Elastic scattering of photons in an intense field and the photoproduction of a pair and a photon

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The mass correction to the amplitude of order α^2 for the elastic scattering of photons in an intense electromagnetic field is found. It is used to find the probability of the photoproduction of a pair and a photon and the mass correction to the photoproduction of a pair.

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1. INTRODUCTION

This article is devoted to an investigation of certain radiative processes of order α^2 in an intense electromagnetic field, i.e., in a field whose intensity is close to the critical value $F\sim F_0=m^2c^3/e\hbar=4.4\times10^{13}$ Oe.[1)] As is well known, nonlinear quantum electrodynamic effects (see, for example, [1-3]) reach their optimal values in such fields and the verification of quantum electrodynamics in the range of field strengths of the order of F_0 constitutes a new aspect of its investigation different from the traditional investigation at small distances.

The field strengths which can be obtained experimentally at the present time are several orders of magnitude smaller than the critical value. However, it is still possible to observe at this time nonlinear effects induced by ultrarelativistic particles with a momentum $p\sim F_0m/F\gg m$. The field strength in the proper frame will be of the order of F_0 for such particles. In addition, independently of the form of the field in the laboratory system, it will be nearly a plane wave field in the proper frame, and if its characteristic wavelength and period are large in comparison with the quantity m/eF which determines the characteristic length and time of the formation of the processes, it can be regarded as a constant crossed field $\mathbf{E}\perp\mathbf{H}$, $E=H\equiv F$. It is precisely such a field which we shall investigate below.

The systematic investigation of effects of order α^2 was initiated by Ritus, who found the polarization[4] and mass (jointly with one of the authors)[5] corrections to the amplitude for the elastic scattering of an electron in a constant crossed field, and also found the probability for electroproduction of a pair, $e\to ee^+e^-$ (also see[6]) and the probability for two-photon emission $e\to e2\gamma$.

The mass correction to the amplitude of the elastic scattering of photons in a constant crossed field is derived in the present article; the imaginary part of this correction contains the nonexchange part of the probability for the photoproduction of a pair and a photon, $\gamma\to e^+e^-\gamma$, and the mass correction to the probability for the photoproduction of a pair, $\gamma\to e^+e^-$. Exchange effects become small in a strong field or at high energies of the incident photons; therefore, in this case the investigated part of the probability for photoproduction of a pair and a photon is the governing factor.

The determined probabilities consist of two terms—incoherent and coherent. The first corresponds to the fact that the regions of formation of the pair production processes and of the subsequent emission of a photon by an electron or positron are separated by a distance much longer than length for the formation of each process separately. The second corresponds to the fact that both processes are being formed in a common region. The incoherent parts of the probability for the photoproduction of a pair and a photon and for the mass correction to the probability for photoproduction of a pair differ in sign, which is natural since the sum of these probabilities, equal to twice the imaginary part of the amplitude of the elastic scattering of photons, must have a common region of formation.

The incoherent part of the probability is proportional to the total observation time T, whereas the coherent part does not depend on T. We must note in this connection that some ambiguity exists in the calculation of the probability component independent of T. This ambiguity is due to the fact that the incoherent part of the probability depends on the shape of the region in which the corresponding process occurs (at the same value of the 4-volume), that is, it essentially depends on the experimental situation. The influence of the shape of the region on the incoherent part leads to the appearance in it of terms that are small in comparison with the terms $\sim T$, but are of the same order as the coherent part. However, these terms, just as the entire incoherent part, are characterized by the presence of two regions of formation, a fact that allows us to distinguish them from the "truly coherent" part, which is uniquely determined.

2. THE MASS CORRECTION TO THE PHOTON ELASTIC-SCATTERING AMPLITUDE

To fourth order in the radiation field the polarization operator is described by the three compact diagrams shown in Fig. 1. The solid lines correspond to electrons interacting with an intense field and the wavy lines correspond to photons. In this article we consider diagrams a and b, which determine the mass correction to the polarization operator whose imaginary part contains in particular the probability for the photoproduction of a pair and a photon in a strong field.

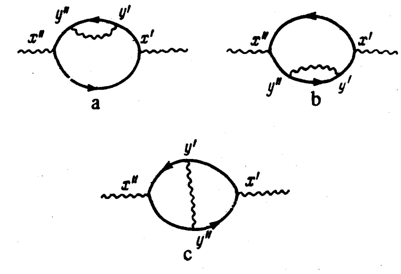

FIG. 1.

In the presence of a crossed field described by the 4-potential

$$A_\mu = a_\mu\varphi,\quad \varphi = kx,\quad k^2 = ak = 0 \tag{1}$$

the mass correction to the polarization operator may be written in the form

$$\pi^{(4)}_{\mu\nu\,\mathrm{mass}}(x'',x') = -ie^2\{\mathrm{Sp}[\gamma_\mu\Delta S^c(x'',x')\gamma_\nu S^c(x',x'')] + \mathrm{Sp}[\gamma_\nu\Delta S^c(x',x'')\gamma_\mu S^c(x'',x')]\}, \tag{2}$$

where

$$S^c(x'',x') = -\frac{i}{(4\pi)^2}e^{i\eta}\int_0^\infty\frac{ds}{s^2}\exp\left\{i\left[\frac{x^2}{4s} - s\left(m^2+\frac{(eFx)^2}{12}\right)\right]\right\} \times\left[m - i\frac{\hat{x}}{2s} + i\frac{se^2\gamma FFx}{3} + i\frac{mse\sigma F}{2} + \frac{e\gamma_5\gamma F^*x}{2}\right] \tag{3}$$

is the causal Green's function in the proper-time representation, [3] and

$$\Delta S^c(x'',x') = -i(x''|S^cMS^c|x') \tag{4}$$

is the mass correction to it, of order α^2, found by Morozov and Ritus. [5] In formulas (3) and (4) η denotes the nondiagonal phase of the Green's function

$$\eta = \frac{e}{2}(ax)(\varphi'+\varphi''),\quad x = x''-x',\quad F^*_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\lambda\sigma}F_{\lambda\sigma},$$

and M denotes the renormalized mass operator of second order in the crossed field. [7,5]

