[1)]We use the units $\hbar = c = 1$, $\alpha = e^2/4\pi = 1/137$, and the notation $p_\mu = (\mathbf{p}, ip_0)$, $pq = \mathbf{p}\cdot\mathbf{q} - p_0 q_0$.

[2)]Here and below all noninvariant quantities are given in a special coordinate system with the axes 1, 2, 3 along the directions of E, H, and E × H, with $l_\pm = l_0 \pm l_3$.

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Unitarity relation and phase shift analysis in a system comprising a resonance and a particle

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Unitarity relations are recorded in explicit form for the amplitude of scattering of a resonance by a particle and for the amplitude for the creation of a resonance and a particle. The solution of these unitarity relations is found, and a representation is obtained for the amplitude for creation B in which all the corrections for rescattering have been taken into account. It is shown that the phase of the amplitude B is a sum of two terms: one of them corresponds to a long-range interaction between the resonance and the particle and an explicit expression is obtained for it; the other term originates from the left (potential) singularities of the amplitude and is the proper phase for the resonance-particle system. The results obtained enable us to carry out correctly a phase analysis for the resonance plus particle system and to improve the procedure employed in the well-known phase analysis of the Illinois group.

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The study of the interaction between a resonance and a particle has recently become particularly important in connection with the phase analysis carried out for the 3π and $K\pi\pi$ systems.[1–3] One of the principal problems in studying the resonance plus particle system is the following: what are the specific features (and the principal difference) of the resonance–particle system compared with the system of two stable particles? For stable particles we can carry out a phase-shift analysis using the interaction in the final state and determining the phase of this interaction from experiment. In such a procedure, representing the amplitude in terms of the scattering phase automatically guarantees two-particle unitarity. In a three particle system, a special case of which is the resonance-plus-particle system, the unitarity relation (UR) is satisfied only when the whole infinite series of rescattering including the exchange of the decay product between the resonance and the particle is taken into account. For this reason it appears at first sight that it is quite a complicated matter to satisfy the unitarity requirements in this system, and that in any case introduction of at least minimal dynamic assumptions is required.

An attempt to take three-particle unitarity into account within the framework of the K-matrix formalism was made recently by Ascoli and Wyld,[4] but this, however, led to a serious deterioration in the quality of fit obtained with phase analysis (and not conversely, as ought to be the case when the correct UR requirements are satisfied). Aitcheson and Golding[5] then noted that taking rescattering into account, as was done by Ascoli and Wyld, satisfies unitarity, but violates the properties of analyticity. The situation was thereby created that nonunitarized solutions should be mistrusted while unitarized solutions violate analyticity.

In this paper we start from rigorous UR for the amplitude for the creation of a resonance and a particle and for the amplitude for the interaction of the resonance with the particle, and with the aid of the solution of these UR we establish the following:

1. Taking unitarity (i.e., all the rescatterings) into account in the amplitude for the creation of a resonance plus a particle reduces to multiplying the nonunitary (initial) amplitude by a phase factor.

2. This factor contains the sum of two phases, the first of which, Φ_r, takes into account discontinuities to the right (in the energy plane) in the amplitude for

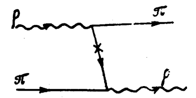

FIG. 1.

the scattering of the resonance by a particle; it is calculated explicitly and its appearance represents the first principal difference from the system of two stable particles.

3. The second part of the phase, Φ_L, is of the same nature as in the case of stable particles; in particular, it must pass through $\pi/2$ in the case of three-particle resonance (of the type A_2).

4. The variation of the phase Φ_r for the $\rho\pi$ system in the range 900–1400 MeV according to preliminary estimates is small ($\approx 10°$), but can cancel part of the resonance phase A_1 (if such is present), since Φ_r falls off with energy for $J^P = I^+$.

5. Consequently, the results of a nonunitarized fitting of the Illinois analysis[1–3] must be corrected by the value of Φ_r in order to obtain the "true" resonance phase Φ_L. In the simplest single-channel case the procedure of extracting Φ_L reduces to subtracting Φ_r from the phase of the nonunitarized fitting, which changes the phases quite insignificantly.

