Effect of target recoil and excitation in bremsstrahlung and pair production processes

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The formulas for the bremsstrahlung and pair production processes in lepton-nuclear collisions are extended to the case when these processes are accompanied by arbitrary excitation of the nucleus (or of any other target with a structure). Exact expressions are obtained, and approximations valid only at relativistic energies are made. It is shown by way of example that the nuclear electromagnetic form-factor is taken into account, the cross section for bremsstrahlung from a muon in a nuclear field decreases by 10–15%.

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I. INTRODUCTION

It has become necessary of late to carry out more accurate calculations of the cross sections of a number of electromagnetic processes, primarily the cross sections for the production of pairs of charged particles when electrons or muons interact with atoms, as well as the bremsstrahlung cross sections. This is caused mainly by two circumstances. First, certain experimental data[1–3] on the interaction of high-energy muons with nuclei in cosmic rays point to the possible existence of additional mechanisms whereby muons lose energy (assuming that the known loss mechanisms have been accounted for with sufficient accuracy). Second, a number of experiments are presently performed and planned for the purpose of verifying the validity of quantum electrodynamics and finding new particles (in particular, heavy leptons) in bremsstrahlung and electromagnetic pair-production processes.[4,5]

The hitherto calculated bremsstrahlung and pair-production cross sections pertain either to the case of a Coulomb field, or to interaction with a free electron (see the review[6]). The form factor of the nucleus is taken into account approximately in the bremsstrahlung cross section,[7,8] but the method by which this account is taken, and accordingly the quantitative results, differ strongly in different papers. Recognizing that the correction that must be introduced into the muon bremsstrahlung cross section because the nucleus is not pointlike can be relatively large, this question calls for a final solution. In addition to the fact that the nucleus is not pointlike, the cross section can also be influenced by other possible target-excitation channels, for example breakup of the nucleus, pion production, etc. (it is assumed, of course, that only leptons and γ rays from the lepton parts of the diagrams of the process are registered in the experiment).

The known formulas for the bremsstrahlung and pair-production cross sections are generalized in the present paper to include the case of an arbitrary interaction with the target. This is attained by describing the target by means of electromagnetic structure functions W_1 and W_2 that depend on the momentun and energy transferred to the target. The formulas derived are universal in the sense that they do not depend on the target mass; all that matters is that there exists for the target a laboratory system (l. s.) in which the target as a whole is at rest. It is in this system that the cross section is calculated. The motion and interaction of the component parts of a complex target (electrons in an atom, etc.) is already accounted for in the dependence of W_1 and W_2 on q^2 and ν (see the following text for the notation), the determination of which is a different problem and is not considered in the present paper (except for particular cases). This approach was proposed in[9], where the differential bremsstrahlung cross section was obtained for the first time in terms of W_1 and W_2 (without integration over phase space). In the present paper we obtain, besides the bremsstrahlung cross section, also the pair-production cross section, and the integration is carried out over those variables with respect to which it can be performed exactly. A transition to the relativistic approximation is made. We consider the particular case where the target is a screened nucleus that interacts elastically, and show that allowance for the nuclear form factor decreases the muon bremsstrahlung cross section by 10–15%.

To conclude this section, we note that we use in the main a covariant-integration technique[6,10,11] which is well known in applications to problems of this type. Therefore, some methodological aspects are either treated very briefly or omitted completely. They can be found in the cited references, particularly in the book of Baĭer, Katkov, and Fadin.[6]

II. PARTICLE PAIR PRODUCTION

1. The process is characterized by two block diagrams (Fig. 1). We consider first the diagram I. Let particles with momenta p_1 and p_2 have a mass μ (muons) and let an e^+e^- pair be produced. The cross section is of the form

$$d\sigma_I=\frac{e^6}{(2\pi)^5}\frac{E}{2[(p_1P)^2-p_1^2P^2]^{1/2}}\frac{1}{q^4k^4}\frac{1}{2}M_{\alpha\beta}{}^{\mu\nu}K_{\alpha\beta}W_{\mu\nu}\frac{d^3p_+}{2\varepsilon_+}\frac{d^3p_-}{2\varepsilon_-}\frac{d^3p_2}{2\varepsilon_2},\quad(1)$$

where $W_{\mu\nu}$ is the hadron tensor describing the lower vertex[9]:

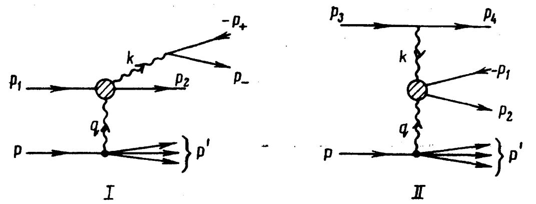

FIG. 1.

$$W_{\mu\nu}=\frac{Z^2e^2}{(2\pi)^3}\left\{W_2\frac{1}{M^2}\left(P-\frac{Pq}{q^2}q\right)_\mu\left(P-\frac{Pq}{q^2}q\right)_\nu+W_1\left(\delta_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)\right\}\frac{M}{E}, \tag{2}$$