It will be convenient henceforth to use the so-called E_p-representation introduced by Ritus, [7] with basis functions

$$E_p(x) = \left[1+\frac{e\hat{k}\hat{a}}{2kp}\varphi\right]\exp\left\{i\frac{epa}{2pk}\varphi^2 - i\frac{e^2a^2}{6pk}\varphi^3 + ipx\right\}. \tag{5}$$

A remarkable property of the E_p-representation is the fact that in this representation the S^c function reduces to the vacuum function

$$S^c(x'',x') = -i\int\frac{d^4p}{(2\pi)^4}\frac{E_p(x'')(m-i\hat{p})\bar{E}_p(x')}{m^2+p^2-i\varepsilon}, \tag{6}$$

and the mass operator is diagonal

$$M(q,p) = ie^2\int d^4x'\,d^4x''\,\bar{E}_q(x'')\gamma_\mu S^c(x'',x')\gamma_\nu E_p(x')D^c_{\mu\nu}(x''-x') = (2\pi)^4\delta(q-p)M(p;F). \tag{7}$$

For the explicit form of the renormalized function $M_R(p;F)$, see Ref. 7, 8. Here we only note that one can represent it in the form of the integral

$$M_R(p;F) = \int_0^\infty\frac{ds'}{s'^2}\int_0^\infty\frac{dt}{t^2}M_{s't}(p;F), \tag{8}$$

where s' and t denote the proper times of the virtual electron and photon, respectively.

Now let us present the form of $\Delta S^c(x'', x')$ in the E_p-representation

$$\Delta S^c(x'',x') = i\int\frac{d^4f}{(2\pi)^4}\frac{E_f(x'')(m-i\hat{f})M_R(f;F)(m-i\hat{f})\bar{E}_f(x')}{(m^2+f^2-i\varepsilon)^2}. \tag{9}$$

For the proper-time representation of $\Delta S^c(x'', x')$, see[5].

Direct calculations show that the entire nondiagonal part $\Delta S^c(x'', x')$ is incorporated in the same phase factor $e^{i\eta}$ which also appears in the function $S^c(x'', x')$. Therefore $\pi^{(4)}_{\mu\nu\,\mathrm{mass}}$ depends only on the difference between the coordinates

$$\pi^{(4)}_{\mu\nu\,\mathrm{mass}}(x'',x') = \pi^{(4)}_{\mu\nu\,\mathrm{mass}}(x''-x')$$

and it is convenient to change to the momentum representation

$$\pi^{(4)}_{\mu\nu\,\mathrm{mass}}(l;F) = \int d^4x\,e^{-ilx}\pi^{(4)}_{\mu\nu\,\mathrm{mass}}(x). \tag{10}$$

It is convenient to perform the integration in (10) by changing to the variables x_1, x_2, $x_\pm = x_0 \pm x_3$ which are natural for a plane-wave field. The integrals over x_1 and x_2 reduce to Gaussian integrals and the integration over x_+ gives a δ-function $2\pi\delta(x_- - 2l_-s_1s/(s_1+s))$ or its derivative so that the remaining integration over x_- is trivial[2)] (s and s_1 denote the proper times of the "dressed" and "bare" electrons, respectively; see the figure).

The resulting expression has an ultraviolet divergence and needs to be renormalized. The renormalized expression for $\pi^{(4)R}_{\mu\nu\,\mathrm{mass}}(l;\,F)$ has the form

$$\pi^{(4)R}_{\mu\nu\,\mathrm{mass}}(l;F) = \pi^{(4)}_{\mu\nu\,\mathrm{mass}}(l;F) - \pi^{(4)}_{\mu\nu\,\mathrm{mass}}(l;0) + \pi^{(4)R}_{\mu\nu\,\mathrm{mass}}(l;0), \tag{11}$$

where $\pi^{(4)R}_{\mu\nu\,\mathrm{mass}}(l;\,0)$ denotes the renormalized mass correction to the vacuum polarization operator. In what follows we shall omit the superscript R.

The quantity $\pi^{(4)}_{\mu\nu\,\mathrm{mass}}(l;\,F)$ has the following tensor structure:

$$\pi^{(4)}_{\mu\nu\,\mathrm{mass}}(l;F) = \pi^{(4)}_{1\,\mathrm{mass}}\frac{L_\mu L_\nu}{L^2} + \pi^{(4)}_{2\,\mathrm{mass}}\frac{L^*_\mu L^*_\nu}{L^2} + \pi^{(4)}_{3\,\mathrm{mass}}\frac{G_\mu G_\nu}{G^2} + \pi^{(4)}_{4\,\mathrm{mass}}\frac{l_\mu l_\nu}{l^2} \tag{12}$$

(cf. [7,8]), where

$$l_\mu,\quad L_\mu = F_{\mu\nu}l_\nu,\quad L^*_\mu = F^*_{\mu\nu}l_\nu,\quad G_\mu = \frac{l^2}{L^2}F_{\mu\nu}F_{\nu\lambda}l_\lambda + l_\mu \tag{13}$$

are four independent orthogonal vectors and the functions $\pi^{(4)}_{i\,\mathrm{mass}}$ depend on two invariant parameters: l^2 and $\varkappa^2 = (eFl)^2/m^6$.

The presence of a term $\sim l_\mu l_\nu$ in Eq. (12) is associated with the fact that we are considering only a portion of the diagrams which contribute to the polarization operator, and which by themselves are not gauge invariant.

