\,\Pi_r(q),\quad \tilde{\omega}(q)=\omega_q+\Delta\omega(q),$$
$$\Delta\omega(q)={}^1/_2\omega_q\,\mathrm{Re}\,\Pi_r(q),$$

$N(q)$ are the generalized occupation numbers connected with the ordinary ones by the formula

$$N_q=\int d\omega\,\Delta(\omega-\tilde{\omega}(q))N(q),$$

$$\Delta(\omega\pm\tilde{\omega}(q))=\frac{1}{\pi}\frac{\gamma(q)/2}{(\omega\pm\tilde{\omega}(q))^2+\gamma^2(q)/4},\quad q=\{\omega,\mathbf{q}\}.$$

A similar representation holds also for the ψ phonon.

We can obtain the following "balance" equation from (3) by using (2) and assuming $\omega\approx\omega_q$:

$$I(q)=-N(q)\gamma(q)+B(q)=0,$$
$$B(q)={}^1/_2\omega_q[\mathrm{Im}\,\Pi_r(q)+{}^1/_2 i\Pi_s(q)], \quad (4)$$

where $I(q)$ is the generalized "collision integral," $B(q)$ describes the "arrival" and $N(q)\gamma(q)$ the "departure." In the case in which $\gamma(q)\sim\Delta\omega(q)\ll\delta\omega$ are the widths of $N(q)$ relative to ω, and change over intervals $\sim\delta\omega$, we can let ω tend to the "mass" surface, defined from the equation $\omega=\tilde{\omega}(q)$ and equal in first approximation to $\tilde{\omega}_q=\omega_q+\Delta\omega_q$, where $\Delta\omega_q\equiv\Delta\omega(\omega_q,\mathbf{q})$, whole $\Delta(\omega\pm\tilde{\omega}(q))$ can be replaced by $\delta(\omega\pm\tilde{\omega}_q)$; here $N(q)$ transforms into N_q and $I(q)$ into I_q—the generalized kinetic collision integral, which takes into account all the possible processes described by the Hamiltonian (1). In our problem, $\gamma(q)\sim\tau^{-1}\ll\delta\omega$,[1] therefore, the kinetic approximation is valid.

We shall seek the stationary distribution N_q by solving the equation

$$I_q=-N_q\gamma_q+B_q=0, \quad (5)$$

where $\gamma_q=\gamma(\tilde{\omega}_q,\mathbf{q})$, $B_q=B(\tilde{\omega}_q,\mathbf{q})$. The diagrams for Π (and they are topologically identical for Π_r and Π_s) can be separated into those containing light lines and those not containing them. Among those containing light lines, it is sufficient to consider only diagrams with a single light line, since the contribution of diagrams containing several light lines contains an additional factor $\sim\tau^{-1}/\omega_0\ll 1$. Account of diagrams with a single light line gives the generation term γ_q^g in γ_q and a term of the form $\gamma_q^g(N_{-q}+1)$ in B_q (here, we have used the assumption that $\Delta\omega\gg\omega_0 v/c$, where $\Delta\omega$ is the width of N_q, v is the group velocity of the φ phonon at $q=q_0$, c is the velocity of light).[1] Representing γ_q in the form

$$\gamma_q=\tau^{-1}+\gamma_q^g+\gamma_q^n, \quad (6)$$

where contributions of all diagrams not containing light lines are designated by γ_q^n, and assuming $N_q=N_{-q}$, we "solve" (5) relative to N_q:

$$N_q=\frac{\gamma_q^g+B_q^n}{\tau^{-1}+2\gamma_q^g+\gamma_q^n}=\frac{{}^1/_2\xi_q+K_q}{1-\xi_q+u_q}. \quad (7)$$

Here $\xi_q=-2\tau\gamma_q^g$, $u_q=\tau\gamma_q^n$, $K_q=\tau B_q^n$, B_q^n are the contributions of diagrams without light lines to B_q. The formula (7) is very convenient for further analysis. It is seen immediately from it that N_q becomes infinite not at $\gamma_q=0$, but when $\gamma_q+\gamma_q^g=0$. Another conclusion from (7) is that the quantity N_q is determined by the denominator of (7), and, since $\gamma_q^g<0$ (generation!), it follows that with increase in $|\gamma_q^g|$ a decrease occurs in the denominator; consequently, the N_q increase, which in turn leads to a growth in the nonlinear quantities, counteracting either the growth of $|\gamma_q^g|$ or the decrease in the denominator. As a result, some sort of self-consistent distribution should be established. It also follows from (7) that the condition of instability of the higher nonlinearities is its smallness in comparison with the denominator (i.e., with N_0^{-1}, where N_0 is the maximum of N_q), and not with the accounted-for nonlinearity, since a cancellation of the principal contributions of the nonlinearities already taken into account can occur in the denominator, due to the negativeness of γ_q^g.

We now estimate the simplest diagrams for Π (Fig. 1, the solid line corresponds to φ, the dashed one to ψ and the wavy one to light). Figure 1 does not show all the diagrams of the given order, but only their types. Diagram 1a does not make a contribution to I_q, but makes a contribution only to $\Delta\omega_q$, and depends weakly on q, its order being $\sim\Gamma_0 N_0\Delta\omega/\omega_0$, where Γ_0 is the width of the ψ phonon, determined by the anharmonism of the third order. The contribution of such a form and order is given by the real part of the diagram 1c for Π_r. The diagram 1b gives the generation contribution $\gamma_q^{g0}=\gamma^g\varphi(2\omega_q)$, $\gamma^g\sim\tilde{J}/4\nu$.[1] Near threshold, $\gamma_q^{g0}\sim\tau^{-1}$, the real part of the corresponding Π_r makes a contribution to $\Delta\omega_q$ of the order τ^{-1}.

The diagram 1c corresponds to the collision integral $I_q^{(3)}$:

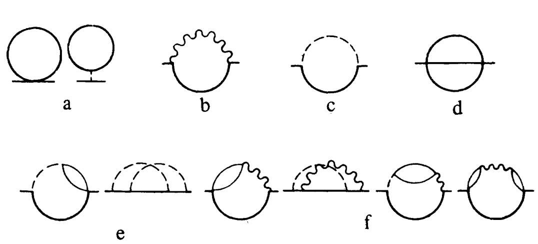

FIG. 1.