$M^{\mu\nu}_{\alpha\beta}$ is the Compton tensor:

$$M_{\alpha\beta}{}^{\mu\nu}=\mathrm{Sp}\left\{\left(\gamma_\alpha\frac{i(\hat{q}+\hat{p}_1)-\mu}{\mu^2\varkappa_1}\gamma_\mu+\gamma_\mu\frac{i(\hat{p}_2-\hat{q})-\mu}{\mu^2\varkappa_2}\gamma_\alpha\right)(i\hat{p}_1-\mu)\right.$$
$$\left.\times\left(\gamma_\beta\frac{i(\hat{p}_2-\hat{q})-\mu}{\mu^2\varkappa_2}\gamma_\nu+\gamma_\nu\frac{i(\hat{q}+\hat{p}_1)-\mu}{\mu^2\varkappa_1}\gamma_\beta\right)(i\hat{p}_2-\mu)\right\}, \tag{3}$$

$$\mu^2\varkappa_1=2kp_2+k^2,\quad \mu^2\varkappa_2=-2kp_1+k^2 \tag{4}$$

and finally, the tensor $K_{\alpha\beta}$ describes pair production:

$$K_{\alpha\beta}=\mathrm{Sp}\,\{\gamma_\alpha(i\hat{p}_++m)\gamma_\beta(i\hat{p}_--m)\}. \tag{5}$$

Expression (1) can be rewritten in the following form, by introducing two additional δ-functions[6]:

$$d\sigma_1=\frac{e^6}{(2\pi)^5}\frac{E}{2[(p_1P)^2-p_1^2P^2]^{1/2}}\frac{1}{q^4k^4}\frac{1}{2}T_{\alpha\beta}M_{\alpha\beta}{}^{\mu\nu}W_{\mu\nu}\frac{d^3p_2}{2\varepsilon_2}\frac{d^3k}{2\omega}dk^2\,d\varepsilon_+{}^L, \tag{6}$$

$$T_{\alpha\beta}=\int K_{\alpha\beta}\delta\left(\varepsilon_+{}^L+\frac{Pp_+}{M}\right)\delta(k-p_+-p_-)\frac{d^3p_+}{2\varepsilon_+}\frac{d^3p_-}{2\varepsilon_-}, \tag{7}$$

where ε_+^L is the positron energy in the l. s. The use of gauge invariance leads to the following expression for the contraction of the tensors in (6):

$$T_{\alpha\beta}M_{\alpha\beta}W_{\mu\nu}=\frac{Z^2e^2}{(2\pi)^3}(T_1W_1M_{\alpha\alpha}+T_2W_2M_{44}^{44}-T_2W_1M_{44}^{\mu\mu}-W_2T_1M_{\alpha\alpha}^{44}), \tag{8}$$

$$T_1=\frac{1}{2}T_{\alpha\beta}\delta_{\alpha\beta}-\frac{1}{2\alpha}T_{\alpha\beta}P_\alpha P_\beta,\quad T_2=-\frac{M^2}{2\alpha}T_{\alpha\beta}\delta_{\alpha\beta}+\frac{3M^2}{2\alpha^2}T_{\alpha\beta}P_\alpha P_\beta, \tag{9}$$
$$\alpha=P^2-\frac{1}{k^2}(kP)^2.$$

In the contraction (8) (but not in the expressions (7) and (9)) we have changed over to the l. s., in which the subsequent calculation of the cross section will be made. The calculation of T_1 and T_2 yields ($k^*=(k^2+\omega^2)^{1/2}$)

$$T_1=\frac{2\pi}{k^*}\left\{\frac{k^2}{2}-m^2-\frac{k^2}{k^{*2}}\left(\varepsilon_+\varepsilon_-+\frac{k^2}{4}\right)\right\},$$
$$T_2=\frac{2\pi}{k^{*3}}k^2\left\{\frac{k^2}{2}-m^2-3\frac{k^2}{k^{*2}}\left(\varepsilon_+\varepsilon_-+\frac{k^2}{4}\right)\right\}. \tag{10}$$

To calculate the cross section it is necessary to know also the components and the contractions of the Compton tensor which enter into (8). Calculating the traces, we obtain ($\nu=q_0$)

$$\frac{1}{8}M_{44}^{44}=\left\{\frac{1}{\mu^4\varkappa_2^2}\left[\frac{q^2k^2}{4}-(\varepsilon_1-\omega)(q^2\varepsilon_1+k^2\varepsilon_2)+4\varepsilon_1\varepsilon_2(\varepsilon_1-\omega)^2\right]\right.$$
$$+\frac{2}{\mu^2\varkappa_2}(\varepsilon_1+\varepsilon_2)(\varepsilon_1-\omega)-\frac{1}{4}\frac{\varkappa_2}{\varkappa_1}+\frac{1}{2\mu^4\varkappa_1\varkappa_2}\left[-\frac{q^2k^2}{2}-q^2(\varepsilon_1{}^2+\varepsilon_2{}^2+\nu\varepsilon_1-\nu\varepsilon_2)\right.$$
$$\left.\left.+k^2(\nu\varepsilon_1-\nu\varepsilon_2-2\varepsilon_1\varepsilon_2)+8\varepsilon_1\varepsilon_2(\varepsilon_1\varepsilon_2-\omega\nu)\right]+\frac{1}{4}\right\}+\left\{\begin{matrix}\varepsilon_1\leftrightarrow-\varepsilon_2\\ \varkappa_1\leftrightarrow\varkappa_2\end{matrix}\right\}$$
$$\frac{1}{8}M_{44}^{\mu\mu}=\left\{\frac{1}{\mu^4\varkappa_2^2}\left[4\mu^2\varepsilon_1(\varepsilon_1-\omega)-\mu^2k^2+\frac{q^2k^2}{2}-2q^2\varepsilon_1(\varepsilon_2-\nu)\right]\right. \tag{11}$$
$$+\frac{2}{\mu^2\varkappa_2}\left(\frac{k^2}{2}-\omega\varepsilon_2\right)-\frac{1}{2}\frac{\varkappa_2}{\varkappa_1}-\frac{1}{\mu^4\varkappa_1\varkappa_2}\left[\frac{k^4}{2}+\mu^2k^2+\frac{k^2q^2}{2}\right.$$
$$\left.\left.-k^2(2\varepsilon_1\varepsilon_2-\omega\nu+\nu^2)+2\mu^2(\omega\nu-2\varepsilon_1\varepsilon_2)+2\varepsilon_1\varepsilon_2q^2+q^2\omega(\varepsilon_1-\varepsilon_2)\right]\right\}+\left\{\begin{matrix}\varepsilon_1\leftrightarrow-\varepsilon_2\\ \varkappa_1\leftrightarrow\varkappa_2\end{matrix}\right\}$$