On the mass surface $l^2=0$ the quantity $\pi^{(4)}_{\mu\nu\,\text{mass}}(l;\,F)$ determines the mass correction to the amplitude for the elastic scattering of a photon with a change of polarization $e\to e'$

$$T^{(4)}_{l'e',le;\,\text{mass}}=-\frac{(2\pi)^4\delta(l'-l)}{2l_0}n_\gamma e_\mu{}'\pi^{(4)}_{\mu\nu\,\text{mass}}(l;F)e_\nu$$
$$=-\frac{(2\pi)^4\delta(l'-l)}{2l_0}n_\gamma\sum_{i=1,2}(e'e_i)(ee_i)\pi^{(4)}_{i\,\text{mass}}(0,\varkappa),\tag{14}$$

where the unit polarization vectors e_1 and e_2 are given by

$$e_{1\mu}=L_\mu/L^2,\quad e_{2\mu}=L_\mu^*/L^2\tag{15}$$

and describe in the "special" coordinate system the photons polarized respectively along $\mathbf{E}$ and $-\mathbf{H}$, and n_γ denotes the density of incident photons.

Thus, the amplitude is solely determined by the functions $\pi^{(4)}_{1,2\,\text{mass}}(0,\varkappa)$ which are given by

$$\pi^{(4)}_{1,2\,\text{mass}}(0,\varkappa)=\frac{\alpha^2m^2}{\pi^2}\int_0^\infty\frac{du}{(1+u)^2}\int_0^\infty\frac{dv}{(1+v)^3}\left\{-\left[\frac{2u(1+2v)}{1+u}(\partial-a\partial')-z^{-1}\right]\right.$$
$$\times\left[\partial^{-1}I_{11}+\frac{2+2u^2+(2\mp3)u}{3u}\partial I_1\right]+2\left[v+z\frac{u(1+2v)}{1+u}(\partial-a\partial')\right]I_{11}$$
$$-\frac{4u}{z'}\frac{1+v}{1+u}\left(1+\frac{v}{3}\right)\left[z-(2-u)\partial^{-1}+\frac{1-u}{3}\partial'^3\partial^{-1}\right.$$
$$\left.+\frac{2+2u^2+(2\mp3)u}{3u}\partial^2\right](\partial-a\partial')\partial'\partial^{-1}I$$
$$-\left[2v(u\mp1)+\frac{2u}{3a}\frac{(1+2v)(u-1)}{1+u}\partial'^2\partial^{-2}\right](\partial-a\partial')aI$$
$$\left.+\frac{1}{z}\left[(2+v)(u\mp1)+2u(1+v)\left(1+\frac{v}{3}\right)a\partial'\partial^{-1}+\frac{u-1}{3a}\partial'^2\partial^{-2}\right]aI\right\}.\tag{16}$$

Here I, I_1, and I_{11} denote the three characteristic integrals

$$I=\int_0^1\frac{dx}{a}\frac{f'(\xi)}{(1+x^3/a^3)^{3/2}},\quad I_1=\int_0^\infty\frac{dx}{x}\left[f(\eta)-\frac{\theta(1-x)f(\xi)}{(1+x^3/a^3)^{1/3}}\right],$$
$$I_{11}=\int_0^\infty\frac{dx}{x}[f_1(\eta)-\theta(1-x)f_1(\xi)],\tag{17}$$

determined by the well known special functions[4]

$$f(x)=i\int_0^\infty d\sigma\exp\left[-i\left(x\sigma+\frac{\sigma^3}{3}\right)\right],\quad f_1(x)=\int_x^\infty dx'[f(x')-x'^{-1}];\tag{18}$$

finally

$$\eta=z+z_\lambda'x/a,\quad \xi=\eta(1+x^3/a^3)^{-1/3},$$
$$z=\left[\frac{(1+u)^2}{u\varkappa}\right]^{2/3}\quad z'=\left[\frac{(1+u)v}{\varkappa}\right]^{2/3},\quad z_\lambda'=z'\frac{v_\lambda}{v},$$
$$a=\left(\frac{1+u}{uv}\right)^{1/3},\quad v_\lambda=v+\lambda\frac{1+v}{v},\tag{19}$$
$$\partial=\frac{\partial}{\partial z},\quad \partial'=\frac{\partial}{\partial z_\lambda'},\quad \partial^{-1}=-\int_z^\infty dz.$$

As is evident from Eqs. (19), the amplitude depends on a small fictitious photon mass μ via the parameter $\lambda=(\mu/m)^2$ which was introduced in order to eliminate the infrared divergence of the mass operator (7).

If λ is assumed to be a very small parameter, the infrared divergence can be isolated into a separate term

$$T^{(4)}_{l'e',le;\,\text{mass}}=\frac{\alpha}{\pi}\ln\lambda T^{(2)}_{l'e',le}+A^{(4)}_{l'e',le},\tag{20}$$

in which $T^{(2)}_{l'e',le}$ denotes the second-order amplitude.[7,8] The form of Eq. (20) is quite understandable. In fact, the infrared divergent part of the vacuum mass operator is equal to $(\alpha/2\pi)\ln\lambda(i\hat{p}+m)$ (see, for example, [9]). Therefore, the isolation of terms $\sim\ln\lambda$ in each of the diagrams a and b (see the figure) amounts to the replacement of a "dressed" electron line by a "bare" line with the factor $(\alpha/2\pi)\ln\lambda$, which ultimately leads to Eq. (20). The λ-dependent part of the amplitude $A^{(4)}_{l'e',le}$ is cumbersome and is not presented here.

The variables u and v have a simple physical meaning: v is the ratio of the proper times s' and t of the virtual electron and of the photon of the mass operator, and u is the ratio of the proper times s and s_1 of the "dressed" s and "bare" virtual electrons

$$v=s'/t,\quad u=s/s_1.\tag{21}$$

On the other hand, if f and f' denote the momenta of the "dressed" and "bare" electrons and l' is the momentum of a virtual photon, then

$$v=l_-'/(f_--l_-),\quad u=f_-'/f_-,\tag{22}$$

subject to the conservation law

$$l_-'=f_--f_-'.\tag{23}$$

The variable x is given by

$$x=y_-/x_-,\tag{24}$$

where y_- and x_- denote the durations of the internal and external interactions (see Fig. 1).