FIG. 2.

$$I_q^{(3)}=2\pi\int\frac{d\mathbf{q}_1}{(2\pi)^3}B_{\mathbf{q},\mathbf{q}_1}\Delta_\psi(\tilde\omega_q+\tilde\omega_{q_1}-\tilde\Omega_{\mathbf{q},\mathbf{q}_1})\{f_{\mathbf{q},\mathbf{q}_1}[N_q+N_{q_1}+1]-N_qN_{q_1}\},\qquad(8)$$
$$B_{\mathbf{q},\mathbf{q}_1}=b^2\omega_q\omega_{q_1}\Omega_{\mathbf{q}+\mathbf{q}_1}/2^3,$$

where $\Delta_\psi(\tilde\omega_q+\tilde\omega_{q_1}-\tilde\Omega_{\mathbf{q},\mathbf{q}_1})$ is the Δ function of the ψ phonon (see (3)), $f_{\mathbf{q},\mathbf{q}_1}$ are the generalized occupation numbers, $\tilde\Omega_{\mathbf{q},\mathbf{q}_1}$ is the spectrum of the ψ phonon, taken at a frequency equal to $\tilde\omega_q+\tilde\omega_{q_1}$.[5] The second component in the curly brackets (8) contributes a term of the form $\gamma_q^{(3)}N_q$ to (5), where $\gamma_q^{(3)}\sim\Gamma_0N_0(\Delta\omega/\omega_0)$ (this term was considered in Ref. 1). For an estimate of the first term, we can use the "balance" equation for $f_{\mathbf{q},\mathbf{q}_1}$:

$$\frac{1}{2}2\pi\int\frac{d\mathbf{q}_1}{(2\pi)^3}B_{\mathbf{q}_1,\mathbf{q}-\mathbf{q}_1}\Delta_\psi(\tilde\Omega_{\mathbf{q}_1,\mathbf{q}-\mathbf{q}_1}-\tilde\omega_{q_1}-\tilde\omega_{\mathbf{q}-\mathbf{q}_1})$$
$$\times\{f_{\mathbf{q}_1,\mathbf{q}-\mathbf{q}_1}[N_{q_1}+N_{\mathbf{q}-\mathbf{q}_1}+1]-N_{q_1}N_{\mathbf{q}-\mathbf{q}_1}\}=0.\qquad(9)$$

It follows from (9) that $f_0\sim N_0\tilde N[\max\{1,\tilde N\}]^{-1}$, and the width of $f_{\mathbf{q},\mathbf{q}_1}\sim\Delta\omega$, where f_0 is the maximum of $f_{\mathbf{q},\mathbf{q}_1}$, $\tilde N\equiv N_0(\Delta\omega/\omega_0)$. For what follows we note that the characteristic combination $N_0(\Delta\omega/\omega_0)$, equal in order of magnitude to the number of phonons in the cell,[1] arises in all the integrated expressions containing N_q (with the exception of the case of contributions to the vertex g). We shall assume the parameter $\tilde N\ll 1$, since in the opposite case we must take into account also the occupation numbers of the ungenerated modes and the problem is considerably more complicated. We can then use the equilibrium Γ_q for the ψ phonon and the spectrum, and for f_0 we get $f_0\sim N_0\tilde N$. Using these estimates, we find that the contribution of the first term of $I_q^{(3)}$ contains three terms with the following properties: the first term $\sim\Gamma_0N_q\tilde N^2$ with width $\sim\Delta\omega$, the second $\sim\Gamma_0N_0\tilde N^2$ with width $\sim\Delta\omega$, the third $\sim\Gamma_0\tilde N^2$ with width $\sim\omega_D$. They are all much less than $\gamma_q^{(3)}N_0$.

The diagram 1d corresponds to the collision integral $I_q^{(4)}$:

$$I_q^{(4)}=\frac{1}{2}a^2\frac{\omega_q}{2}\int\prod_{i=1}^{3}\frac{d\mathbf{q}_i}{(2\pi)^3}\left(\frac{\omega_{q_i}}{2}\right)(2\pi)^4\delta(\tilde\omega_q+\tilde\omega_{q_3}-\tilde\omega_{q_1}-\tilde\omega_{q_2})$$
$$\times\delta(\mathbf{q}+\mathbf{q}_3-\mathbf{q}_1-\mathbf{q}_2)[N_qN_{q_1}N_{q_2}+N_{q_3}N_{q_1}N_{q_2}-N_qN_{q_1}N_{q_3}$$
$$-N_qN_{q_2}N_{q_3}+N_{q_1}N_{q_2}-N_qN_{q_3}].\qquad(10)$$

Because of the non-decay nature of the φ phonon only that part is written down which describes the scattering of the φ phonons by one another. The first four terms in the square brackets in (10) have the same order: $\sim N_0\Gamma_0(\Gamma_0/\omega_0)\tilde N^2$, their width $\sim\Delta\omega$; the fifth term $\sim\Gamma_0(\Gamma_0/\omega_0)\tilde N^2$, its width $\sim\omega_D$; the sixth term $\sim N_0\Gamma_0(\Gamma_0/\omega_0)\tilde N$, its width $\sim\Delta\omega$. The diagram 1e contains corrections to diagram 1c and terms similar to (10) (here we assume that the second order cubic interaction in (1) is of the same order as the fourth interaction, i.e., $b^2\sim a$). And finally, the diagrams 1f give corrections to γ_q^{g0} and are of the order of $\tau^{-1}N_0\Gamma_0/\omega_0$ and $\tau^{-1}(N_0\Gamma_0/\omega_0)^2$ and width $\sim\Delta\omega$. More complicated diagrams contain additional powers of the small parameters Γ_0/ω_0 and $N_0\Delta\omega/\omega_0$ and therefore will not be needed by us in what follows (except diagrams of the type of Fig. 2).

We now discuss the role of the diagrams enumerated above and their corresponding contributions. The renormalization of the spectrum $\Delta\omega_q$, which depends smoothly on q, is equivalent to a shift in the pump center $\omega_0\to\omega_0-\Delta\omega_{q_0}$. Since we are interested in the behavior of quantities over intervals $\sim\Delta\omega$, it follows that at $\Delta\omega_q\ll\Delta\omega$, renormalization of the spectrum cannot be taken into account. For this reason, we have not considered the contribution of the diagram 1b to $\Delta\omega_q$. In what follows, we shall write simply ω_q everywhere in place of $\tilde\omega_q$. At the end of the section, we shall discuss when it is necessary to take $\Delta\omega_q$ into account. A second remark relates to the contributions to γ_q^n. It is easy to see that all the contributions to γ_q^n depend weakly on q and reduce to powers of the concentration of the φ and ψ phonons for narrow N_q. In this case we can assume γ_q^n to be a constant, γ_q, in Eq. (7) at $q\approx q_0$; this constant is to be determined along with the concentrations after finding N_q. The form of N_q is determined by γ_q^g and B_q^n. The contributions to B_q^n are of two types: weakly and strongly dependent on q. To the first belong those in which the number of significant integrations is not less than the number of "sharp" functions under the integral sign, for example, the third term in the square brackets in (8) and the fifth term in square brackets in (10). There is an additional smallness $\sim N_0^{-1}$ in such contributions at $q\approx q_0$, in comparison with the remaining terms of B_q^n, and they lead to the appearance of a "background" in the distribution function

$$N_q^{\mathrm{ph}}=B_{q\,\mathrm{ph}}^n/(\tau_q^{-1}+2\gamma_q^g+\gamma_q^n),\qquad(11)$$

where $B_{q\mathrm{ph}}^n$ is the part of B_q^n that is weakly dependent on q. To the second belong such terms in which the number of essential integrations is less than the number of "sharp" functions under the integral, for example, the second term in the square brackets in (8) and the fourth in (10). It is clear from physical considerations (this also follows from the estimates given for these contributions) that their width is at least no less than $\Delta\omega$. The problem of the effect of these contributions on the shape of N_q must be considered separately.