$$\frac{1}{8}M_{\alpha\alpha}^{\mu\mu}=\left\{\frac{1}{\mu^4\varkappa_2^2}(2\mu^2-q^2)(2\mu^2-k^2)-\frac{2}{\mu^2\varkappa_2}(2\mu^2-k^2-q^2)\right.$$
$$\left.-\frac{\varkappa_2}{\varkappa_1}+\frac{1}{\mu^4\varkappa_1\varkappa_2}[4\mu^4-(k^2+q^2)^2]\right\}+\{\varkappa_1\leftrightarrow\varkappa_2\}.$$

The expression for $M^{44}_{\alpha\alpha}$ is obtained from $M^{\mu\mu}_{44}$ by making the substitutions $q^2\leftrightarrow k^2$, $\omega-\nu$, $\leftrightarrow\varkappa_1\leftrightarrow\varkappa_2$.

We now make in (6) the following change of variables:

$$\frac{d^3k}{2\omega}\frac{d^3p_2}{2\varepsilon_2}\to\frac{\pi\mu^4}{4p_1\sqrt{\Delta}}d\omega\,d\nu\,dq^2\,d\varkappa_2\,d\varkappa_1, \tag{12}$$

$$\begin{aligned}\Delta=&-\mu^4\varkappa_1{}^2(p_1{}^2+\omega^2-2\omega\varepsilon_1)-\mu^4\varkappa_2{}^2(p_1{}^2+\nu^2+2\nu\varepsilon_1)+\mu^2\varkappa_1[4\mu^2\omega\nu\\&+2\omega q^2(\omega-\varepsilon_1)+k^2(q^2+2\omega\nu-2\varepsilon_1\nu)]+\mu^2\varkappa_2[4\mu^2\omega\nu+2\omega q^2(\nu+\varepsilon_1)\\&+k^2(q^2+2\nu^2+2\nu\varepsilon_1)]+\mu^4\varkappa_1\varkappa_2(q^2-2p_1{}^2+2\omega\varepsilon_1-2\nu(\varepsilon_1+\omega)+k^2)\\&-\mu^6\varkappa_1\varkappa_2(\varkappa_1+\varkappa_2)-q^2\omega^2(q^2+4\mu^2)+k^2(-q^4-\nu^2k^2+4p_1{}^2q^2-k^2q^2\\&-4\omega\varepsilon_1q^2-2\omega\nu q^2+4\nu\varepsilon_1q^2-4\nu^2\mu^2)\equiv c\varkappa_1{}^2+b\varkappa_1+a.\end{aligned} \tag{13}$$

Taking (6), (8) and (12) into account, we obtain the following expression for the cross section in the l. s.:

$$\frac{d\sigma}{d\omega d\varepsilon_+}=\frac{Z^2\alpha^4\mu^4}{16\pi^3p_1{}^2}\int\frac{1}{k^4q^4}(T_1W_1\mathscr{M}_{\alpha\alpha}^{\mu\mu}+T_2W_2\mathscr{M}_{44}^{44}$$
$$-T_2W_1\mathscr{M}_{44}^{\mu\mu}-T_1W_2\mathscr{M}_{\alpha\alpha}^{44})\,dk^2\,d\nu\,dq^2, \tag{14}$$
$$\mathscr{M}_{\alpha\beta}^{\mu\nu}=\int M_{\alpha\beta}^{\mu\nu}\frac{d\varkappa_2\,d\varkappa_1}{\sqrt{\Delta}}. \tag{15}$$

2. We now discuss the question of the integration limits. We integrate in the following sequence: with respect to $\varkappa_1$, and then with respect to $\varkappa_2$, q^2, ν, and k^2. The limits with respect to $\varkappa_1$ are determined by the zeroes of the function Δ.[6] We determine the limits with respect to $\varkappa_2$ and q^2 by starting from two requirements: in the l. s., the vector **k** is directed along $\mathbf{p}_1$ while the vector $\mathbf{p}_2$ is directed along $\mathbf{P}'$. This leads respectively to the following equations ($q^*=(q^2+\nu^2)^{1/2}$):

$$\varkappa_2^{(1),(2)}=\frac{2}{\mu^2}(\omega\varepsilon_1\mp k^*p_1)+\frac{k^2}{\mu^2},$$
$$\varkappa_2{}^\pm=\frac{2}{\mu^2}\left(\frac{q^2}{2}+\varepsilon_2\nu\pm q^*p_2\right). \tag{16}$$

The integration region specified by these equations is shown in Fig. 2. The points of intersection of the straight lines $\varkappa_2^{(1),(2)}$ and the curves $\varkappa_2^\pm$ can be easily obtained from (16):

$$q_{1,3}^2=(p_1-k^*\mp p_2)^2-\nu^2,\quad q_{2,4}^2=(p_1+k^*\mp p_2)^2-\nu^2. \tag{17}$$

Figure 2 shows the case when $q_2<q_3$, i.e., $k^*<p_2$. If, however, $k^*>p_2$, then $q_2>q_3$ and the integration region is deformed in obvious fashion.

