Drift-kinetic equation for a tenuous plasma in the presence of high-frequency fields

V. P. Milant'ev

Patrice Lumumba University
(Submitted June 17, 1976)
Zh. Eksp. Teor. Fiz. 72, 159–169 (January 1977)

An averaging method is used to obtain an approximate kinetic equation for a collisionless plasma in the drift approximation in the presence of a high-frequency field in the form of a quasimonochromatic wave, both in the absence of resonance conditions and under cyclotron-resonance conditions.

PACS numbers: 52.25.Dg

1. The drift-kinetic approximation refers to the approximate equation obtained from the "exact" kinetic equation by averaging in the case when the plasma-particle gyroscopic radius is small. It is necessary to distinguish in this case between the drift approximation and the magnetohydrodynamic approximation. In the single-particle theory, which determines the characteristics of the kinetic equation, this corresponds respectively to the cases of weak and strong electric fields.[1,2] In the drift theory it is assumed that

$$1/|\Omega|T\sim a/L=\varepsilon\ll 1, \qquad (1.1)$$

$$v_E\sim\varepsilon v\sim\varepsilon L/T. \qquad (1.2)$$

$$\Omega=-\frac{eB_0}{mc},\quad T\sim\left|\mathbf{B}_0\Big/\frac{\partial\mathbf{B}_0}{\partial t}\right|,\quad L\sim\left|\frac{\mathbf{B}_0}{\nabla\mathbf{B}_0}\right|.$$

v is the particle characteristic velocity, a is its gyroscopic radius, v_E is the electric drift velocity, and ε is a small parameter. The drift-kinetic equation under conditions (1.1) and (122) was obtained, for example by Sivukhin[2] and Hazeltine.[3] In the magnetohydrodynamic theory, the condition (1.2) is replaced by the assumption

$$v_E\sim v. \qquad (1.3)$$

In this case, the corresponding averaged kinetic equations were obtained in a number of studies,[4–7] which have led to the known Chew–Goldberger–Low system of equations as well as to a number of corrections to them.

In connection with problems involving heating and stabilization of plasma, as well as others, interest in a plasma situated in a magnetic field on which high-frequency (RF) fields are superimposed has increased of late (see, e.g.,[8,9]). Simplified kinetic equations for a tenuous plasma in the presence of RF fields are known for some particular cases.[10–12] This paper presents a derivation of averaged kinetic equations for a plasma situated in a strong magnetic field in the presence of RE fields in the form of arbitrary quasi-monochromatic waves. The drift approximation (1.1) and (1.2) is used. The characteristic space-time scales of the RF fields are assumed to be close to the corresponding values of the inhomogeneous magnetic field $\mathbf{B}_0$, so that the "frequency" $\nu=-\omega+k_1v_\parallel$ of the RF field (see (2.6b)) and the displacement $a_\sim$ of the particle in the RF field satisfy the relations of the type (1.1)

$$1/|\nu|T\sim a_\sim/L\sim\varepsilon. \qquad (1.4)$$

The magnetic field of the RF wave is assumed to be small in comparison with the strong field $\mathbf{B}_0$. In the absence of resonances, $s_1\Omega+s_2\nu\neq 0$, where s_1 and s_2 are certain prime numbers, it turned out that the RF field has no effect whatever on the drift-kinetic equations of zeroth and first order in the parameter ε. Its influence manifests itself only in the second-order approximation. The second-approximation equation (4.1) below was obtained for a longitudinal wave propagating along a homogeneous magnetic field, as was Eq. (5.1) in the case of a quasi-stationary RF field. It is also assumed in the averaging that there are no resonances $s_1\Omega+s_2\nu+s_3\omega_j$, where ω_j are the frequencies of the plasma natural oscillations.

An averaged equation under the resonance conditions $s_1\Omega+s_2\nu\approx 0$, which is considered in first order in the expansion parameter, was obtained in general form. In this case the drift-kinetic equation turns out to be even of zeroth-approximation in the cyclotron-resonance region.