3. SOLUTION WITHOUT ACCOUNT OF THE "ARRIVAL" TERMS

In this section, we shall solve Eq. (7) for the case in which we do not have to take B_q^n into account. The limits of applicability of this solution will be discussed at the end. It must then be expected that N_q will be a "sharp" function; therefore γ_q^n will be assumed to be constant and Eq. (7) takes the form

$$N_q=\frac{1}{2}\frac{\xi_q}{1-\xi_q+u}=\frac{1}{2}\frac{\tilde\xi_q}{1-\tilde\xi_q},\qquad u=\tau\gamma^n,\qquad \tilde\xi_q=\frac{\xi_q}{1+u}.\qquad(12)$$

We shall seek a solution with $N_0\sim\omega_0/\Gamma_0$, $\Delta\omega\sim\Delta\nu$. In this case, we must take into account all terms in the denominator of (12) with accuracy to Γ_0/ω_0. We shall elucidate which diagrams must be taken into account for

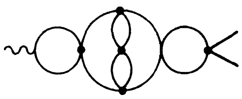

FIG. 3.

γ_q^ξ. It is seen from the preceding section that the corrections to g (Fig. 1f) are expanded in powers of $(N_0\Gamma_0/\omega_0)$, therefore we must take into account all diagrams of the type of Fig. 2. The fact that the expansion parameter here is $N_0\Gamma_0/\omega_0$ [and not $(\Gamma_0/\omega_0)N_0\Delta\omega/\omega_0$] is connected with the smallness of the momentum of the light. It is easy to understand that summation of diagrams of the type of Fig. 2 reduces to summation of "chains." This can be proven rigorously by using the diagram technique of the Appendix; here it turns out that in the loops it suffices to taken into account only sections of the rs and as types (i.e., sections with the corresponding lines) and that the "right" chain is the complex conjugate of the "left." Since the momentum entering the chain is assumed to be zero, then only the harmonics with $l=0$ would be left in the 4-phonon vertices entering this "chain;" therefore, we can immediately assume these vertices to be constants. Then summation of the chains reduces to the sum of the geometric progression of the loops. Since we need terms in the denominator with accuracy to $\Gamma_0/\omega 0$, it is easier (and more compact) to do this by writing the complete rs and as loops in the sum and not only their part $\sim\Gamma_0 N_0/\omega_0$, and this is done in what follows in order to keep the needed accuracy. Diagrams of the type of Fig. 3, which have the order $(N_0\Gamma_0/\omega_0)^n\tilde{N}(\Gamma_0/\omega_0)$ will not be taken into account, since we assume that $\tilde{N}\ll 1$. As a result, we obtain the following equation:

$$N_q=\frac{1}{2}\frac{\tilde{\xi}_q^0/|f_q|^2}{1-\tilde{\xi}_q^0/|f_q|^2}=\frac{1}{2}\frac{\tilde{\xi}_q^0}{|f_q|^2-\tilde{\xi}_q^0},\tag{13}$$

where

$$f_q=1-\frac{\bar{a}}{2}\Pi_\Gamma(2\omega_q,0)=1-\frac{\lambda}{\pi}\int_0^{\omega_D}\frac{(N_{\omega'}+1/2)\Phi_{\omega'}d\omega'}{\omega_q-\omega'+i\gamma_q/2}$$

$$\Pi_\Gamma(2\omega_q,0)=\int\frac{d^4q_1}{(2\pi)^4}D_r(\omega_1,q_1)D_s(2\omega_q-\omega_1,-q_1),$$

$$\Phi_\omega=\frac{\omega^2}{\omega_0^2}\frac{v}{v(\omega)}\frac{q^2(\omega)}{q_0^2},\quad\lambda=\frac{\bar{a}\omega_0^2q_0^2}{2^5\pi v}\sim\frac{\Gamma_0}{\omega_0},$$

$\bar{a}$ is the zero harmonic of the 4-phonon vertex at a total energy $\approx 2\omega_0$, and at a transferred energy ≈ 0; $q(\omega)$ is the momentum, $v(\omega)$ the group velocity as a function of the energy; $\tilde{\xi}_q^0=\tilde{\xi}\varphi_q$, and we assume φ_q to be Lorentzian:

$$\varphi_q=\frac{(\Delta\nu/2)^2}{(\omega_q-\omega_0)^2+(\Delta\nu/2)^2}.$$

Since we expect that $\gamma_q\ll\Delta\omega$ and are interested in the behavior of f_q over intervals $\sim\Delta\omega$, we shall assume γ_q in f_q to be an infintesimally small constant ε, which shows that ω_q is located on the physical sheet. It further turns out to be more convenient to solve the equation for f_q, which is obtained by substitution of (13) in the definition of f_q:

$$f_\omega=1-\frac{\lambda}{2\pi}\int_0^{\omega_D}\frac{\Phi_{\omega'}d\omega'}{\omega-\omega'+i\varepsilon}\frac{|f_{\omega'}|^2}{|f_{\omega'}|^2-\tilde{\xi}\varphi_{\omega'}}.\tag{14}$$

Here and in what follows, we shall write ω instead of ω_q.

It follows from the definition of f_ω that when the integral converges, i.e., when N_ω is finite, f_ω can be regarded as an analytic function of the complex ω, given by this integral representation. It follows from (14) that f_ω has a cut $(0,\omega_D)$; moreover, this function can have zeros on the real axis outside the cut, about the end ω_D, where f_ω changes strongly (these zeros are insignificant). Using these properties, we solve Eq. (14). We consider the function

$$g_\omega=f_\omega+\tilde{\xi}\varphi_\omega/f_\omega.\tag{15}$$

It has the same analytical properties as the function f_ω and has poles at the points where f_ω has zeros and where the poles of φ_ω are located $(\omega=\omega_0\pm i\Delta\nu/2)$. The jump of g_ω at the cut is equal to $\Delta g_\omega=2i\,\mathrm{Im}g_{\omega+}$, where

$$\mathrm{Im}\,g_{\omega_+}=\mathrm{Im}\,f_{\omega_+}\frac{|f_\omega|^2-\tilde{\xi}\varphi_\omega}{|f_\omega|^2},\quad\omega_+=\omega+i\varepsilon.$$

But, according to (14),

$$\mathrm{Im}\,f_{\omega_+}=\frac{\lambda}{2}\Phi_\omega\frac{|f_\omega|^2}{|f_\omega|^2-\tilde{\xi}\varphi_\omega}$$

and consequently, $\mathrm{Im}g_{\omega+}=\frac{1}{2}\lambda\Phi_\omega$. If we now write down the Cauchy formula for $g_\omega-1$ and extend the contour to infinity, then the integral over the infinite arc will be equal to zero and there remains only the integral over the cut and the poles. As a result, by neglecting the residues at the zeros of f_ω (they are small, $\sim(\Delta\nu/\omega_0)^2$), we obtain:

$$g_\omega=1+\lambda C_\omega+\tilde{\xi}\frac{\Delta\nu/2}{2i}\left(\frac{1}{f_0(\omega-\omega_0-i\Delta\nu/2)}-\text{c.c.}\right),\tag{16}$$

where f_0 is the value of f_ω at $\omega=\omega_0+i\Delta\nu/2$,

$$C_\omega=\frac{1}{2\pi}\int_0^{\omega_D}\frac{\Phi_{\omega'}d\omega'}{\omega'-\omega-i\varepsilon}.$$