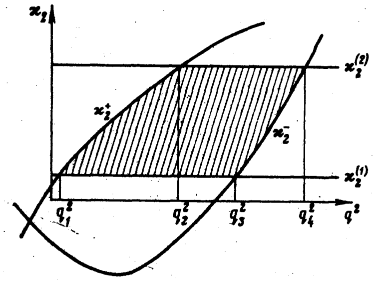

FIG. 2.

If the interaction with the target is elastic, it is necessary to put $\nu = -q^2/2M$ in (16). We then obtain

$$\varkappa_2^{\pm} = \frac{2}{\mu^2}\left[\frac{q^2}{2} - \frac{q^2}{2M}\left(\varepsilon_1 - \omega - \frac{q^2}{2M}\right) \pm \left\{\left(q^2 + \frac{q^4}{4M^2}\right)\left[\left(\varepsilon_1 - \omega - \frac{q^2}{2M}\right)^2 - \mu^2\right]\right\}^{1/2}\right]. \tag{18}$$

The $\varkappa_2^{\pm}$ curves converge at $q_c^2 = 2M(\varepsilon_1 - \omega - \mu)$, with

$$\varkappa_2^+(q_c^2) = \varkappa_2^-(q_c^2) = \frac{2}{\mu^2}(M-\mu)(\varepsilon_1 - \omega - \mu). \tag{19}$$

The maximum value of $\varkappa_2^+$ is given by the expression

$$\varkappa_{2\,\max}^+ = \frac{2M}{\mu^2}(\varepsilon_1 - \omega - \mu). \tag{20}$$

The form of the region of integration with respect to $\varkappa_2$ and q^2 depends on the actual values of ω, k^2, and M and is determined from Eqs. (16) and (18)–(20). Figure 3 shows schematically the region for the case $\varkappa_2^{(1)} < \varkappa_2^{\pm}(q_c^2) < \varkappa_2^{(2)}$ and $\varkappa_{2\max}^+ > \varkappa_2^{(2)}$. The case $M = m < \mu$ is considered in[11]. In that case $\varkappa_2^{\pm}(q_c^2) < 0$, i.e., the point where $\varkappa_2^+$ and $\varkappa_2^-$ converge leaves the integration region. This means that at $M < \mu$ the vector $\mathbf{p}_2$ cannot be antiparallel to $\mathbf{P}'$.

The limiting values of q^2 in the elastic case are easiest to determine directly from the conservation laws, since the configuration of the momenta is simplest at the limiting points—all the particles move along one straight line. In particular, if $\mathbf{k}$ is parallel to $\mathbf{p}_1$, then, by solving this sum of equations

$$\begin{aligned} \varepsilon_1 + M &= \varepsilon_2 + \omega + E', \\ p_1 &= p_2 + k^* + P', \end{aligned} \tag{21}$$

we obtain

$$\varepsilon_2 = (2\gamma + \mu^2 + M^2)^{-1} \times \{(\varepsilon_1 + M - \omega)(\gamma + \mu^2) \pm (p_1 - k^*)(\gamma^2 - \mu^2 M^2)^{1/2}\}, \tag{22}$$

$$\gamma = -{}^1/_2 k^2 + M(\varepsilon_1 - \omega) - \omega\varepsilon_1 + p_1 k^*, \tag{23}$$

After which the values of $q_{1,4}^2$ are obtained from the relation

$$q_{1,4}^2 = 2M(\varepsilon_1 - \varepsilon_2 - \omega) = -2M\nu.$$

The values of $q_{2,3}^2$ are obtained from the same formulas (22) and (23), in which it is necessary to make the substitution $k^* \to -k^*$. There is a possibility of obtaining unphysical (complex) solutions corresponding to the case when the straight line $\varkappa_2^{(2)}$ lies higher than the curve $\varkappa_2^+$. Equating the radicand in (22) to zero, we obtain the corresponding boundary equation with respect to k^2 and ω:

$$-{}^1/_2 k^2 + M(\varepsilon_1 - \omega) - \omega\varepsilon_1 - k^* p_1 = M\mu. \tag{24}$$

One more boundary equation for k^2 and ω is obtained from the condition $q_1^2 \geqslant q_4^2$, or $\varkappa_{2\max}^+ \geqslant \varkappa_2^{(1)}$:

$$-{}^1/_2 k^2 + M(\varepsilon_1 - \omega) - \omega\varepsilon_1 + k^* p_1 = M\mu. \tag{25}$$

Equations (16), (18)–(20), and (22)–(25) suffice for a complete determination of the region of integration with respect to $\varkappa_2$ and q^2 in the case of elastic interaction with the target.