The integrals (17) have the following "original" representations:

$$I=\int_0^\infty d\tau\exp\left\{-i\left(z_\lambda'\tau+\frac{\tau^3}{3}\right)\right\}\int_{a\tau}^\infty d\sigma\exp\left\{-i\left(z\sigma+\frac{\sigma^3}{3}\right)\right\},$$
$$I_1=-i\int_0^\infty\frac{d\tau}{\tau}\exp\{-iz_\lambda'\tau\}\left[\left(\exp\left\{-i\frac{\tau^3}{3}\right\}-1\right)\right.$$
$$\left.\times\int_{a\tau}^\infty d\sigma\exp\left\{-i\left(z\sigma+\frac{\sigma^3}{3}\right)\right\}-\int_0^{a\tau}d\sigma\exp\left\{-i\left(z\sigma+\frac{\sigma^3}{3}\right)\right\}\right],$$
$$I_{11}=-\int_0^\infty\frac{d\tau}{\tau}\exp\{-iz_\lambda'\tau\}\left[\left(\exp\left\{-i\frac{\tau^3}{3}\right\}-1\right)\right.\tag{25}$$
$$\left.\times\int_{a\tau}^\infty\frac{d\sigma}{\sigma}\exp\left\{-i\left(z\sigma+\frac{\sigma^3}{3}\right)\right\}-\int_0^{a\tau}\frac{d\sigma}{\sigma}e^{-iz\sigma}\left(\exp\left\{-i\frac{\sigma^3}{3}\right\}-1\right)\right],$$

where the variables σ and τ are related to the previously introduced variables in the following way:

$$\tau=\sigma x/a, \quad \sigma=m^2 s \varkappa^{2/3}/[u(1+u)]^{2/3}. \tag{26}$$

We note that in the representation (17) the vacuum part of the mass operator leads to integrals over x from 1 to ∞, and in the representation (25) it leads to integrals over σ from 0 to $a\tau$.

For $\lambda^{1/2} \ll \varkappa \ll 1$ we have

$$\begin{aligned}\pi^{(4)}_{1,2\,\mathrm{mass}}(0,\varkappa)=&-\frac{\alpha^2m^2}{\pi^2}\left\{\frac{\varkappa^2}{900}\left[\frac{20471}{36}\mp 173+10\ln\lambda(11\mp3)\right]\right.\\&+i\frac{\pi}{8}\left(\frac{3}{2}\right)^{1/2}\varkappa e^{-8/3\varkappa}\left[(3\mp1)\left(-\ln\frac{\varkappa}{2\lambda^{1/2}}+1-C+\frac{5}{2}\ln\frac{2}{3}\right.\right.\\&\left.\left.\left.+\frac{5}{2\cdot3^{1/2}}\ln(2+3^{1/2})\right)+(1\mp1)\right]\right\},\end{aligned} \tag{27}$$

where C is Euler's constant. For $\varkappa \gg 1$ we have

$$\pi^{(4)}_{1,2\,\mathrm{mass}}(0,\varkappa)=-\frac{\alpha^2m^2}{\pi^3}3^{1/2}\frac{5\mp1}{28}(-1+i3^{1/2})\Gamma^4\left(\frac{2}{3}\right)(3\varkappa)^{2/3}\ln(\lambda\varkappa^{1/2}), \tag{28}$$

or

$$T^{(4)}_{l'e',le;\,\mathrm{mass}}=\frac{\alpha}{\pi}\ln(\lambda\varkappa^{1/2})T^{(2)}_{l'e',le}. \tag{29}$$

For $\varkappa \gg 1$ we present only the major term with asymptotic behavior $\sim\varkappa^{2/3}\ln\varkappa$ since the next term $\sim\varkappa^{2/3}$ in the expansion contains a rather cumbersome coefficient which is expressed in terms of the generalized hypergeometric series ${}_3F_2$ with argument equal to unity. For large values of $\varkappa$ the real and imaginary parts of the mass correction to the amplitude turn out to be $\sim\alpha^2\varkappa^{2/3}\times\ln\varkappa$, whereas the amplitude $T^{(2)}\sim\alpha\varkappa^{2/3}$,[7,8] so that the obtained result is valid as long as the condition $\alpha\ln\varkappa\ll1$ is fulfilled. It is natural to regard $\alpha\ln\varkappa$ as the expansion parameter of the polarization operator. We note that the fourth-order mass operator increases more rapidly and has an expansion parameter $\alpha\chi^{1/3}\ln\chi$,[4] where $\chi^2=(eFp)^2/m^6$.

The doubled imaginary part of the amplitude consists of the nonexchange part of the probability for the production of a pair and a photon and the mass correction to the probability for the photoproduction of a pair

$$2\,\mathrm{Im}\,T^{(4)}_{l'e',le;\,\mathrm{mass}}=VT(W^{(4)}_{e^+e^-\gamma\,\mathrm{noex}}+W^{(4)}_{e^+e^-\,\mathrm{mass}}). \tag{30}$$

Sections 3 and 4 are devoted to the derivation of the probability $W^{(4)}_{e^+e^-\gamma\,\mathrm{noex}}$ and the correction $W^{(4)}_{e^+e^-\,\mathrm{mass}}$ from the amplitude $T^{(4)}_{l'e',le;\mathrm{mass}}$.