Knowing g_ω, it is easy to find f_ω by inverting (15):

$$f_\omega=1/2[g_\omega\pm(g_\omega^2-4\tilde{\xi}\varphi_\omega)^{1/2}],\tag{17}$$

where we must take the positive branch (this follows from consideration of the asymptotic behavior as $\omega\to\infty$). From (17), we get for N_ω,

$$N_\omega=\frac{\mathrm{Im}\,f_{\omega+}}{\lambda\Phi_\omega}-\frac{1}{2}=-\frac{1}{4}+\frac{1}{2\lambda}\left[\frac{(X_\omega^2+Y_\omega^2)^{1/2}-X_\omega}{2}\right]^{1/2},\tag{18}$$

where

$$X_\omega=\mathrm{Re}\,(g_\omega^2-4\tilde{\xi}\varphi_\omega),\tag{18a}$$

$$Y_\omega=\mathrm{Im}\,(g_\omega^2-4\tilde{\xi}\varphi_\omega).\tag{18b}$$

We set $\Phi_\omega=1$ in (18) because of its smoothness, since we are interested in $\omega \approx \omega_0$. It follows from (18b) that Y_ω is always $\sim\lambda$, but X_ω can change, depending on ω and $\tilde{\xi}$, from values ~ 1 to values ~ -1. We can then show immediately from (18) that

$$N_\omega \sim \frac{1}{\lambda}\frac{Y_\omega}{X_\omega^{1/2}} \sim \lambda^0, \quad \Delta\omega\sim\Delta\nu \quad \text{at} \quad X_\omega \sim 1,$$
$$N_\omega \sim \lambda^{1/2}, \ \Delta\omega \sim \Delta\nu\lambda^{1/4} \quad \text{at} \quad X_\omega \lesssim \lambda, \tag{19}$$
$$N_\omega \sim |X_\omega|^{1/2}/\lambda, \ \Delta\omega\sim\Delta\nu \quad \text{at} \quad X_\omega \sim -1.$$

It remains to determine f_0. Letting ω in (17) approach $\omega_0 + i\Delta\nu/2$, we obtain the identity $f_0 = f_0$ in first-order approximation (here we should have $\mathrm{Im} f_0^{-1} < 0$, which also follows from the definition of f_ω, at $\lambda > 0$), while the value of the first derivative f_0' becomes "linked" in second-order approximation and so on. Therefore, we use other methods. By definition, f_ω has on the physical sheet no branch points other than zero and ω_D. But it follows from (17) that at such ω, when $g_\omega^2 = 4\tilde{\xi}\varphi_\omega$, additional branch points appear for f_ω. To avoid this, it is necessary either that there be no zeros for the function $g_\omega^2 - 4\tilde{\xi}\varphi_\omega$ on the physical sheet or, if there are any, they should be paired. We determine f_0 from these conditions.

We introduce the new variable $x = 2(\omega - \omega_0)/\Delta\nu$. Then the equation $g_\omega^2 - 4\tilde{\xi}\varphi_\omega = 0$ reduces to the following equation for x (with accuracy to terms $\sim\lambda$):

$$(x^2+1-\tilde{\xi})^2 + 2\lambda C_x(x^2+1)[\tilde{\xi}(R+Jx)+x^2+1+\tilde{\xi}]$$
$$+\tilde{\xi}(R+Jx)[\tilde{\xi}(R+Jx)+2(x^2+1+\tilde{\xi})]=0, \tag{20}$$

where $R = \mathrm{Re} f_0^{-1} - 1$ and $J = \mathrm{Im} f_0^{-1}$. We can reduce this equation to two fourth degree algebraic equations. The function C_x has the cut

$$\left(\frac{-\omega_0}{\Delta\nu/2}, \frac{\omega_D - \omega_0}{\Delta\nu/2}\right).$$

We consider its analytic continuation from the upper edge of the cut. On this sheet, C_x is a smooth function and it can be assumed to be constant,

$$C \equiv C(0+i\varepsilon) = C_1 + iC_2,$$

in the solution of Eq. (19), since the roots of this equation $\sim (1-\tilde{\xi})^{1/2}$, and C_x changes to $x \sim \omega_D/\Delta\nu \gg (1-\tilde{\xi})^{1/2}$. As a result, we obtain a fourth degree algebraic equation. A second equation is obtained upon analytic continuation from the upper edge of the cut (it is distinguished from the first by the sign of $C_2 = \mathrm{Im} C$).

We shall solve the equation obtained by analytic continuation from the upper edge of the cut (the roots of the second equation are the complex conjugates of the roots of the first). This equation is the same as (20), except that C is present instead of C_x. It has four roots; therefore, it is necessary that the roots with a positive imaginary part be double. To estimate the double roots, we must set its derivative equal to zero. As a result, denoting the left side of (20) with $C_x = C$ by $P_4(x)$, we obtain a set of two equations:

$$P_4(x) = 0, \tag{21a}$$
$$P_4'(x) = 0 \tag{21b}$$

(the prime indicates the derivative with respect to x). The solution of this set (generally a system with complex coefficients) should give the location of the zeros and constants R and J as functions of $\tilde{\xi}$ and λ. Such a solution is difficult in the general case; therefore, we shall solve this set for several ranges of the parameter $\tilde{\xi}$.

1) $\lambda^{1/2} \ll 1 - \tilde{\xi} \sim 1$. In this case, $\tilde{\xi}$ is located far from the threshold value $\xi_0 = 1$; therefore, $N_0 \lesssim 1, R, J \sim \lambda$, and the value of the roots $x_i \sim 1$. Equation (21a) can be written in the form

$$(x^2+1-\tilde{\xi})^2 + 2[\lambda C(x^2+1) + \tilde{\xi}(R+Jx)](x^2+1+\tilde{\xi}) = 0. \tag{22}$$

We shall solve it by expansion in powers of λ:

$$x_i = x_i^{(0)} + x_i^{(1)} + x_i^{(2)} + \ldots, \quad \text{where} \quad i = 1,2,3,4. \tag{23}$$

Substitution of (23) in (22) gives

$$x_{1,3}^{(0)} = \pm i(1-\tilde{\xi})^{1/2}, \quad x_2 = x_1, \quad x_4 = x_3, \tag{24a}$$

$$x_i^{(1)2} = -\frac{\tilde{\xi}}{x_3^{(0)2}}(\lambda C + R + Jx_i^{(0)}). \tag{24b}$$

In order that splitting of the roots x_1 and x_2 not occur, it is necessary that the right side of (24b) vanish for $i = 1$ and 2. We then obtain

$$R = -\lambda C_1, \ J = -\lambda C_2/(1-\tilde{\xi})^{1/2}. \tag{25}$$

We write out the first nonvanishing corrections to the roots;

$$x_{1,2}^{(1)} = 0, \quad x_{1,2}^{(2)} = -\frac{\tilde{\xi}}{2x_1^{(0)2}}[2\lambda C x_1^{(0)} + \tilde{\xi}J]$$
$$= \frac{\tilde{\xi}}{2(1-\tilde{\xi})}\left\{-\lambda C_2\left[\frac{\tilde{\xi}}{(1-\tilde{\xi})^{1/2}} + (1-\tilde{\xi})^{1/2}\right] + i\cdot 2\lambda C_1(1-\tilde{\xi})^{1/2}\right\},$$
$$x_{3,4}^{(1)} = \pm\left[-\frac{\tilde{\xi}}{x_3^{(0)2}}[\lambda C + R + Jx_3^{(0)}]\right]^{1/2} = \pm\left(\frac{\tilde{\xi}\lambda C_2}{1-\tilde{\xi}}\right)^{1/2}(1+i).$$

Equation (25) can be verified by direct substitution of $N_\omega = \frac{1}{2}\tilde{\xi}\varphi_\omega[1-\tilde{\xi}\varphi_\omega]^{-1}$ in f_ω.