Below we describe the method on the example of a system of three spinless particles, two of which are identical. The case of different masses was investigated earlier,[6] and we follow the notations of that paper and of the paper of Ascoli and Wyld.[4] The set of connected graphs for the amplitude $3 \to 3$ can be expressed in terms of the resonance–particle amplitude $C^J(\sigma_\alpha, \sigma'_\beta, s)$:

$$w = \sum_{\alpha,\beta=1,2} \sum_{JMLS,L'S'} Z_{LS}^{JM}(\Omega_\alpha, \overline{\Omega_\alpha}) C^J_{LS,L'S'}(\sigma_\alpha, \sigma_\beta', s)$$

$$\times Z_{L'S'}^{JM*}(\Omega'_\beta, \overline{\Omega_\beta}') \tilde{t}_S(\sigma_\alpha) \tilde{t}_{S'}(\sigma_\beta') k^L(s, \sigma_\alpha) k^{L'}(s, \sigma_\beta'),$$

$$Z_{LS}^{JM}(\Omega_\alpha, \overline{\Omega_\alpha}) = \sum_{\lambda+\lambda'=M} C^{JM}_{L\lambda,S\lambda'} Y_{L\lambda}(\Omega_\alpha) \psi_{S\lambda'}(\overline{\Omega_\alpha}). \tag{1}$$

In (1) the quantity $\tilde{t}_S(\sigma)$ is a part of the two-particle t-matrix, which includes the propagator for the two-meson resonance with spin S and the vertex constant with a formfactor; therefore $\tilde{t}_S(\sigma)$ is up to a constant factor equal to the quantity $R_{LS}(\sigma, W)$ from[4].

The UR for $C^J(s)$ were derived earlier[6] in the case of single-channel scattering with spin $S=0$. In order not to write awkward expressions we shall exhibit only the general structure of the UR in our case, defining below all the quantities as matrices with respect to the indices L and S, and assuming summation over repeated discrete indices and integration over the continuous σ:

$$\Delta_s \hat{C}^J(s)/2i = J_r{}^J + \hat{C}^J(s_+) \hat{\varkappa}^J(s) \hat{C}^J(s_-). \tag{2}$$

The UR (2) is presented for the quantity $C^J_{LS,L'S'}(\sigma_-, \sigma'_+, s)$, where the $\pm$ signs denote the signs of the imaginary additions to the quantities σ, σ' and where in $\hat{C}^J$ a summation has been carried out over the topologically different graphs for identical particles. Equation (2) differs from (2.6) of the paper by Ascoli and Wyld[4] since the latter refers to the quantity $T^J_{LS,L'S'}$, which contains unconnected graphs with a δ-function singularity. Moreover, we have defined $C^J(s)$ having separated out all the essential dependence on σ and σ', so that $C^J(s)$ is a smooth function of these variables and can be taken outside the integral sign in (2) at the point $\sigma'' = \sigma''' = \sigma_R = M_R^2$, where the $\tilde{t}(\sigma)$ in $\varkappa(s)$ have a peak. Assuming the external σ and σ' also to be equal to σ_R, we obtain the quasi-two-particle UR for the quantity $C^J_{LS,L'S'}(\sigma_R^-, \sigma_R^+, s)$, with which we will be dealing in future, and these UR again are of the form (2), but now without integration over σ.

The UR (2) differ from the two-particle UR, firstly, by the presence of the inhomogeneous term J_r, which is equal to the discontinuity with respect to s of the partial wave of the pole graph in Fig. 1. This term leads to the singularities of $C^J(s)$ lying on the unitary cut $s \geqslant (3m)^2$ at the points s_1 and s_2 which are characteristic for the process of rescattering of a resonance by a particle. Secondly, $\varkappa^J(s)$ plays the role of the phase volume of the resonance and the particle, but consists of a sum of two terms, $J_d + J_c$, obtained from the separations shown in Fig. 2. $\varkappa^J(s)$ is always positive, but depends strongly on the quantum numbers J, L, S through the term J_c. For certain J, for example for $J^P = 0^-$, in the $\rho\pi$ system a considerable cancellation occurs in the sum $J_d + J_c$. In the limit of a small width Γ $J_c \to 0$ and J_d tends to the two-particle phase volume.