3. PHOTOPRODUCTION OF A PAIR AND A PHOTON

The nonexchange part of the probability for the photoproduction of a pair and a photon can be represented in the form

$$\begin{aligned}VTW^{(4)}_{e^+e^-\gamma\,\mathrm{noex}}=&\frac{e^2n_\gamma}{2l_0}\int d^4x'\,d^4x''\,e^{il(x'-x'')}e_\mu e_\nu\\&\times\{\mathrm{Sp}[\gamma_\mu\Delta S^{(+)}(x'',x')\gamma_\nu S^{(-)}(x',x'')]+\mathrm{Sp}[\gamma_\nu\Delta S^{(-)}(x',x'')\gamma_\mu S^{(+)}(x'',x')]\},\end{aligned} \tag{31}$$

where $S^{(+)}(S^{(-)})$ denotes the positive-frequency (negative-frequency) Green's function,

$$\Delta S^{(\pm)}(x'',x')=-i\int\frac{d^4f}{(2\pi)^4}\frac{E_f(x'')(m-i\hat f)M^{(\pm)}(f;F)(m-i\hat f)\bar E_f(x')}{(m^2+f^2+i\varepsilon)(m^2+f^2-i\varepsilon)}, \tag{32}$$

$$\begin{aligned}M^{(\pm)}(f,q)=&ie^2\int d^4x'\,d^4x''\,\bar E_f(x')\gamma_\mu S^{(\pm)}(x',x'')\\&\times\gamma_\nu E_q(x'')D^{(\pm)}_{\mu\nu}(x'-x'')=(2\pi)^4\delta(f-q)M^{(\pm)}(f;F).\end{aligned} \tag{33}$$

The proper-time representations for the positive-frequency (negative-frequency) Green's functions differ from expression (3) in that the limits of integration over s are excluded from $-\infty$ to $+\infty$ and by the appearance of the factor

$$\pm\theta(\pm x_-/s)\varepsilon(s).$$

The proper-time representation $M^{(\pm)}(f;F)$ differs from expression (8) by an extension of the limits of integration with respect to s' and t from $-\infty$ to $+\infty$, by the appearance of the factors

$$\pm\theta(\pm f_-)\theta(s'/t)\varepsilon(t)$$

and by discarding the terms arising from renormalization. The latter circumstance is associated with the fact that replacement of the causal functions S^c and D^c in Eq. (7) by positive- or negative-frequency functions actually means the replacement of the mass operator by its doubled imaginary part,[5] which is not subject to renormalization.

The transition from ΔS^c to $\Delta S^{(\pm)}$ contains one more substitution

$$\begin{aligned}(m^2+f^2-i\varepsilon)^2\to&-(m^2+f^2+i\varepsilon)^{-1}(m^2+f^2-i\varepsilon)^{-1}\\&=\left(\frac{P}{m^2+f^2}\right)'-2\pi^2\delta^2(m^2+f^2).\end{aligned} \tag{34}$$

Substitution of the term $(P/(m^2+f^2))'$ into (32) consequently leads to an expression differing from (14) by replacement of the functions I, I_1, I_{11} by the real functions $2R$, $2R_1$, $2R_{11}$ and determining the coherent part of the probability characterized by the fact that the processes of pair production and emission have a common region of formation.

The functions $2R$, $2R_1$, and $2R_{11}$ are obtained if the limits of integration with respect to σ and τ are extended from $-\infty$ to $+\infty$ in the "original" representations (25) of the integrals, I, I_1, and I_{11}, respectively, if the terms arising from renormalization are discarded, and if the factor $(2i)^{-1}\varepsilon(\sigma-a\tau)$ is introduced:

$$R=\frac{1}{4i}\int_{-\infty}^{\infty}d\sigma\int_{-\infty}^{\infty}d\tau\,\varepsilon(\sigma-a\tau)\exp\left\{-i\left[z_\lambda'\tau+\frac{\tau^3}{3}+z\sigma+\frac{\sigma^3}{3}\right]\right\}, \tag{35}$$

$$R_1=-i\partial'^{-1}R, \quad R_{11}=-\partial^{-1}R_1. \tag{36}$$

The "original" representations for the functions R_1 and R_{11} differ from the similar representation for R by the factors $-i(\tau-i\varepsilon)^{-1}$ and $-(\tau-i\varepsilon)^{-1}(\sigma-i\delta)^{-1}$. Going around the points $\sigma=0$ and $\tau=0$ from below corresponds to Pauli-Villars regularization of singular functions.[10] However, the following representations are more convenient

$$R=\int_0^1 \frac{dx}{a}\frac{\Phi'(\xi)}{(1+x^3/a^3)^{2/3}}+\Upsilon(z_\lambda')\Phi(z),$$

$$R_1=\int_0^\infty \frac{dx}{x}\left[\Phi(\eta)-\frac{\theta(1-x)\Phi(\xi)}{(1+x^3/a^3)^{1/3}}\right]+\Upsilon_1(z_\lambda')\Phi(z), \tag{37}$$

$$R_{11}=\int_0^\infty \frac{dx}{x}[\Phi_1(\eta)-\theta(1-x)\Phi_1(\xi)]+\Upsilon_1(z_\lambda')\Phi_1(z),$$

where $\Phi(z)=\operatorname{Im} f(z)$ denotes the Airy function,

$$\Upsilon(z_\lambda')=\operatorname{Re} f(z_\lambda'),\quad \Phi_1(z)=\int_z^\infty dx\,\Phi(x),\quad \Upsilon_1(z_\lambda')=\int_{z_\lambda'}^\infty dx[\Upsilon(x)-x^{-1}],$$

see Ref. 4.

Substitution of the term $-2\pi^2\delta(m^2+f^2)$ from (34) into Eq. (32) leads to the incoherent part of the probability $W^{(4)\,\text{incoh}}_{e^+e^-\gamma}$ which is characterized by separated regions for the formation of the processes associated with pair production and emission. In this connection it is necessary to replace one of the δ-functions by

$$\delta(m^2+f^2=0)=T/4\pi f_0=L_-/4\pi f_-, \tag{38}$$

where T and L_- denote the intervals of variation of the time x_0 and of the coordinate x_-.