2) $|1-\tilde{\xi}| \sim \lambda^{1/2}$. In this case, $\min X_\omega \sim \lambda$; it then follows from (19) that $N_0 \sim \lambda^{1/2}$, $\Delta\omega \sim \Delta\nu\lambda^{1/4}$, and from the definition of f_ω, that $R\sim\lambda$, $J\sim\lambda^{3/4}$ and the value of the root must be $\sim\lambda^{1/4}$. In this region of $\tilde{\xi}$, the set (21) can be written in the form

$$(x^2+1-\tilde{\xi})^2 + 4(Jx+R+\lambda C) = 0, \tag{26a}$$
$$(x^2+1-\tilde{\xi})x + J = 0. \tag{26b}$$

We can obtain an equation for the double root from Eq. (26b):

$$x_{1,2} = -\frac{A+B}{2} + i\sqrt{3}\frac{A-B}{2}, \tag{27}$$

where

$$A = \left(-\frac{J}{2}+\sqrt{Q}\right)^{1/3}, \quad B = \left(-\frac{J}{2}-\sqrt{Q}\right)^{1/3}, \quad Q = \left(\frac{1-\tilde{\xi}}{3}\right)^3 + \left(\frac{J}{2}\right)^2.$$

Two other roots remain in the lower halfplane, since Eq. (26a) does not have real roots and therefore its roots cannot cross the real axis. The equation for R and J, which contains no radicals, can be obtained by making up, from the system (26), an equation with smaller powers. In short, we obtain the following equation:

$$\bar{R}^4+\mu^2\bar{R}^3+\frac{1}{3}(\mu^4-2\varkappa)\bar{R}^2+\mu^2\frac{15\varkappa+\mu^4}{3^3}\bar{R}+\frac{\varkappa}{36}\left(4\varkappa+\frac{\mu^4}{3}\right)=0, \quad (28a)$$

$$J^4-\left(\frac{2}{3}\right)^3\mu(2\mu^2+9\bar{R})J^2-\frac{2^4}{3^3}[\mu^2(\bar{R}^2-\varkappa)+4\bar{R}(\bar{R}^2-3\varkappa)]=0, \quad (28b)$$

where

$$\bar{R}=R+\lambda C_1, \quad \mu=1-\tilde{\xi}, \quad \varkappa=(\lambda C_2)^2.$$

The general solution of this system is cumbersome. A simple solution exists for $|1-\tilde{\xi}|\ll\lambda^{1/2}$. In this case, (28a) takes the form

$$(\bar{R}^2-\varkappa/3)^2=0 \quad \text{or} \quad \bar{R}=\pm\lambda C_2/\sqrt{3}. \quad (29)$$

One must take the minus sign. We then get for J from (28b)

$$J=-\left[\left(\frac{2}{\sqrt{3}}\right)^3\lambda C_2\right]^{1/4}. \quad (30)$$

The following expressions are obtained for the roots:

$$x_{1,2}=-{}^1\!/_2|J|^{1/3}(1-i\sqrt{3}),$$
$$x_{3,4}={}^1\!/_2|J|^{1/3}[1\pm\sqrt{6}-i(\sqrt{3}\mp\sqrt{2})]. \quad (31)$$

As follows from (29)–(31), in this region $\tilde{\xi}(|1-\tilde{\xi}|\ll\lambda^{1/2})$, none of the quantities depend on $\tilde{\xi}$ and have the orders of magnitude expected from the estimates.

3) $\lambda^{1/2}\ll\tilde{\xi}-1\ll1$. In this region, the equations have the same form as in (26), except that λC can be omitted in (26a). The system becomes real and in this region emergence of the roots on the real axis becomes possible (with accuracy to powers of λ). At the same time, two equations are now insufficient to determine the roots and the two parameters R and J. We therefore expect additional degeneracy of the roots upon their emergence on the real axis. Two cases are possible: the approach of the roots $x_{1,2}$ to x_4 or of x_3 to x_4. The roots and R and J are determined in identical fashion in both cases and can be found by several methods. We use Eq. (28), neglecting $\varkappa$ in it and assuming $\bar{R}=R$. We have from (28a)

$$R(R+\mu^2/3)^3=0, \quad R=0, \quad -\mu^2/3. \quad (32)$$

The solution $R=0$ is not suitable at $\tilde{\xi}-1>0$. For J and the roots, we get

$$J=-2[(\tilde{\xi}-1)/3]^{3/2},$$
$$x_{1,2}=x_4=-[(\tilde{\xi}-1)/3]^{1/2},\ x_3=-3x_1=[3(\tilde{\xi}-1)]^{1/2}. \quad (33)$$

We note that $x_4=x_{1,2}$ is automatically satisfied.

4) $\tilde{\xi}-1\sim1$. At such $\tilde{\xi}$, the minimum of X_ω can become ~-1. This follows from the fact that $X_\omega=P_4(x)(x^2+1)^{-2}$ and we expect that the structure of $P_4(x)=(x-x_1)^3(x-x_3)$, $x_1<x_3, x_1, x_3\sim1$ is preserved.

Therefore $P_4(x)$ can take on values ~-1 at $x_1<x<x_3$. Then, from (19) and the definition of f_ω, we have $N_0\sim\lambda^{-1}$, $\Delta\omega\sim\Delta\nu$, $R\sim J\sim1$. The set of equations in this case takes the form

$$P_4(x)=x^4+A_1x^3+B_1x^2+C_1x+D_1=0, \quad (34a)$$
$$P_4'(x)=0; \quad (34b)$$
$$A_1=2\tilde{\xi}J, \quad B_1=2(1-\tilde{\xi})+(\tilde{\xi}J)^2+2\tilde{\xi}R,$$
$$C_1=2[\tilde{\xi}^2RJ+\tilde{\xi}(1+\tilde{\xi})J], \quad D_1=(1-\tilde{\xi})^2+(\tilde{\xi}R)^2+2\tilde{\xi}R(1+\tilde{\xi}).$$

We can obtain two second degree equations from this system. Their structure should be of the form $(x-x_1)^2=0$. Therefore, equating the coefficients in these equations, we obtain equations for R and J:

$$\tilde{R}^3+3(2+\tilde{\xi})\tilde{R}^2+3(2+\tilde{\xi})^2\tilde{R}+(\tilde{\xi}+8)(1-\tilde{\xi})^2=0, \quad (35a)$$

$$J^2=4\frac{2(1-\tilde{\xi})^2+\tilde{R}(3+3\tilde{\xi}+\tilde{R})}{\tilde{\xi}^2(2+\tilde{\xi}+\tilde{R})}, \quad (35b)$$

where $\tilde{R}=\tilde{\xi}R$. There is one real solution for Eq. (35a):

$$R=(3\tilde{\xi}^{1/3}-\tilde{\xi}-2)/\tilde{\xi}. \quad (36)$$

We then get for J and the roots:

$$J=-2\frac{[\tilde{\xi}^{1/3}-1]^{3/2}}{\tilde{\xi}}, \quad x_1=-\frac{1}{2}J\left[\frac{(\tilde{\xi}^{1/3}-1)^3-\tilde{\xi}^2(3\tilde{\xi}^{1/3}+\tilde{\xi}-1)}{(\tilde{\xi}^{1/3}-1)^3-\tilde{\xi}^2(3\tilde{\xi}^{1/3}-2\tilde{\xi}-1)}\right],$$
$$x_3=-3x_1-2\tilde{\xi}J. \quad (37)$$