Let us return to the general case. The kinematic limits of integration with respect to ν and k^2 are obviously

$$-(\varepsilon_1 - \omega - \mu) < \nu < 0, \tag{26}$$

$$(p_+ - p_-)^2 - (\varepsilon_+ + \varepsilon_-)^2 < k^2 < (p_+ + p_-)^2 - (\varepsilon_+ + \varepsilon_-)^2. \tag{27}$$

It is now necessary to integrate with respect to $\varkappa_1$ and $\varkappa_2$. Integration with respect to $\varkappa_1$ between the zeroes of Δ gives rise to the integrals

$$\int \frac{d\varkappa_1}{\sqrt{\Delta}} = \frac{\pi}{\sqrt{-c}}, \quad \int \frac{d\varkappa_1}{\varkappa_1\sqrt{\Delta}} = -\frac{\pi}{\sqrt{-a}}, \tag{28}$$

where $a(\varkappa_2)$ and $c(\varkappa_2)$ are defined by Eq. (13). These two integrals suffice because the symmetries of expressions (11) and of the regions of integration with respect to substitutions $\varkappa_1 \leftrightarrow \varkappa_2$, and $p_1 \leftrightarrow -p_2$ allow us to confine ourselves to integration of only the terms written out in (11). The remainder is obtained by making the substitution $p_1 \leftrightarrow -p_2$. It follows from (11) and (28) that the problem of integrating with respect to $\varkappa_2$ reduces to the evaluation of five integrals:

$$J_1 = \pi\int\frac{d\varkappa_2}{\varkappa_2^2\sqrt{-c}}, \quad J_2 = -\pi\int\frac{d\varkappa_2}{\varkappa_2\sqrt{-a}}, \quad J_3 = -\pi\int\frac{\varkappa_2\, d\varkappa_2}{\sqrt{-a}},$$
$$J_4 = \pi\int\frac{d\varkappa_2}{\varkappa_2\sqrt{-c}}, \quad J_5 = \pi\int\frac{d\varkappa_2}{\sqrt{-c}}. \tag{29}$$

Since $\sqrt{-c}$ and $\sqrt{-a}$ are simple functions of $\varkappa_2$, all these integrals can be easily evaluated. The result is expressed in terms of $\varkappa_2$, $\sqrt{-a}$, and $\sqrt{-c}$ at the limits of the integration region

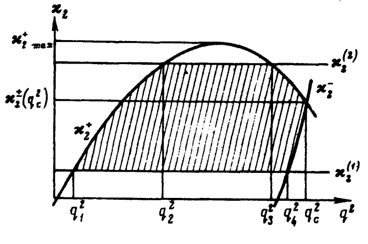

FIG. 3.

$$\{-a(\varkappa_2^{\pm})\}^{1/2}=2[(q^2+\nu^2)^{1/2}(p_2^2+\omega\varepsilon_2-k^2/2)\pm p_2(\nu^2+\nu\varepsilon_1+q^2/2)], \quad (30)$$

$$\{-a(\varkappa_2^{(1),(2)})\}^{1/2}=2[p_1(\varepsilon_1\omega+\omega\nu+k^2/2)\pm(-p_1^2-\varepsilon_1\nu+q^2/2)k^*], \quad (31)$$

$$\{-c(\varkappa_2^{\pm})\}^{1/2}=\mu^2[(q^2+\nu^2)^{1/2}\pm p_2], \quad [-c(\varkappa_2^{(1),(2)})]^{1/2}=\mu^2(p_1\mp k^*). \quad (32)$$

3. So far, the analysis was exact. We proceed now to the relativistic approximation. We assume that $\varepsilon_{1,2}^2 \gg \mu^2$; $\varepsilon_\pm^2 \gg m^2$; k^2, q^2, $\nu^2 \ll \varepsilon_1^2$, and confine ourselves to the case $\nu^2 \ll \omega^2$. Finally, we consider in this article only the region $\omega^2 \gg \mu^2$ (actually it is sufficient to have ω exceed μ by several times). In this region of ω, as follows from (17) and (23), we have $q_2^2 \gg \mu^2$, so that we shall use the following simple limits for the integration with respect to q^2 and $\varkappa_2$:

$$q_1^2 \leqslant q^2 < \infty, \quad \varkappa_2^{(1)} < \varkappa_2 < \varkappa_2^{+}. \quad (33)$$

Because of this, all the subsequent expressions can be used both for the general case and for the elastic case (in the latter case it is necessary only to replace ν by $-q^2/2M$ everywhere and to leave out the integration with respect to ν).

For the integrals (29) we obtain in the relativistic approximation the following expressions:

$$J_1=\frac{\pi}{2\varepsilon_2^2\delta\Gamma}\left(1-\frac{1}{y}\right)-\frac{\pi}{2\varepsilon_2^3}\ln y,$$

$$J_2=-\frac{\pi}{\mu^2\omega\Gamma}\frac{1}{4[\zeta(1+\zeta)]^{1/2}}\ln\left\{\left[1+4\zeta+4(\zeta(1+\zeta))^{1/2}-\left(\frac{\zeta}{1+\zeta}\right)^{1/2}\right]\right.$$
$$\left.\times\left[\left(1-\frac{8\zeta}{y}+\frac{16\zeta(1+\zeta)}{y^2}\right)^{1/2}+4\frac{[\zeta(1+\zeta)]^{1/2}}{y}-\left(\frac{\zeta}{1+\zeta}\right)^{1/2}\right]^{-1}\right\}\equiv-\frac{\pi}{\mu^2\omega\Gamma}$$
$$\ln x, \quad J_3=-\frac{\pi\omega\Gamma}{\mu^2\varepsilon_1^2}\{[y^2-8\zeta y+16\zeta(1+\zeta)]^{1/2}-1-4\zeta\}-\frac{\pi\omega q^2}{\mu^4\varepsilon_1^2}\ln y\equiv-\frac{\pi}{\varepsilon_1^2}J_3',$$

$$J_4=\frac{\pi}{\mu^2\varepsilon_2}\ln y, \quad J_5=2\pi\frac{\delta\Gamma}{\mu^4}(y-1). \quad (34)$$