In the general case the phase volume can be represented in the form

$$\varkappa^J(s) = [s - (M_R + m)^2]^{1/2} F^J(s) + \rho_3(s), \tag{3}$$

where $F^J(s)$ has no singularities to the right of $(M_R + m)^2$ and $\rho_3(s)$ is small. The first term on the right hand side in (3) represents the phase volume of the resonance and the particle, while the second term corresponds to the contribution of the three-particle background, which differs from zero below the threshold for the decay into a resonance plus particle.

We can formally write the solution of Eq. (2) in the single-channel case in the form

$$C^J(s) = \frac{1}{2i\varkappa(s)} \left\{ \exp\left[-\frac{i\varkappa(s)}{2\pi} \int_{s_1}^{s_2} \frac{\ln[1 - 4J_r\varkappa(s')]}{\varkappa(s')(s'-s)} ds' + 2i\nu(s) \right] - 1 \right\}, \tag{4}$$

where the function $\nu(s)$ has the property $\nu(s_+) = -\nu(s_-)$ and is otherwise arbitrary. However in order for $C^J(s)$ to be analytic in the neighborhood of $s = (3m)^2$, where $\varkappa(s)$ vanishes, we require the condition $\varkappa(s_+) = -\varkappa(s_-)$,

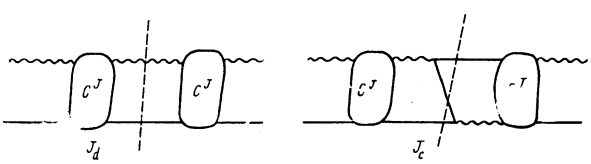

FIG. 2.

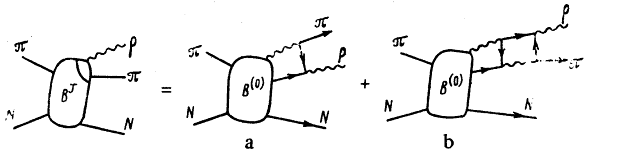

FIG. 3.

which strictly speaking is not satisfied by $\varkappa(s)$. However, the principal part of the phase volume (3), $[s-(M_R+m)^2]^{1/2}=\varkappa_2(s)$ satisfies this requirement, and therefore in future we shall everywhere use the representation (4) where strictly speaking we interpret $\varkappa(s)$ as $\varkappa_2(s)$. We note that in the final result it is apparently possible to utilize $\varkappa(s)$ itself since this does not violate the analytic properties of the amplitude.

We now can in analogy with (1) write the amplitude for the creation of a resonance and a particle in the form

$$v=\sum_{\beta=1,2}\sum_{JLSM}B_{LS}{}^J(\sigma_\beta',s)\hat{t}_S(\sigma_\beta')Z_{LS}^{*JM}(\Omega_\beta',\bar{\Omega}_\beta'). \tag{5}$$

Then the UR for the quantity $B^J_{LS}(\sigma_R^+,s)\equiv\hat{B}^J(s)$ has the form

$$\Delta_s\hat{B}^J(s)/2i=\hat{B}^J(s_+)\hat{\varkappa}^J(s)\hat{C}^J(s_-). \tag{6}$$

The single-channel solution of (6) is given by the solution of the so-called homogeneous Hilbert problems[7]:

$$B^J(s)=B^{(0)}(s)\exp\left\{-\frac{\Lambda(s)}{2\pi i}\int\limits_{(3m)^2}^{\infty}\frac{ds'\ln[1-2i\varkappa(s')C^J(s_-')]}{\Lambda(s')(s'-s)}\right\}. \tag{7}$$