It is easy to establish the fact that the asymptotic forms of (36) and (37) at $\tilde{\xi}-1\ll1$ are identical with the solutions (32) and (33). At $\tilde{\xi}\gg1$, we get from (36) and (37):

$$R=-1, \quad J=-2/\tilde{\xi}^{1/2}, \quad x_1=-1/\tilde{\xi}^{1/2}, \quad x_3=4\tilde{\xi}^{1/2}. \quad (38)$$

In the obtained solutions, all the quantities are functions of $\tilde{\xi}=\xi/(1+u)$, where u can be represented in the form $\alpha\tau\bar{N}$,

$$\bar{N}=\int_0^{\omega_D}d\omega\,N_\omega,$$

α is a dimensionless coefficient, the value of which depends on the contribution to γ^n to which it corresponds; for $\gamma^{(3)}$, we have $\alpha\sim\Gamma_0/\omega_0$, [1] for $\gamma^{(4)}$, $\alpha\sim(\Gamma_0/\omega_0)^2$. For the final solution of the problem, we must express $\tilde{\xi}$ in terms of ξ. For this purpose, we must find the dependence of $\bar{N}$ on $\tilde{\xi}$ and substitute in the definition of $\tilde{\xi}$. It is simplest to do this by considering the asymptotic form of f_ω as $\omega\to\infty$. It follows from the definition of f_ω that

$$f_\omega\to1-\frac{\lambda}{\pi\omega}\overline{N+{}^1\!/_2}, \quad \text{where} \quad \overline{N+{}^1\!/_2}=\int_0^{\omega_D}d\omega(N_\omega+{}^1\!/_2)\Phi_\omega. \quad (39)$$

Equating (39) with the asymptotic form of (17), we obtain

$$\bar{N}=-\pi\frac{\tilde{\xi}J\Delta\nu/2}{\lambda}. \quad (40)$$

The equation for $\tilde{\xi}$ takes the form

$$\xi=\tilde{\xi}\left(1+\beta\frac{2\tilde{\xi}|J(\tilde{\xi})|}{\lambda}\right),\quad \beta=\frac{\pi}{2}\alpha\tau\frac{\Delta\nu}{2}. \tag{41}$$

It is not simple to invert the relation (41) in the general case, but (41) allows us to find the values of ξ at which $\tilde{\xi}$ is found in the regions $1-\tilde{\xi}\gg\lambda^{1/2}$, $|1-\tilde{\xi}|\ll\lambda^{1/2}$, and $\tilde{\xi}-1\gg\lambda^{1/2}$. Using the formulas for $J(\tilde{\xi})$, we obtain

$$\xi=\tilde{\xi}\left(1+\frac{\beta\tilde{\xi}}{(1-\tilde{\xi})^{1/2}}\right),\quad 1-\tilde{\xi}\gg\lambda^{1/2}, \tag{42a}$$

$$\xi=\tilde{\xi}\left[1+2\tilde{\xi}\beta\lambda^{-1/4}\left(\left(\frac{2}{\sqrt{3}}\right)^3C_2\right)^{1/4}\right],\quad |1-\tilde{\xi}|\ll\lambda^{1/2}, \tag{42b}$$

$$\xi=\tilde{\xi}\left[1+4\beta\frac{(\tilde{\xi}^{1/2}-1)^{1/2}}{\lambda}\right],\quad \tilde{\xi}-1\gg\lambda^{1/2}. \tag{42c}$$

Equation (42a) corresponds to the region of applicability of the theory advanced in Ref. 1. It follows from (42) that the excess of ξ over $\tilde{\xi}$ depends on the relation between β and the different powers of λ.

Some additional data on the form of N_q in the spectrum can be obtained at $\lambda^{1/2}\ll\tilde{\xi}-1\ll1$ and at $\tilde{\xi}-1\sim1$. In the first case, near the minimum $X_\omega\approx P_4(x)$; therefore, the maximum N_ω corresponding to the minimum X_ω is found at the point $\bar{\omega}=\omega_0+\bar{x}\Delta\nu/2$, where $\bar{x}=2[(\tilde{\xi}-1)/3]^{1/2}$; here, $\Delta\omega_{q_0}\sim(\Gamma_0/\omega_0)N_0\Delta\omega\ll\Delta\omega$; therefore, the renormalization can be disregarded. In the second case, $\bar{\omega}=\omega_0-\Delta\omega_{q_0}+\bar{x}\Delta\nu/2$; therefore, renormalization must be taken into account. The location of the maximum N_ω is identical with that of minimum $X_\omega=P_4(x)/(x^2+1)^2$ and is found at the point $\bar{\omega}=\omega_0-\Delta\omega_{q_0}+\bar{x}\Delta\nu/2$, where

$$\bar{x}=\frac{2(x_1x_3-1)}{3x_1+x_3}+\left[\frac{4(x_1x_3-1)^2}{(3x_1+x_3)^2}+\frac{x_1+3x_3}{3x_1+x_3}\right]^{1/2},$$

and x_1 and x_3 are given by Eqs. (37). We note that in the region $\tilde{\xi}-1\gg\lambda^{1/2}$ the shape of N_ω no longer depends on λ. Thus, N_ω at $\lambda^{1/2}\ll1-\tilde{\xi}\sim1$ has an asymmetric shape with $N_0\sim1/(1-\tilde{\xi})$ and $\Delta\omega\sim\Delta\nu(1-\tilde{\xi})^{1/2}$, becoming narrower with increasing $\tilde{\xi}$. At $1-\tilde{\xi}\sim\lambda^{1/2}$ strong rearrangement of N_ω takes place, it loses its symmetry, and the narrowing ceases; at $\tilde{\xi}-1\gg\lambda^{1/2}$, some asymmetric distribution takes place with $\Delta\omega\to\sim\Delta\nu$, $N_0\to\sim\lambda^{-1}$, the shape of which does not depend on λ.

We now discuss the form of the function γ_q:

$$\gamma_q=\tau^{-1}(1+u-{}^1/_2\xi_{q'}/|f_q|^2)=\tau^{-1}(1+u)(1-{}^1/_2\tilde{\xi}_{q'}/|f_q|^2).$$

Its characteristic width is $\sim\Delta\omega$; in this case it remains everywhere greater than zero and $\sim\tau^{-1}(1+u)$.