We have used here the notation

$$\delta=\frac{\mu^2\omega}{2\varepsilon_1\varepsilon_2}, \quad \Gamma=1-\frac{k^2\varepsilon_1\varepsilon_2}{\mu^2\omega^2},$$
$$y=\frac{1}{\delta\Gamma}[(q^2+\nu^2)^{1/2}+\nu], \quad \zeta=\frac{q^2}{4\mu^2\Gamma}. \quad (35)$$

Substituting these expressions in (11) and (15) we obtain, retaining only the highest-order terms,

$$\frac{1}{8}\mathscr{M}_{44}{}^{44}=\frac{8\pi}{\mu^6\omega\Gamma}\varepsilon_1^2\varepsilon_2^2\left(1-\frac{1}{y}-\frac{\delta^2\Gamma}{\mu^2}\ln y-\ln x\right),$$

$$\frac{1}{8}\mathscr{M}_{\alpha\alpha}{}^{44}=\frac{4\pi}{\mu^4\omega\Gamma}\varepsilon_1\varepsilon_2\left(2-\frac{k^2}{\mu^2}\right)\left(1-\frac{1}{y}-\frac{\delta^2\Gamma}{\mu^2}\ln y-\ln x\right)$$
$$+\pi(\varepsilon_1^2+\varepsilon_2^2)\left\{\frac{J_3'}{2\varepsilon_1^2\varepsilon_2^2}+\frac{2q^2}{\pi\mu^4}J_2-\frac{4\delta\nu}{\mu^6\omega}\ln y\right\},$$

$$\frac{1}{8}\mathscr{M}_{44}{}^{\mu\mu}=\frac{\pi}{\mu^2}\left(2-\frac{q^2}{\mu^2}\right)\varepsilon_1\varepsilon_2\left[\left(1-\frac{1}{y}\right)\frac{1}{\delta\Gamma}\left(\frac{1}{\varepsilon_1^2}+\frac{1}{\varepsilon_2^2}\right)+\frac{4}{\pi}J_2\right] \quad (36)$$
$$-\frac{2}{\mu^4}(q^2\omega^2-2\varepsilon_1\varepsilon_2k^2)J_2+\frac{4\pi}{\mu^6}\delta(\varepsilon_1^2+\varepsilon_2^2)\ln y,$$

$$\frac{1}{8}\mathscr{M}_{\alpha\alpha}{}^{\mu\mu}=\frac{\pi}{2\mu^4}(2\mu^2-k^2)(2\mu^2-q^2)\frac{1}{\delta\Gamma}\left(1-\frac{1}{y}\right)\left(\frac{1}{\varepsilon_1^2}+\frac{1}{\varepsilon_2^2}\right)$$
$$+\frac{2}{\mu^4}[4\mu^4-(q^2+k^2)^2]J_2+\pi\left(\frac{1}{\varepsilon_1^2}+\frac{1}{\varepsilon_2^2}\right)J_3'-\frac{4\pi\delta}{\mu^6}(2\mu^2-k^2-q^2)\ln y.$$

The limits of integration with respect to k^2 and q^2 in this approximation are

$$-4\varepsilon_+\varepsilon_- < k^2 < -m^2\frac{\omega^2}{\varepsilon_+\varepsilon_-}, \quad (\delta\Gamma)^2-2\nu\delta\Gamma < q^2 < \infty, \quad (37)$$

and since in our case $\omega > \mu$, we can neglect k^2 everywhere in comparison with ω^2 (it follows from the form of the cross section that the main contribution is made by the k^2 near the upper limit of integration).

4. We proceed now to calculation of the calculation made to the cross section by diagram II. We write down in analogy with (1)

$$d\sigma_{II}=\frac{e^6}{(2\pi)^5}\frac{E}{2[(p_3P)^2-p_3^2P^2]^{1/2}}\frac{1}{q^4k^4}\frac{1}{2}L_{\alpha\beta}M_{1\alpha\beta}^{\mu\nu}W_{\mu\nu}\frac{d^3p_4}{2\varepsilon_4}\frac{d^3p_1}{2\varepsilon_1}\frac{d^3p_2}{2\varepsilon_2} \quad (38)$$

(see Fig. 1). It was expedient here to change the notation somewhat. The particles with momenta p_1 and p_2 are now the (e^+e^-) pair, and the particles with momenta p_3 and p_4 are muons. In formula (38), $M_{1\alpha\beta}^{\mu\nu}$ is a Compton tensor that differs from (3) only in the substitutions $p_{1\mu} \to -p_{1\mu}$, $k_\mu \to -k_\mu$, $\mu \to m$ and in the common sign;

$$L_{\alpha\beta}=\mathrm{Sp}\,\gamma_\alpha(ip_3-m)\gamma_\beta(ip_4-m); \quad (39)$$

The tensor $W_{\mu\nu}$ has the same meaning as before.