2. It is convenient to use the Vlasov equation in the form of a continuity equation in a cylindrical coordinate system in velocity space[6]:

$$\frac{\partial f}{\partial t}+\nabla\mathbf{v}f+\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}v_\perp a_\perp f+\frac{\partial}{\partial v_\parallel}a_\parallel f+\frac{1}{v_\perp}\frac{\partial}{\partial\theta_1}a_\theta f=0. \qquad (2.1)$$

In the approximation (1.2), the particle velocity vector can be resolved in terms of the local unit vectors $\mathbf{e}_1=\mathbf{B}_0/B_0$, $\mathbf{e}_2$, and $\mathbf{e}_3$, which are connected with the force lines of the field $\mathbf{B}_0$:

$$\mathbf{v}=v_\parallel\mathbf{e}_1+v_\perp(\mathbf{e}_2\cos\theta_1+\mathbf{e}_3\sin\theta_1). \qquad (2.2)$$

Here $v_\parallel$ and $v_\perp$ are respectively the longitudinal and transverse components of the velocity vector relative to the force lines of the field $\mathbf{B}_0$, and θ_1 is the phase of the particle cyclotron rotation. The accelerations $a_\perp=\dot v_\perp$, $a_\parallel=\dot v_\parallel$, $a_\theta=v_\perp\dot\theta_1$ are determined from the equation of motion

$$\frac{d\mathbf{v}}{dt}=\frac{e}{m}\mathbf{E}+\frac{e}{mc}[\mathbf{v}\times\mathbf{B}] \qquad (2.3)$$

by projection on the directions

$\mathbf{e}_1, \mathbf{e}_2\cos\theta_1+\mathbf{e}_3\sin\theta_1, \quad -\mathbf{e}_2\sin\theta_1+\mathbf{e}_3\cos\theta_1.$

The fields **E** and **B** are assumed to consists of two parts: slowly varying fields $\mathbf{E}_0(\mathbf{r}, t)$ and $\mathbf{B}_0(\mathbf{r}, t)$, and a rapidly alternating RF field of rather general form

$$\mathbf{E}_\sim=\sum_{1\leq i\leq 3}\mathbf{e}_iE_i\cos(\theta_2+\varphi_i),$$
$$\mathbf{B}_\sim=\sum_{1\leq i\leq 3}\mathbf{e}_iB_i\cos(\theta_2+\chi_i). \tag{2.4}$$

The quantities $\mathbf{e}_i$, E_i, B_i, φ_i, and χ_i are assumed to be slowly varying functions of the coordinates and of the time. The phase θ_2 of the RF wave, assumed to vary rapidly, is defined by the relation[13,14]

$$\frac{d\theta_2}{dt}=\frac{\partial\theta_2}{\partial t}+\frac{d\mathbf{r}}{dt}\nabla\theta_2=\nu+v_\perp(k_2\cos\theta_1+k_3\sin\theta_1), \tag{2.5}$$

where

$$\omega(\mathbf{r}, t)=-\partial\theta_2/\partial t, \quad \mathbf{k}(\mathbf{r}, t)=\nabla\theta_2, \tag{2.6a}$$

are respectively the "frequency" and the "wave vector" of the quasi-monochromatic wave (2.4), and

$$\nu=-\omega+k_1v_\parallel, \quad k_i=\mathbf{e}_i\mathbf{k}. \tag{2.6b}$$

The fields $\mathbf{E}_\sim$ and $\mathbf{B}_\sim$ are not independent. From Maxwell's induction equation it follows that[14]

$$\mathbf{B}_\sim=\mathbf{h}_0+\varepsilon\mathbf{h}_1+\varepsilon^2\mathbf{h}_2+\ldots, \tag{2.7}$$
$$\mathbf{h}_0=\frac{c}{\omega}[\mathbf{k}\times\mathbf{E}_\sim], \quad \mathbf{h}_1=\frac{1}{\omega}\frac{\partial\hat{\mathbf{h}}_0}{\partial t}+\frac{c}{\omega}\operatorname{rot}\hat{\mathbf{E}}_\sim,\ldots \tag{2.7a}$$