We discuss the self-consistency of the solution separately for the case in which sticking into ψ phonons is possible and for the case in which it is impossible. In the first case, the conditions for a self-consistent solution have the form

$$N_0^{-1}\gg\tau\Gamma_0\tilde{N}^2,\quad \tilde{N}\ll1,\quad \Delta\omega\gg\gamma_q,\quad \omega_0v/c. \tag{43}$$

At $N_0\sim(\omega_0/\Gamma_0)^{1/2}$, $\Delta\omega_0\sim\lambda^{1/2}\Delta\nu$, these conditions reduce to the following:

$$\tau^{-1}\omega_0\gg(\Delta\nu)^2,\quad \omega_0(\Gamma_0/\omega_0)^{1/4}\gg\Delta\nu\gg(\Gamma_0/\omega_0)^{-1/4}[\tau^{-1}(1+u),\ \omega_0v/c].$$

If we use the values (see Ref. 1): $\tau^{-1}\sim10^9$ sec^{-1}, $\omega_0\sim10^{14}$ sec^{-1}, $\Delta\nu\sim\Gamma_0\sim10^{11}$ sec^{-1}, $v\sim10^5$ cm/sec, and $c\sim10^{10}$ cm/sec, then it is easy to establish the fact that all the inequalities are satisfied (although some only in order of magnitude), and $N_0\sim10^{3/2}$, $\Delta\omega\sim10^{11}$ sec^{-1}, $u\sim1$, $\xi\approx1+O(\beta)$, $\beta\sim0.1$.

In the second case, only the first condition changes:

$$N_0^{-1}\gg\tau\Gamma_0(\Gamma_0/\omega_0)\tilde{N}^2,\quad \tilde{N}\ll1,\quad \Delta\omega\gg\gamma_q,\ \omega_0v/c. \tag{44}$$

At $N_0\sim\omega_0/\Gamma_0$ and $\Delta\omega\sim\Delta\nu$, these conditions reduce to

$$\tau^{-1}\Gamma_0\gg(\Delta\nu)^2;\quad \Gamma_0\gg\Delta\nu\gg\tau^{-1}(1+u),\quad \omega_0v/c.$$

These conditions are not satisfied for the values given above. But, as has been noted in Ref. 1, the parameters entering into the condition can change considerably. Therefore, if we take $\omega_0\sim10^{13}$ sec^{-1}, $\Gamma_0\sim10^{11}$ sec^{-1}, $v/c\sim10^{-5}$, $\tau^{-1}\sim10^8$ sec^{-1}, and $\Delta\nu\sim10^9$ sec^{-1}, then all the conditions are satisfied (almost at the limit). In this case, $N_0\sim10^2$, $\Delta\omega\sim10^9$ sec^{-1}, $u\sim0.1$, $\xi\approx\tilde{\xi}+O(\beta/\lambda)$, $\beta/\lambda\sim0.1$. It can be shown that at these values, the conditions of the first case are also satisfied.

In the solutions given above, we have assumed $\lambda>0$, but if we take it into account that the second-order contribution of the cubic interaction to the 4-phonon interaction at the considered negative values of momenta and energy, then the case $\lambda<0$ is possible. In this case, $J>0$, $R>0$, and N_ω has the form that is the mirror image relative to ω_0. Still one more remark pertains to the fact that the form of N_ω at $|\tilde{\xi}-1|\ll1$ does not depend on the shape of φ_ω (if φ_ω is not Lorentzian). This follows from the fact that in this region $\tilde{\xi}\Delta\nu\gg\Delta\omega$ and since the larger N_ω are concentrated in the region $\Delta\omega$, then φ_q^{-1} in (13) can be expanded about ω_0 and only a term $\sim z_q(\Delta\nu)$ remains, where

$$z_q(\Delta\nu)=\left(\frac{\omega_q-\omega_0}{\Delta\nu/2}\right)^2,$$

but this is equivalent to a transition to a Lorentzian φ_q with $\Delta\tilde{\nu}=\Delta\nu A^{-1/2}$, where $A=(\varphi^{-1})'$ is the derivative of φ_q^{-1} at $\omega_q=\omega_0$:

$$N_q=\frac{{}^1/_2\tilde{\xi}}{|f_q|^2\varphi_q^{-1}-\tilde{\xi}}\approx\frac{{}^1/_2\tilde{\xi}}{|f_q|^2[1+Az_q(\Delta\nu)]-\tilde{\xi}}=\frac{{}^1/_2\tilde{\xi}\tilde{\varphi}_q}{|f_q|^2-\tilde{\xi}\tilde{\varphi}_q},$$

where

$$\tilde{\varphi}_q=[1+z_q(\Delta\tilde{\nu})]^{-1}.$$

We have assumed that $\varphi_q=\varphi(z_q(\Delta\nu))$. This always takes place for φ_q that is symmetric in ω_0.

4. DISCUSSION

We discuss first the bounding mechanism—the corrections to the vertex of the interaction of the light with phonons. This mechanism is connected, on the one hand, with the mechanism of "effective pumping," which is operative in theories with monochromatic pumping.[2]

It is easy to see that if we average over the pumping, then, of the diagrams that describe the "effective pumping," the greatest contribution is made by diagrams of the type of Fig. 2. On the other hand, the action of the corrections to the vertex g recalls the action of the "intermediate particle," i.e., the "particle" which has a structure with a gap $\Omega_0 \approx \nu_0$ and which interacts with the light (for example, the optical phonon in Ref. 5 or the uniform precession of the magnetization in ferromagnets[2]). In this case, the pumping of phonons takes place according to the scheme: light → "intermediate particle" → phonon. The latter is not accidental. If N_q were fixed, then at $N_0 \sim \lambda^{-1}$, it would be necessary, when finding the 4-phonon vertex at a small total momentum and a total energy $\approx 2\omega_0$, to sum the "chain" from Fig. 2. In the resultant equation, N_q would play the role of the effective density of states and the pole of the 4-phonon vertex (or zero of f_q) on the unphysical sheet (if there were one) would correspond to an "intermediate particle" (see also Ref. 6). When generation takes place, however, it is necessary to determine N_q also. This can lead to a change in the analytic properties of f_q (or the Green's function of the "intermediate particle"). In our case, there are no zeros on the unphysical sheet, but branch points do appear (they correspond to the roots x_3 and x_4 of Eq. (20)). There is also an analogy between f_ω and the dielectric permittivity $\varepsilon(\omega, 0)$.

We note that our solution turned out to be possible because of the satisfaction of the condition $\Delta\omega \gg \gamma_q$, which would allow us to transform to the kinetic equation (5) and neglect γ_q in f_q. The stability of the obtained solution will be studied in another paper.

We now discuss several generalizations of the considered theory. It can be generalized with insignificant changes to the case of generation of different particles. There is significant difficulty in taking account of the anisotropy. Although the mechanism of corrections to the vertex will be operative in this case, the equation for f_q will contain an integral over the angles and, because of the strong nonlinearity of the equation, all the harmonics will turn out to be coupled. It can only be expected that in the case of weak anisotropy the qualitative picture remains the same. We note that the achievement of large N_0 strongly increases the nonlinear absorption coefficient $K_n \sim K_l N_0$, where K_l is the linear coefficient. Therefore, thin crystals are needed for the case of generation through direct absorption[1] (thickness $d < 1/K_n$), in order not to have to take into account the change in the intensity along the sample. This condition is not required in the generation via beats of two laser beams with optical frequencies.