Here $B^{(0)}(s)$ and $\Lambda(s)$ have no singularities along the cut $s\geqslant(3m)^2$ and are otherwise arbitrary. For the sake of simplicity we choose $\Lambda(s)=s-s_0$ and substitute into (7) $C^J(s)$ in the form (4). Then $B^J(s)$ takes on the form

$$B^J(s)=B^{(0)}(s)\exp(i\Phi(s)),\quad \Phi(s)=\Phi_r+\Phi_L, \tag{8}$$

$$\Phi_r(s)=\frac{i(s-s_0)}{4\pi}\int\limits_{s_1}^{s_2}\frac{ds'\ln[1-4J_r\varkappa(s')]}{\varkappa(s')(s'-s)}\left(\frac{\rho(s)}{s-s_0}-\frac{\rho(s_+')}{s_+'-s_0}\right), \tag{9}$$

$$\Phi_L(s)=-\frac{i(s-s_0)}{\pi}\int\limits_{(3m)^2}^{\infty}\frac{ds'\,\nu(s')}{(s'-s)(s'-s_0)} \tag{10}$$

Above we have introduced the definition

$$\rho(s)=\frac{s-s_0}{\pi}\int\limits_{(3m)^2}^{\infty}\frac{ds'\varkappa(s')}{(s'-s)(s'-s_0)}. \tag{11}$$

We discuss the properties of the representation (8). Firstly, it can be seen from (9) that $\Phi_r(s)$ has no singularities at $s_+=s_1$, s_2 (although $C^J(s)$ has singularities at those points), and this is in agreement with the analysis carried out by a number of authors.[8] Secondly, the phase $\Phi_L(s)$ is analogous to the phase arising in the rescattering of two stable particles, if their scattering phase is $\nu(s)$. The real part of $\Phi_L(s)$ is exactly equal to $\nu(s)$ as it ought to be. Thirdly, $\exp(i\Phi_r)$ describes the series for the rescattering shown in Fig. 3, or more accurately all the singularities of this series to the right of the threshold $s=(M_R+m)^2$ while the left singularities of the rescattering series are collected in $\Phi_L(s)$.

In order to establish the correspondence between the first diagrams for rescattering in Fig. 3 and $B_J(s)$ we consider (7), having substituted into it the expansion of $C^J(s)$ in terms of the number of rescatterings:

$$C^J(s)=C_1{}^J(s)+C_2{}^J(s)+\ldots, \tag{12}$$

where C_1^J corresponds to the diagram of Fig. 1

$$C_1{}^J=\frac{1}{\pi}\int\limits_{s_1}^{s_2}\frac{ds'\,J_r(s')}{s'-s}. \tag{13}$$

In analogy with (12) the expression (7) can be represented in the form of the sum

$$B^J(s)=B^{(0)}(s)+B_1{}^J(s)+B_2{}^J(s)+\ldots, \tag{14}$$

where $B^{(0)}(s)$ is the amplitude without rescattering (for example, the diagram corresponding to the Deck mechanism for the creation of a $\rho\pi$-system). B_1^J corresponds to the triangular diagram of Fig. 3a. For it we obtain

$$B_1{}^J=B^{(0)}(s)\frac{\Lambda(s)}{\pi}\int\limits_{(3m)^2}^{\infty}\frac{ds'\varkappa(s')C_1{}^J(s_-')}{\Lambda(s')(s'-s)}. \tag{15}$$

From this it can be seen that (15) is the dispersion representation of the triangular diagram, where the role of the left vertex is played by the quantity $\Lambda^-(s)$. Here $\varkappa(s)=J_d(s)$, while the other term in $\varkappa(s)$, equal to $J_c(s)$, corresponds to the dispersion integral for the more complex rescattering diagram in Fig. 3b. The analysis for the subsequent terms of the expansion is carried out in a similar manner, and from it follows the third property noted above.