The symbol $\hat{f}$ stands here for[1]

$$\hat{f}=\int_0^\theta d\theta(f-\bar{f})-\overline{\int_0^\theta d\theta(f-\bar{f})}, \quad \bar{f}=\frac{1}{2\pi}\int_0^{2\pi}d\theta f. \tag{2.7b}$$

Taking the expansion (2.7) into account, the right-hand side of (2.3), and with it the accelerations $a_\perp$, $a_\parallel$, and a_θ, are also represented in the form of expansions in powers of ε. In the presence of an RF field, the distribution function is of the type $f=f(t, \mathbf{r}, \mathbf{v}, \theta_2)$. The operators $\partial/\partial t$ and ∇ must therefore be defined respectively as

$$\frac{\partial}{\partial t}\to\frac{\partial}{\partial t}-\omega\frac{\partial}{\partial\theta_2}, \tag{2.8}$$
$$\nabla\to\nabla+\mathbf{k}\frac{\partial}{\partial\theta_2}. \tag{2.9}$$

After subtracting the accelerations $a_\perp$, $a_\parallel$, and a_θ from (2.3) and taking into account the relations (2.8) and (2.9), we can rewrite the kinetic equation (2.1) in the form

$$-\left(\Omega\frac{\partial}{\partial\theta_1}+\nu\frac{\partial}{\partial\theta_2}\right)f=\varepsilon Lf, \tag{2.10}$$

where the operator L is defined by

$$\begin{aligned}L=&D_0+A_0\frac{\partial}{\partial\theta_1}+\cos\theta_1\left(D_1+A_1\frac{\partial}{\partial\theta_1}+C_1\frac{\partial}{\partial\theta_2}\right)\\&+\sin\theta_1\left(D_2+A_2\frac{\partial}{\partial\theta_1}+C_2\frac{\partial}{\partial\theta_2}\right)+\cos 2\theta_1\left(D_3+A_3\frac{\partial}{\partial\theta_1}\right)\\&+\sin 2\theta_1\left(D_4+A_4\frac{\partial}{\partial\theta_1}\right)+\cos\theta_2\left(D_5+A_5\frac{\partial}{\partial\theta_1}\right)\\&+\sin\theta_2\left(D_6+A_6\frac{\partial}{\partial\theta_1}\right)+\cos\theta_+\left(D_7+A_7\frac{\partial}{\partial\theta_1}\right)\\&+\sin\theta_+\left(D_8+A_8\frac{\partial}{\partial\theta_1}\right)+\cos\theta_-\left(D_9+A_9\frac{\partial}{\partial\theta_1}\right)+\sin\theta_-\left(D_{10}+A_{10}\frac{\partial}{\partial\theta_1}\right)\end{aligned} \tag{2.11}$$

Here

$$\theta_\pm=\theta_1\pm\theta_2, \quad C_{1,2}=v_\perp k_{2,3}, \quad F_i=eE_i/m,$$
$$D_0=\frac{\partial}{\partial t}+v_\parallel\mathbf{e}_1\nabla-\frac{v_\parallel v_\perp}{2}\nabla\mathbf{e}_1\frac{\partial}{\partial v_\perp}+\left(F_{01}+\frac{v_\perp^2}{2}\nabla e_1\right)\frac{\partial}{\partial v_\parallel}. \tag{2.12}$$

The explicit forms of the other operators D_i and A_i are quite cumbersome and will not be written out here (see the expressions for some of the first operators in Volkov's paper[9]).