It is convenient to carry out the integration over coordinates in (31) according to the same scheme as in (10). In this connection it is necessary to remember that $\delta(0)$ is a function of f_- or of x_- if the proper-time representations are used for the S-functions. Therefore, in those places where $\delta(0)$ is multiplied by a δ'-function (see the text after formula (10)), it must be differentiated as an ordinary function first in the calculation of $\Delta S^{(\pm)}$ and then in the calculation of the probability $W^{(4)\,\text{incoh}}_{e^+e^-\gamma}$.

Those probability terms in which $\delta(0)$ was not differentiated can be obtained from the amplitude with the aid of a simple prescription. Namely, it is necessary to carry out the following substitutions

$$(\partial-a\partial')I\to\tau z'^{1/2}\Phi(z')\Phi(z),\quad (\partial-a\partial')I_1\to\tau z'^{1/2}\Phi_1(z')\Phi(z),$$
$$(\partial-a\partial')I_{11}\to\tau z'^{1/2}\Phi_1(z')\Phi_1(z),$$

where $\tau=eFL_-/m$, and to discard the remaining terms, since in the E_p-representation they contain the product $(m^2+f^2)\,\delta(m^2+f^2)$. No analogs of the terms arising from the differentiation of $\delta(0)$ are contained in the amplitude, and it is necessary to calculate them separately. As a result we obtain an answer which, after several integrations by parts, reduces to the form

$$W^{(4)\,\text{incoh}}_{e^+e^-\gamma\parallel,\perp}=-\tau\frac{\alpha^2m^2n_\gamma}{\pi^2l_0\varkappa}\int_0^\infty\frac{du}{1+u}\int_0^\infty\frac{dv}{(1+v)^2}\left\{\left[\Phi_1(z)-\frac{u^2\mp u+1}{u}\frac{\Phi'(z)}{z}\right]\right.$$
$$\times\left[\frac{1+2v}{1+v}\Phi_1(z')+2\left(1+\frac{v}{3}\right)\frac{\Phi'(z')}{z'}\right]$$
$$\left.-\frac{\varkappa z^{1/2}}{(1+u)(1+v)}av(u\mp1)\Phi(z)\Phi(z')\right\}. \tag{39}$$

The subscripts $\parallel$ and $\perp$ denote photons polarized along **E** and $-$**H**. Since the incoherent part does not contain an infrared divergence, we have assumed $\lambda=0$.

Expression (39) can also be obtained directly (cf. Eq. (15) of[4]) if the probability $n_e'^{-1}W_\gamma^{(2)}(\chi',\zeta)$ for the emission of a polarized electron[11] is integrated over the spectrum $dW^{(2)}_{e^+e^-\parallel,\perp}(\varkappa,u,\zeta)$ of the electrons appearing after pair production during the entire time up to the moment of emission, followed by averaging over the total observation time and summation over the spin of the intermediate electron. In addition it is necessary to multiply this expression by two since the emission of a positron gives the same contribution to the desired probability:

$$W^{(4)\,\text{incoh}}_{e^+e^-\parallel,\perp}=T\sum_\zeta\int_0^\infty du\,\frac{dW^{(2)}_{e^+e^-\parallel,\perp}(\varkappa,u,\zeta)}{du}\,\frac{W_\gamma^{(2)}(\chi',\zeta)}{n_e'},\quad \chi'=\frac{\varkappa}{1+u}. \tag{40}$$

The agreement between expressions (40) and (39) is not only a check on formula (39) but to some degree also a check on the total amplitude.

We note that in the evaluation of the incoherent part of the probability $W^{(4)}_{2\gamma\,\text{incoh}}$ for two-photon emission in[5], the function $\delta(0)=T/4\pi f_0$ mentioned above was erroneously assumed to be constant in the course of the differentiation. This led to the absence of the following terms inside the curly brackets of Eq. (54) in[5]:

$$-\frac{u^2}{2z}\Phi_2(z)\left[(1+2v)\Phi_1(z')+2\left(1+\frac{v}{3}\right)(1+v)\frac{\Phi'(z')}{z'}\right] \tag{41}$$

and to its agreement with the product of the probabilities for one-photon emission integrated over the proper time of the electrons, but not over the laboratory time which would have been natural. Taking account of the omitted terms leads to agreement of $W^{(4)}_{2\gamma\,\text{incoh}}$ with the product of the probabilities for one-photon emission integrated over the laboratory time, and differing from Eq. (55) in[5] by the replacement p_0'/p_0-1; compare with Eq. (40) and with Eq. (15) in[4]. In this connection the asymptotic formula (54) in[5] for $\chi\gg1$ is replaced by

$$W^{(4)}_{2\gamma\,\text{incoh}}=\tau\frac{\alpha^2m^2n_e}{p_0}\left[\frac{49}{135}\Gamma\left(\frac{1}{3}\right)(3\chi)^{2/3}+\gamma\frac{8}{15\chi}\right]. \tag{42}$$

We note that the utilization of the representation (38) for $\delta(0)$ is associated with a specific form of the region in which the process takes place. Namely, it was assumed that the difference between the centers of the regions of formation of the two processes varies within the interval from $-L_-/2$ to $L_-/2$. For a different form of the region (having the same value for the 4-volume) the representation for $\delta(0)$ may have the form $\delta(0)=(L_-+b)/4\pi f_-$, where b is small in comparison with the L_- correction, being unimportant for the calculation of the major term of the probability $\sim L_-$ but having the same order (with respect to L_-) as the coherent part. In other words, one can say that the terms in the probability which do not depend on L_- are not uniquely determined and depend essentially on the experimental situation. However, the presence of two regions of formation is characteristic for the terms in the probability which depend on the shape of the region, i.e., on b, and this allows one to uniquely isolate the "truly coherent" part.

Thus, one can write the nonexchange part of the probability for the photoproduction of a pair and a photon in the form

$$W^{(4)}_{e^+e^-\gamma\,\text{noex}\,\|,\perp}=W^{(4)\,\text{incoh}}_{e^+e^-\gamma\,\|,\perp}-\frac{\alpha^2m^2n_\gamma}{\pi^2l_0}\int\limits_0^\infty\frac{du}{(1+u)^2}\int\limits_0^\infty\frac{dv}{(1+v)^3}\{R,R_1,R_{11}\},\quad(43)$$

where $\{R, R_1, R_{11}\}$ in the second, coherent term denotes the expression inside the curly brackets in formula (16) in which the functions I, I_1, I_{11} are replaced by the functions R, R_1, R_{11}.