We note that the nonanalyticity of $\varkappa(s)$ does not affect the answer: $B^J(s)$ defined by (8) and (9) has the correct analytical properties for any arbitrary $\varkappa(s)$, and not only for $\varkappa(s)=\varkappa_2(s)$; moreover, correspondence with the rescattering series is obtained when the complete $\varkappa(s)$ is substituted. Nevertheless, if one can utilize the numerical smallness of $\rho_3(s)$ in (3) and neglect it, and also replace $F^J(s)$ by the constant value F_0 then the expression for Φ_r takes on a more lucid form (we put $y=s-(M_R+m)^2$):

$$\Phi_r(y)=\frac{1}{4\pi}\int\limits_{y_1}^{y_2}\frac{dy'}{(y')^{1/2}}\ln[1-4F_0(y')^{1/2}J_r]\left(\frac{1}{(y')^{1/2}+(y)^{1/2}}-\frac{1}{(y')^{1/2}+(y_0)^{1/2}}\right). \tag{16}$$

The phase Φ_r is characteristic specifically of the system resonance–particle and reflects the physical fact that the change interaction corresponding to the diagram in Fig. 1 has an infinite range. In this sense Φ_r is analogous to the Coulomb phase, and the remaining part of the phase Φ_L is analogous to the nuclear phase in the case of interference between Coulomb and nuclear forces. This analogy also manifests itself in the fact that the "nuclear phase" Φ_L contains the contribution of the left cuts from the rescattering graphs (which can themselves lead to bound and virtual states, cf., [9]). Together with the contribution from the exchange of particles, as in the usual nuclear phase, there is also present the contribution of the Coulomb forces.

We now discuss the application of our results under conditions of phase-shift analysis as compared with the unitarization procedure in the work of Ascoli and Wyld.[4] As the authors themselves have noted, unitarization in[4] reduces to the summation of the rescattering series with the propagators in Fig. 3 being replaced by δ-functions. Then, naturally, the analytical properties of $B^J(\sigma, s)$ are violated and as a result of this singularities appear at $s = s_1, s_2$, i.e., at those threshold values of the energy s when the formation of a resonance is possible simultaneously in two subsystems (13) and (23) of three particles. Taking into account the width of the resonance, the singularities s_1 and s_2 are shifted to the second sheet but there remains a strong dependence of $B^J(\sigma, s)$ on s in the neighborhood of these points. The forcing of such an incorrect dependence on to the physical amplitude B^J, apparently, is what leads to the deterioration of the quality of the fit.

The results of our analysis show that unitarization leads (in the single-channel case) to the appearance of the additional phase Φ_r, which should not affect the procedure of phase analysis, in which the total phase $\Phi_r + \Phi_L$ is determined. At the same time the phase ν, appearing in Φ_L is determined by the real analytic solution of N/D equations taking into account arbitrary exchange mechanisms (arbitrary left cuts), and therefore must pass through $\pi/2$, if there exists a resonance in the $\rho\pi$ system.

In principle the phase Φ_r can cancel a part of the resonance dependence of Φ_L. For example, in the state $J^P = 1^+$ of the $\rho\pi$-system, where the existence of a $A1$-resonance is assumed, the phase Φ_r is positive and falls off with energy (while Φ_L increases in the resonance case). But the total change in Φ_r over the range 900 MeV $\leq s^{1/2} \leq$ 1400 MeV is according to preliminary estimates $\sim 10°$ and therefore is not likely to explain the absence of resonance behavior in the total phase $\Phi_r + \Phi_L$.

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Properties of heavy leptons and their manifestation in the $e^+e^- \to$ hadrons reaction

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The manifestations of heavy leptons in various characteristics of e^+e^- annihilation are examined. The properties of the heavy-lepton decays and their contributions to $R(s)$ and $\epsilon(s)$ are discussed with allowance for event-selection effects. The inclusive spectra of the secondary particles are also discussed. It is shown that the characteristic properties expected for the heavy leptons are consistent with the data on anomalous muon production.

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1. INTRODUCTION

According to current ideas (e.g. [1–5]) the reaction $e^+e^- \to$ hadrons is a rather complicated process in the energy region $2.4 \lesssim s^{1/2} \lesssim 7.4$ GeV, there being at least two components:

a) In the region $s^{1/2} \gtrsim 2.4$ GeV there is apparently a