In Eq. (2.10) there has been introduced explicitly a small parameter in accordance with the conditions (1.4). By virtue of the expansion (2.7), the operators D_i and A_i ($i\geq 5, \ldots, 10$) are also expanded in powers of ε. The operator L then takes the form

$$L=L_0+\varepsilon L_1+\varepsilon^2L_2+\ldots \tag{2.13}$$

3. We seek a solution of the kinetic equation (2.10) in the form of an expansion in powers of ε:

$$f=f_0+\varepsilon f_1+\varepsilon^2f_2+\ldots \tag{3.1}$$

In the zeroth approximation we then have

$$-\left(\Omega\frac{\partial}{\partial\theta_1}+\nu\frac{\partial}{\partial\theta_2}\right)f_0=0,$$

i.e., the distribution function f_0 does not depend explicitly on the phase shifts θ_1 and θ_2:

$$f_0=f_0(t, \mathbf{r}, v_\parallel, v_\perp). \tag{3.2}$$

In the next order of the approximation we have

$$-\left(\Omega\frac{\partial}{\partial\theta_1}+\nu\frac{\partial}{\partial\theta_2}\right)f_1=L_0f_0. \tag{3.3}$$

By definition, the functions f_i ($i\geq 1$) are periodic (with period 2π in each argument) and can be written in the form

$$f_i=\bar{f}_i+\tilde{f}_i, \tag{3.4}$$

where

$$\bar{f}_i=\frac{1}{(2\pi)^2}\iint_0^{2\pi}f_i\,d\theta_1\,d\theta_2$$

is the "dc" component of f_i, and $\tilde{f}_i=f_i-\bar{f}_i$ is its "alternating" part, which can be described in the general case by the formula

$$\tilde{f}_1=\sum_{p,q} f_1^{(pq)}\exp\{i(p\theta_1+q\theta_2)\}. \qquad (3.5)$$

Then, averaging (3.3), we obtain

$$\overline{L}_0 f_0=0$$

or, taking (2.11) into account,

$$\overline{L}_0 f_0=D_0 f_0=0. \qquad (3.6)$$

This is the zeroth-approximation drift-kinetic equation, which is of the same form as in the absence of the RF field (see[6] with $v_E=0$). The operator D_0 is defined by (2.12). In the case of a constant magnetic field, Eq. (3.6) takes the simpler form

$$D_0 f_0=\left(\frac{\partial}{\partial t}+v_\parallel\nabla_1+F_{01}\frac{\partial}{\partial v_\parallel}\right)f_0=0. \qquad (3.7)$$

This equation describes the distribution of the centers of the Larmor circles as $B_0\to\infty$ (see, e.g.,[5] with $v_E=0$), $\nabla_1=\mathbf{e}_1\nabla$, and $F_{01}=\mathbf{e}_1\cdot\mathbf{F}_0$. It follows from (3.7) that the distribution function f_0 is independent of the transverse coordinates and of the transverse velocity. The dependence of these variables can be only "slow," i.e., in place of (3.2) we must write

$$f_0=f_0(t,\ \mathbf{e}_1\mathbf{r},\ v_\parallel,\ \varepsilon\mathbf{e}_2\mathbf{r},\ \varepsilon\mathbf{e}_3\mathbf{r},\ \varepsilon v_\perp). \qquad (3.8)$$

If no account is taken of the slow dependence on the transverse variables, then the dependence on these variables drops out completely in the higher-order approximations. It must therefore be assumed from the very outset that

$$f=f(t,\ \mathbf{r},\ v_\parallel,\ v_\perp,\ \theta_1,\ \theta_2,\ \varepsilon\mathbf{e}_2\mathbf{r},\ \varepsilon\mathbf{e}_3\mathbf{r},\ \varepsilon v_\perp). \qquad (3.9)$$

Then all the operators L_i in (2.12) must be regarded to be of the form

$$L_i=L_i^{(0)}+\varepsilon L_i^{(1)}, \qquad (3.10)$$

where the operators $L_i^{(1)}$ contain the operations of differentiation with respect to the "slow" transverse variables

$$\frac{\partial}{\partial\varepsilon v_\perp},\quad \mathbf{e}_2\frac{\partial}{\partial\varepsilon\mathbf{r}},\quad \mathbf{e}_3\frac{\partial}{\partial\varepsilon\mathbf{r}}.$$

Thus, the kinetic equation (2.10) must be written in the form

$$-\left(\Omega\frac{\partial}{\partial\theta_1}+\nu\frac{\partial}{\partial\theta_2}\right)f=\varepsilon\{L_0^{(0)}+\varepsilon(L_0^{(1)}+L_1^{(0)})+\ldots\}f. \qquad (3.11)$$

Equations (3.3) and (3.6) remain unchanged in this case, but the operators L_0 and D_0 must be replaced by $L_0^{(0)}$ and $D_0^{(0)}$.