APPENDIX

We now set forth the formulation of the Keldysh technique[3] and obtain one important property of the Green's function in a representation in which a retarded (r), an advanced (a) and a statistical s Green's function are employed (the r-representation). We apply the entire analysis to the example of a scalar field with an interaction Hamiltonian

$$H_{\text{int}} = \frac{\lambda}{4!} \int \varphi^4(\mathbf{x})\, d\mathbf{x}.$$

According to Ref. 3, to calculate quantities of the type

$$F(x_1, \ldots, x_n) = \operatorname{Tr}\{\rho T(\varphi(x_1)\ldots\varphi(x_n))\},$$

where $\varphi(x_i)$ are Heisenberg operators and $x = \{t, \mathbf{x}\}$, we can transform to the interaction representation; here

$$F(x_1, \ldots, x_n) = \operatorname{Tr}\{\rho_0 T_c(\varphi(x_1)\ldots\varphi(x_n) S_c\} \equiv \langle T_c(\varphi(x_1)\ldots\varphi(x_n) S_c\rangle_0,$$

where T_c is the time ordering operator along the contour c, which runs from $t = -\infty$ to $t = +\infty$ and then returns to $t = -\infty$,

$$S_c = T_c \exp\left(-i \int H_{\text{int}}(x)\, d^4x\right),$$

S is a matrix calculated along this contour, and the fields $\varphi(x_1)$ are already taken in the interaction representation.

We introduce the index $i = 1, 2$ for the field $\varphi(x)$ in the Heisenberg representation and determine the ordering operator T_K which acts, in addition to the time, also on these indices, so that the fields with $i = 2$ are always located in time after the fields with $i = 1$, whereby the fields with $i = 1$ are chronological, and the fields with $i = 2$ are anti-chronological.

After changing to the interaction representation, it is easy to see that

$$F_{i_1\ldots i_n}(x_1, \ldots, x_n) \equiv \operatorname{Tr}\{\rho T_K(\varphi_{i_1}(x_1)\ldots\varphi_{i_n}(x_n))\} = \langle T_K(\varphi_{i_1}(x_1)\ldots\varphi_{i_n}(x_n) S_K)\rangle_0,$$

where

$$S_K = T_K \exp\left(-i\frac{\lambda}{4}\int_{-\infty}^{\infty} \gamma_{iklm}\varphi_i\varphi_k\varphi_l\varphi_m\, d^4x\right),$$

$$\gamma_{iklm}\begin{cases} 1 & \text{for } i=k=l=m=1, \\ -1 & \text{for } i=k=l=m=2, \\ 0 & \text{for all other } i, k, l, m, \end{cases}$$

and the operator T_K in the interaction representation also acts on the field $\varphi_i(x)$ in the same way as is defined for the Heisenberg representation. Expanding the S_K matrix in powers of λ and using the Wick theorem, we obtain the diagram expansion for the quantities $F_{i_1\ldots i_n}(x_1, \ldots, x_n)$. It is identical, on the one hand, with the Keldysh expansion, and on the other, with the ordinary diagram technique for a multicomponent field with nondiagonal (in the index i) single-particle Green's functions, and only connected diagrams are left then.

We introduce the single-particle Green's functions:

$$D_{ik}(x, x') = -i \operatorname{Tr}\{\rho T_K(\varphi_i(x)\varphi_k(x'))\} = -i\langle T_K(\varphi_i(x)\varphi_k(x') S_K\rangle_0.$$

They are identical with those introduced by Keldysh[3]:

$$D_{ik}(x, x') = \begin{Bmatrix} D^c & D^+ \\ D^- & \tilde{D}^c \end{Bmatrix}.$$

Formally, $\varphi_i(x)$ and γ_{iklm} can be regarded respectively as a vector and a tensor in some linear space (two-dimensional, since $i=1, 2$). Then the transformation to the r representation is equivalent to some orthogonal transformation in this space of the fields $\varphi_i(x)$ and the tensor γ_{iklm}:

$$\tilde{D}_{ik}(x, x')=U_{ii'}U_{kk'}D_{i'k'}(x, x')=-i\langle T_K(\tilde{\varphi}_i(x)\tilde{\varphi}_k(x')\tilde{S}_K\rangle_0,$$

$$\tilde{\varphi}_i=U_{ii'}\varphi_{i'},\quad U_{ik}=(U^{-1})_{ki}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix},\qquad \text{(A.1)}$$

$$\tilde{D}_{ik}=\begin{Bmatrix}0, & D_r\\ D_a, & D_s\end{Bmatrix},\quad \tilde{\gamma}_{iklm}=U_{ii'}U_{kk'}U_{ll'}U_{mm'}\gamma_{i'k'l'm'},$$

and all $\tilde{\gamma}$ with an odd number of indices $i=1$ are equal to one another and to $\frac{1}{2}$, and all the remainder are equal to zero,

$$\tilde{S}_K=T_K\exp\left(-i\frac{\lambda}{4!}\int_{-\infty}^{\infty}\tilde{\gamma}_{iklm}\tilde{\varphi}_i\tilde{\varphi}_k\tilde{\varphi}_l\tilde{\varphi}_m\,d^4x\right)=S_K,$$

i.e., S_K is invariant.

It follows from these formulas that we can construct a diagram expansion immediately in the r representation (or in any other representation obtained from the original orthogonal transformation). In the case in which the interaction is nonlocal, the tensor vertex is obtained by multiplication of a form factor by the structure tensor γ_{iklm}. All this holds also for the case in which the interaction has the form $\int\varphi^n d\mathbf{x}$; here the corresponding structure tensor $\gamma_{ikl\ldots}$ of n-th rank appears. In particular, in the r representation, the components of this tensor with odd number of indices $i=1$ are equal to $2^{1-n/2}$, and the remainder are equal to zero.

With the help of this formulation, it is easy to obtain the following important property of the Green's function in the r representation.

We consider the n-particle Green's function

$$\tilde{F}_{1\ldots1}(x_1,\ldots,x_n)=\mathrm{Tr}\,\{\rho T_K(\tilde{\varphi}_1(x_1)\ldots\tilde{\varphi}_1(x_n))\}.$$

According to (A.1), it is equal to

$$\tilde{F}_{1\ldots1}(x_1,\ldots,x_n)=2^{-n/2}\mathrm{Tr}\{\rho T_K[\varphi_1(x_1)-\varphi_2(x_1)]\ldots[\varphi_1(x_n)-\varphi_2(x_n)]\}.\qquad \text{(A.2)}$$

Using the definition of the operator T_K in (A.2), we can show that

$$\tilde{F}_{1\ldots1}(x_1,\ldots,x_n)=0.\qquad \text{(A.3)}$$

In particular, this leads to the equality $\tilde{D}_{11}(x,x')=0$ obtained by Keldysh.[3] If we now define the vertices by the formula

$$\tilde{F}_{l_1\ldots l_n}(x_1,\ldots,x_n)=\int\prod_{i=1}^{n}dx_i'\tilde{D}_{l_ik_i}(x_i,x_i')\tilde{\Gamma}_{k_1\ldots k_n}(x_1',\ldots,x_n'),$$

it then follows from (A.3) that $\tilde{\Gamma}_{2\ldots2}(x_1,\ldots,x_n)=0$, i.e., the vertices for which all the indices are equal to 2, are equal to zero in the r representation. This property can be proved by perturbation theory. It is very important in the solution of problems with large occupation numbers and in the study of the transition to the classical case. We note that the complete vertices, which have an even number of indices equal to unity in the r representation, are no longer equal to zero in contrast to the "bare" vertices. All the vertices are symmetric relative to the one-dimensional permutation of indices and arguments.

The author thanks I. B. Levinson for fruitful discussions.

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Translated by R. T. Beyer