Let us present the asymptotic expressions for $W^{(4)\,\text{incoh}}_{e^+e^-\gamma}$ associated with small and large values of $\varkappa$:

$$W^{(4)\,\text{incoh}}_{e^+e^-\gamma\,\|,\perp}=\tau\frac{\alpha^2m^2n_\gamma}{l_0}\begin{cases}\frac{14}{243}(7\mp1)\Gamma^2\left(\frac{2}{3}\right)(3\varkappa)^{2/3}, & \varkappa\gg1\\ \frac{5(3\mp1)}{32\cdot2^{1/2}}\varkappa e^{-8/3\varkappa}, & \varkappa\ll1.\end{cases}\quad(44)$$

For large values of $\varkappa$ this probability does not depend on the electron mass, i.e., it possesses the property of scale invariance.

The asymptotic expression for $W^{(4)\,\text{coh}}_{e^+e^-\gamma\,\text{noex}}$ is as follows:

$$W^{(4)\,\text{coh}}_{e^+e^-\gamma\,\text{noex}\,\|,\perp}=\frac{\alpha^2m^2n_\gamma}{\pi^2l_0}\times\begin{cases}-\frac{3}{14\pi}(5\mp1)\Gamma^4\left(\frac{2}{3}\right)(3\varkappa)^{2/3}\ln\frac{\varkappa}{\lambda^{3/2}}, & \varkappa\gg1,\\ \frac{\pi}{8}\left(\frac{3}{2}\right)^{1/2}(3\mp1)\varkappa e^{-8/3\varkappa}\left[-3\ln\left[\frac{\varkappa}{2}\left(\frac{3}{2\lambda}\right)^{1/2}\right]+1\right.\\ \left.+C+\ln2+\frac{5}{2\cdot3^{1/2}}\ln(2+3^{1/2})\right], & \varkappa\ll1.\end{cases}\quad(45)$$

Since the exchange effects are small in comparison with the nonexchange effects at high energies or large fields, $W^{(4)}_{e^+e^-\gamma\,\text{noex}}$ for large values of $\varkappa$ coincides with the total probability $W^{(4)}_{e^+e^-\gamma}$ and is given by the asymptotic expressions (44) and (45).

The coherent term in this probability turns out to be negative at large values of $\varkappa$ and decreases the total probability. In spite of the fact that it increases more strongly than the incoherent part with increasing values of $\varkappa$, the probability $W^{(4)}_{e^+e^-\gamma}$ of course remains positive since the length L_- entering into the incoherent term should be larger than the length of formation Δx_-, which is given by $\Delta x_-\sim(m/eF)\varkappa^{1/3}$ for this process at $\varkappa\gg1$. Thus, $\tau\gtrsim\varkappa^{1/3}$. On the other hand, τ cannot become too large since according to perturbation theory the probability for the photoproduction of a pair and a photon must remain smaller than the probability for the photoproduction of a pair in the α-approximation: $\tau\lesssim\alpha^{-1}\varkappa^{1/3}$.

We also note that the coherent term of $W^{(4)}_{e^+e^-\gamma}$ continues to contain the photon mass, but the electron mass does not appear in it,

$$\alpha^2m^2\varkappa^{2/3}\ln\frac{\varkappa}{\lambda^{3/2}}=1/2\alpha^2(eFl)^{2/3}\ln\frac{(eFl)^2}{\mu^6}$$

which corresponds to minimal violation of scale invariance.

4. MASS CORRECTION TO THE PROBABILITY FOR THE PHOTOPRODUCTION OF A PAIR

One can represent the mass correction to the pair photoproduction probability in the form

$$VTW^{(4)}_{e^+e^-\,\text{mass}}=2\,\text{Re}\frac{e^2n_\gamma}{2l_0}\int d^4x'\,d^4x''\,e^{il(x'-x'')}e_\mu e_\nu$$
$$\times\{\text{Sp}[\gamma_\mu\Delta S'^{(+)}(x'',x')\gamma_\nu S^{(-)}(x',x'')]+\text{Sp}[\gamma_\nu\Delta S'^{(-)}(x',x'')\gamma_\mu S^{(+)}(x'',x')]\},\quad(46)$$

where

$$\Delta S'^{(+)}(x'',x')=i(x''|S^{(+)}MS^c|x'),\quad\Delta S'^{(-)}(x',x'')=i(x'|S^cMS^{(-)}|x'').\quad(47)$$

The transition from ΔS^c to $\Delta S'^{(\pm)}$ contains the substitution

$$(m^2+f^2-i\varepsilon)^{-2}\to\pm2\pi i\theta(\pm f_-)\delta(m^2+f^2)(m^2+f^2-i\varepsilon)^{-1}$$
$$=\mp\theta(\pm f_-)[i\pi\delta'(m^2+f^2)+2\pi\delta^2(m^2+f^2)],\quad(48)$$

compare with[5]. The last term in (48) leads to a $\delta(0)$ singularity in the probability and forms its incoherent part. It is not difficult to see that this part differs from the incoherent part of the probability $W^{(4)}_{e^+e^-\gamma}$ only in sign.