When account is taken of (3.6), Eq. (3.3) can be represented in the form

$$-\left(\Omega\frac{\partial}{\partial\theta_1}+\nu\frac{\partial}{\partial\theta_2}\right)\tilde{f}_1=\tilde{L}_0^{(0)}f_0=\sum_{p,q} l_0^{(pq)}\exp\{i(p\theta_1+q\theta_2)\}, \qquad (3.12)$$

where $\tilde{L}_0^{(0)}$ is the "alternating part" of the operator $L_0^{(0)}$. The right-hand side in (3.12) is given by (3.5). We then obtain from (3.12)

$$\tilde{f}_1=-\sum_{p,q}\frac{l_0^{(pq)}\exp\{i(p\theta_1+q\theta_2)\}}{i(p\Omega+q\nu)}\equiv-\check{L}_0^{(0)}f_0. \qquad (3.13)$$

The haček $\vee$ marks a quantity defined by the foregoing identity. To determine the dc component of the function f_1 it is necessary to consider Eq. (3.11) in second-order approximation:

$$-\left(\Omega\frac{\partial}{\partial\theta_1}+\nu\frac{\partial}{\partial\theta_2}\right)f_2=L_0^{(0)}f_1+L_1^{(0)}f_0+L_0^{(1)}f_0. \qquad (3.14)$$

This yields, after averaging with allowance for (3.8),

$$\overline{L_0^{(0)}f_1}=-\overline{L_0^{(1)}f_0}, \qquad (3.15)$$

or by virtue of relations (3.4) and (2.11),

$$D_0^{(0)}\overline{f}_1=-\overline{\tilde{L}_0^{(0)}\tilde{f}_1}-D_0^{(1)}f_0. \qquad (3.16)$$

We have also taken account here of the fact that

$$\overline{L}_i f_0=0 \text{ where } i\geqslant1. \qquad (3.17)$$

Relation (3.16) is the first-order approximation of the drift kinetic equation. By virtue of (2.11) one could expect this equation to depend on the RF field. Rather cumbersome calculations, however, show that the RF corrections vanish and the drift-kinetic equation (3.16) can be written in the form of a Liouville equation in the phase space $\mathbf{r}$, $v_\parallel$, $v_\perp$,[2], where $\dot{\mathbf{r}}$, $\dot{v}_\parallel$, and $\dot{v}_\perp$ are determined by the known first-approximation formulas of particle drift motion.[1,2] Thus, the influence of the RF field comes into play in the second order in the parameter ε. For the alternating part of the second-approximation distribution function we obtain

$$\tilde{f}_2=-\check{L}_0^{(0)}\overline{f}_1-\check{\tilde{L}}_0^{(0)}\tilde{f}_1-\check{L}_1^{(0)}f_0-\check{L}_0^{(1)}f_0. \qquad (3.18)$$

To find the equation for the dc component of the function f_2 it is necessary to consider Eq. (3.11) in third-order approximation

$$-\left(\Omega\frac{\partial}{\partial\theta_1}+\nu\frac{\partial}{\partial\theta_2}\right)f_3=(L_2^{(0)}+L_1^{(1)})f_0+(L_1^{(0)}+L_0^{(1)})f_1+L_0^{(0)}f_2. \qquad (3.19)$$

From this we obtain, after averaging, the sought second-order approximation of the drift-kinetic equation

$$D_0^{(0)}\overline{f}_2=-\overline{\tilde{L}_0^{(0)}\tilde{f}_2}-\overline{(L_0^{(1)}+L_1^{(0)})\tilde{f}_1}-D_0^{(1)}\overline{f}_1. \qquad (3.20)$$

The explicit form of this equation is exceedingly complicated. We shall consider some particular cases.