We now make the change of variable

$$\frac{d^3p_1}{2\varepsilon_1}\frac{d^3p_2}{2\varepsilon_2}\to\pi\frac{m^4}{4k\sqrt{\Delta_1}}d\varepsilon_1\,d\nu\,dq^2\,d\varkappa_1'\,d\varkappa_2', \quad \frac{d^3p_4}{2\varepsilon_4}\to\frac{\pi}{2p_3}d\omega\,dk^2, \quad (40)$$

where $\varkappa_{1,2}'$, Δ_1 differ from $\varkappa_{1,2}$, Δ in the substitutions $k_\mu \to -k_\mu$, $p_{1\mu} \to -p_{1\mu}$, $\mu \to m$. From a comparison of (40) and (12) we obtain the following substitution rules:

$$\mathscr{M}_{1\alpha\beta}^{\mu\nu}=\int\frac{d\varkappa_1'\,d\varkappa_2'}{\sqrt{\Delta_1}}M_{1\alpha\beta}^{\mu\nu}=-\mathscr{M}_{\alpha\beta}^{\mu\nu}(\varepsilon_1\to-\varepsilon_1,\,\omega\to-\omega,\,\mu\to m), \quad (41)$$

$$\int\mathscr{M}_{1\alpha\beta}^{\mu\nu}W_{\mu\nu}\frac{dq^2}{q^4}=-\int\mathscr{M}_{\alpha\beta}^{\mu\nu}W_{\mu\nu}\frac{dq^2}{q^4}(\varepsilon_1\to-\varepsilon_1,\,\omega\to-\omega,\,\mu\to m). \quad (42)$$

Substitutions similar to those given above lead to the result

$$\frac{d\sigma_{II}}{d\omega\,d\varepsilon_1}=\frac{Z^2\alpha^4m^4}{32\pi^2k^*p_3^2}\int\frac{1}{k^4q^4}\{W_1L_1\mathscr{M}_{1\alpha\alpha}^{\mu\mu}+W_2L_2\mathscr{M}_{144}^{44}$$
$$-W_2L_1\mathscr{M}_{1\alpha\alpha}^{44}-W_1L_2\mathscr{M}_{144}^{\mu\mu}\}dk^2\,d\nu\,dq^2, \quad (43)$$

where

$$L_1=-4\left\{\frac{k^2}{2}-\mu^2-\frac{k^2}{k^{*2}}\left(-\varepsilon_3\varepsilon_4+\frac{k^2}{4}\right)\right\},$$
$$L_2=-4\frac{k^2}{k^{*2}}\left\{\frac{k^2}{2}-\mu^2-3\frac{k^2}{k^{*2}}\left(-\varepsilon_3\varepsilon_4+\frac{k^2}{4}\right)\right\}. \quad (44)$$

It follows from (41) and (42) that to calculate the cross section $d\sigma_{II}$ in the relativistic approximation we can use the earlier formulas (34)–(36) but with the substitutions $\varepsilon_1 \to -\varepsilon_1$, $\omega \to -\omega$, $\mu \to m$. The limits of integration with respect to k^2 in this approximation are

$$\mu^2\omega^2/\varepsilon_3\varepsilon_4 < k^2 < 4\varepsilon_3\varepsilon_4. \quad (45)$$

III. BREMSSTRAHLUNG

1. The diagram of the process is shown in Fig. 4. The cross section is written in the form

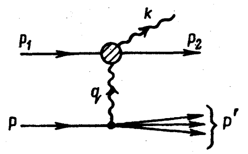

FIG. 4.

$$d\sigma=\frac{e^4}{(2\pi)^2}\frac{E}{2[(p_1P)^2-p_1^2P^2]^{1/2}}\frac{1}{q^4}\frac{1}{2}(-\delta_{\alpha\beta})M^{\mu\nu}_{2\alpha\beta}W_{\mu\nu}\frac{d^3p_2}{2\varepsilon_2}\frac{d^3k}{2\omega},\qquad(46)$$

where $M^{\mu\nu}_{2\alpha\beta}$ is a tensor that differs from (3) only in that now $k^2=0$. Gauge invariance yields

$$-\delta_{\alpha\beta}M^{\mu\nu}_{2\alpha\beta}W_{\mu\nu}=-\frac{Z^2e^2}{(2\pi)^3}(-M^{44}_{2\alpha\alpha}W_2+M^{\mu\mu}_{2\alpha\alpha}W_1).\qquad(47)$$

We change variables in accordance with (12) (with the only difference that $\mu^2\varkappa_1=2kp_2$, $\mu^2\varkappa_2=-2kp_1$) and write down the cross section in the l. s.:

$$d\sigma=\frac{\alpha^3\mu^4Z^2}{8\pi p_1^2q^4}(\mathcal{M}^{44}_{2\alpha\alpha}W_2-\mathcal{M}^{\mu\mu}_{2\alpha\alpha}W_1)\,d\omega\,d\nu\,dq^2.\qquad(48)$$