Substitution of the first coherent term from (48) leads to the result that the transition from the amplitude $T^{(4)}_{l'e',\,le;\,\text{mass}}$ to the coherent part $W^{(4)}_{e^+e^-\text{mass}}$ consists of an extension of the limits of integration over σ from $-\infty$ to $+\infty$ in the "original" representations of the functions $\pi^{(4)}_{1,2\,\text{mass}}$ (see Eqs. (16) and (25)), discarding of terms stemming from the vacuum part of the mass operator, and taking the imaginary part. This is equivalent to the replacement of the functions I, I_1, and I_{11} in Eq. (16) by the real functions $-2\gamma(z'_\lambda)\Phi(z)$, $-2\gamma_1(z'_\lambda)\times\Phi(z)$, and $-2\gamma_1(z'_\lambda)\Phi_1(z)$. Thus we obtain

$$W^{(4)}_{e^+e^-\,\text{mass}}=-W^{(4)\,\text{incoh}}_{e^+e^-\gamma}+\frac{\alpha^2m^2n_\gamma}{\pi^2l_0}\int\limits_0^\infty\frac{du}{(1+u)^2}\int\limits_0^\infty\frac{dv}{(1+v)^3}\{\Upsilon\Phi,\Upsilon_1\Phi,\Upsilon_1\Phi_1\},\quad(49)$$

where $\{\ \}$ denotes the expression in curly brackets in formula (16) with the appropriate replacement of the functions I, I_1, and I_{11}.

The sum $W^{(4)}_{e^+e^-\gamma\,\text{noex}}+W^{(4)}_{e^+e^-\text{mass}}$ satisfies the unitarity relationship (30) as long as the representations (37) exist. The incoherent terms cancel each other which is natural since any compact part of the elastic scattering amplitude possesses a single region of formation.

Let us present the asymptotic expression for $W^{(4)\text{coh}}_{e^+e^-\text{mass}}$:

$$W^{(4)\,\text{coh}}_{e^+e^-\,\text{mass}\,\|,\perp}=\frac{\alpha^2m^2n_\gamma}{\pi^2l_0}\times\begin{cases}(3\varkappa)^{2/3}\left\{\frac{1}{4\pi}\Gamma^4\left(\frac{2}{3}\right)\left[(5\mp1)\left(\ln\frac{\varkappa}{3^{1/2}\lambda^{3/2}}-C+\frac{\pi}{3^{1/2}}\right)-\frac{125}{7}\pm\frac{13}{4}\right]\right.\\ \left.-\frac{\pi^2}{27\cdot3^{1/2}}\left[\frac{49}{10}\Gamma\left(\frac{2}{3}\right)-\frac{5(2\mp1)}{27}\Gamma^2\left(\frac{1}{3}\right)\right]\right\}, & \varkappa\gg1,\\ \frac{\pi}{4}\left(\frac{3}{2}\right)^{1/2}\varkappa e^{-8/3\varkappa}\left[\left(\ln\frac{\varkappa}{2(3\lambda)^{1/2}}-C\right)(3\mp1)+\frac{1\mp1}{2}\right], & \varkappa\ll1.\end{cases}\quad(50)$$

We note that the coherent part of the probability $W^{(4)}_{e^+e^-\text{mass}}$ contains the electron mass and does not possess minimally violated gauge invariance.

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[1)]We use the units $\hbar = c = 1$, $\alpha = e^2/4\pi = 1/137$, and the notation $p_\mu = (\mathbf{p}, ip_0)$, $pq = \mathbf{p}\cdot\mathbf{q} - p_0 q_0$.

[2)]Here and below all noninvariant quantities are given in a special coordinate system with the axes 1, 2, 3 along the directions of E, H, and E × H, with $l_\pm = l_0 \pm l_3$.

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Unitarity relation and phase shift analysis in a system comprising a resonance and a particle

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Unitarity relations are recorded in explicit form for the amplitude of scattering of a resonance by a particle and for the amplitude for the creation of a resonance and a particle. The solution of these unitarity relations is found, and a representation is obtained for the amplitude for creation B in which all the corrections for rescattering have been taken into account. It is shown that the phase of the amplitude B is a sum of two terms: one of them corresponds to a long-range interaction between the resonance and the particle and an explicit expression is obtained for it; the other term originates from the left (potential) singularities of the amplitude and is the proper phase for the resonance-particle system. The results obtained enable us to carry out correctly a phase analysis for the resonance plus particle system and to improve the procedure employed in the well-known phase analysis of the Illinois group.

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The study of the interaction between a resonance and a particle has recently become particularly important in connection with the phase analysis carried out for the 3π and $K\pi\pi$ systems.[1–3] One of the principal problems in studying the resonance plus particle system is the following: what are the specific features (and the principal difference) of the resonance–particle system compared with the system of two stable particles? For stable particles we can carry out a phase-shift analysis using the interaction in the final state and determining the phase of this interaction from experiment. In such a procedure, representing the amplitude in terms of the scattering phase automatically guarantees two-particle unitarity. In a three particle system, a special case of which is the resonance-plus-particle system, the unitarity relation (UR) is satisfied only when the whole infinite series of rescattering including the exchange of the decay product between the resonance and the particle is taken into account. For this reason it appears at first sight that it is quite a complicated matter to satisfy the unitarity requirements in this system, and that in any case introduction of at least minimal dynamic assumptions is required.

An attempt to take three-particle unitarity into account within the framework of the K-matrix formalism was made recently by Ascoli and Wyld,[4] but this, however, led to a serious deterioration in the quality of fit obtained with phase analysis (and not conversely, as ought to be the case when the correct UR requirements are satisfied). Aitcheson and Golding[5] then noted that taking rescattering into account, as was done by Ascoli and Wyld, satisfies unitarity, but violates the properties of analyticity. The situation was thereby created that nonunitarized solutions should be mistrusted while unitarized solutions violate analyticity.

In this paper we start from rigorous UR for the amplitude for the creation of a resonance and a particle and for the amplitude for the interaction of the resonance with the particle, and with the aid of the solution of these UR we establish the following:

1. Taking unitarity (i.e., all the rescatterings) into account in the amplitude for the creation of a resonance plus a particle reduces to multiplying the nonunitary (initial) amplitude by a phase factor.

2. This factor contains the sum of two phases, the first of which, Φ_r, takes into account discontinuities to the right (in the energy plane) in the amplitude for