The tensor $\mathcal{M}_2$ has a meaning analogous to that of the tensor $\mathcal{M}$(15). All the formulas (16)–(25) with $k^2=0$ and $k^*=\omega$ are valid for the integration with respect to $\varkappa_1$ and $\varkappa_2$. In the relativistic approximation we obtain, using relations (36) with $k^2=0$,

$$\begin{aligned}d\sigma=\frac{Z^2\alpha^3\mu^4}{\pi p_1^2q^4}\Big\{W_2\Big[&\frac{8\pi}{\mu^4\omega}\varepsilon_1\varepsilon_2\Big(1-\frac{1}{y_1}-\frac{\delta^2}{\mu^2}\ln y_1-\ln x'\Big)\\&+\pi(\varepsilon_1^2+\varepsilon_2^2)\Big(\frac{\omega}{2\varepsilon_1^2\varepsilon_2^2\mu^2}(y_1-1)-\frac{2q^2}{\mu^6\omega}\ln x'\\&-\frac{4\delta\nu}{\mu^6\omega}\ln y_1\Big)\Big]-\mathcal{M}^{\mu\mu}_{2\alpha\alpha}W_1\Big\}\,d\omega\,d\nu\,dq^2,\end{aligned}\qquad(49)$$

where we have used the notation (see (34) and (35))

$$y_1=\frac{1}{\delta}[(q^2+\nu^2)^{1/2}+\nu],\quad \ln x'=\ln x(\zeta\to\zeta',y\to y_1),\quad \zeta'=\frac{q^2}{4\mu^2}.\qquad(50)$$

Only the zeroth term of the expansion of J_3 (see (34)) in powers of ζ' has been retained in (49), since the factor that contains J_3 in (49) is significant only in the region of small ζ', when the higher-order terms are cancelled out. Retaining only the terms linear in ζ', we obtain

$$\begin{aligned}d\sigma=\frac{Z^2\alpha^3\mu^4}{8\pi p_1^2q^4}\Big\{W_2\frac{16\pi}{\mu^4\omega}\Big[&4\varepsilon_1\varepsilon_2\Big(-\frac{\delta^2}{\mu^2}\ln y_1+\frac{2}{3}\zeta'\Big(1+\frac{3}{y_1^2}-\frac{4}{y_1^3}\Big)\Big)\\+\frac{1}{\mu^2}(\varepsilon_1^2+\varepsilon_2^2)&\Big(\delta^2(y_1-1)-q^2\Big(1-\frac{1}{y_1}\Big)-2\delta\nu\ln y_1\Big)\Big]-\mathcal{M}^{\mu\mu}_{2\alpha\alpha}W_1\Big\}\,d\omega\,d\nu\,dq^2.\end{aligned}\qquad(51)$$

This expression should be used for the numerical calculations in the case of small q^2, and the general formula (49) must be used for $q^2 \gtrsim \mu^2$.

In the particular case $W_2=-\delta(\nu)$, $W_1=0$ (i.e., $\nu=0$), we obtain from (51) the known expression[12]

$$\begin{aligned}d\sigma(q^2\ll\mu^2)=\frac{4Z^2\alpha}{p_1^2\omega}\Big(r_0\frac{m}{\mu}\Big)^2\Big[&(\varepsilon_1^2+\varepsilon_2^2)(\delta-q)^2\\&-\frac{2}{3}\varepsilon_1\varepsilon_2\Big(q^2-\delta^2\ln\frac{q}{\delta}+3\delta^2-4\frac{\delta^3}{q}\Big)\Big]d\omega\frac{dq}{q^3}.\end{aligned}\qquad(52)$$

2. We consider briefly in conclusion the particular case when the target is a zero-spin nucleus screened by electrons and heavy enough to be able to neglect recoil ($\nu\approx0$). We then have (F_n and F_a are the nuclear and atomic form factors)

$$W_2=-[F_n(q^2)-F_a(q^2)]^2\delta(\nu),\quad W_1=0.\qquad(53)$$

If $F_n=1$ then, integrating (49) with respect to q from δ to ∞, we obtain

$$d\sigma\approx4Z^2\alpha\Big(r_0\frac{m}{\mu}\Big)^2\frac{d\omega}{\omega}\Big\{1+\frac{\varepsilon_2^2}{\varepsilon_1^2}-\frac{2}{3}\frac{\varepsilon_2}{\varepsilon_1}\Big\}\Phi(\delta,Z),\qquad(54)$$

$$\Phi(\delta,Z)\approx\int_\delta^\mu(q-\delta)^2[1-F_a(q^2)]^2\frac{dq}{q^3}+1.\qquad(55)$$

To estimate the extent to which the cross section is influenced by the form factor of the nucleus, we use a simple model with a "step-function" nuclear form factor

$$F_n(q^2<q_c^2)=1,\quad F_n(q^2>q_c^2)=0.$$

In such a model we obtain the same formula (54), but with $\Phi(\delta, Z)$ replaced by $\Phi(\delta,Z)-\Delta(q_c)$, where $\Delta(q_c)$ is equal to

$$\Delta(q_c)=\frac{a}{2}\ln\frac{a+1}{a-1}+\frac{1}{2}\ln\frac{a^2-1}{4},\quad a=\Big(1+\frac{4\mu^2}{q_c^2}\Big)^{1/2}.\qquad(56)$$

This correction does not depend on the degree of screening. If we put $q_c=a\mu Z^{-1/3}$, then a comparison of the calculations obtained from (54)–(56) with the results of the more accurate calculations yields $a\approx1.9$ in the region $Z\sim10$–100. By "more exact" calculations we mean here a numerical integration of (49) with the functions (53). At $a=1.9$ the correction Δ decreases the cross section in the total-screening region by 10–15%.

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