4. Consider the averaged plasma equation in the case of a longitudinal quasi-monochromatic wave propagating along a homogeneous magnetic field. In this case $F_2=F_3=0$ and $k_2=k_3=0$. If we choose the direction of the magnetic field to be the z axis, then

$$F_1=F_1(z,t),\quad \omega=\omega(z,t),\quad k=k_1(z,t).$$

The zeroth approximation equation takes the form (3.7). According to (3.8) we have

$$\nabla_2 f_0=\nabla_3 f_0=\partial f_0/\partial v_\perp=0.$$

The first-approximation equation (3.16) then coincides in form with (3.7). The second-approximation drift-kinetic equation is

$$D_0\bar{f}_2=\frac{\partial}{\partial v_\parallel}\left\{\left(D_0\frac{I}{2v^2}\right)\frac{\partial f_0}{\partial v_\parallel}\right\}-\frac{\partial}{\partial v_\parallel}\left(\frac{I}{v^2}\nabla_1 f_0\right). \tag{4.1}$$

Here $I=e^2E_1^2/2m^2\equiv e^2I_1/m^2$. The operator D_0 is defined by formula (3.7). The first term in (4.1) is of the "diffusion" type, in analogy with the quasilinear approximation for a given external field, [15] with a diffusion coefficient

$$\mathscr{D}=D_0 I/2v^2. \tag{4.2}$$

Expanding the expressions in (4.1) and making the subsitution $V_\parallel=v_\parallel-2Ik/v^3$, so that $V_\parallel$ describes the entire average velocity along the magnetic field, we readily get

$$\frac{\partial f}{\partial t}+V_\parallel\frac{\partial f}{\partial z}+\left\{F_{01}-\frac{\partial}{\partial z}\frac{I}{2v^2}-\left(\frac{Ik}{v^3}\right)'+\frac{3Ik^2F_{01}}{v^4}\right\}\frac{\partial f}{\partial V_\parallel}$$
$$=\left\{\frac{1}{2}\left(\frac{I}{v^2}\right)'-\frac{IkF_{01}}{v^3}\right\}\frac{\partial^2 f}{\partial V_\parallel^2}-\frac{I}{v^2}\frac{\partial^2 f}{\partial z\,\partial V_\parallel}, \tag{4.3}$$

where $(\ldots)'$ stands for $(\partial/\partial t+V_\parallel\partial/\partial z)(\ldots)$. The characteristics of the left-hand side of (4.3) coincide exactly with the corresponding averaged equations of motion of a charged particle in the field of a longitudinal quasi-monochromatic wave. [14] From (3.13) and (3.18) follow expressions for the alternating components of the distribution function:

$$\tilde{f}_1=-\frac{F_1}{v}\sin(\theta_2+\varphi_1)\frac{\partial f_0}{\partial v_\parallel}, \tag{4.4}$$

$$\tilde{f}_2=-\frac{F_1}{v}\sin(\theta_2+\varphi_1)\frac{\partial\tilde{f}_1}{\partial v_\parallel}-\frac{1}{v}D_0\left\{\frac{F_1}{v}\cos(\theta_2+\varphi_1)\frac{\partial f_0}{\partial v_\parallel}\right\}$$
$$-\frac{I}{2v}\cos 2(\theta_2+\varphi_1)\frac{\partial}{\partial v_\parallel}\left(\frac{1}{v}\frac{\partial f_0}{\partial v_\parallel}\right). \tag{4.5}$$

We determine now the averaged macroscopic quantities: the density

$$n=\int(f_0+\tilde{f}_1+\tilde{f}_2)\,d\mathbf{v}, \tag{4.6a}$$

the average velocity

$$u_\parallel=\frac{1}{n}\int v_\parallel(f_0+\tilde{f}_1+\tilde{f}_2)\,d\mathbf{v}. \tag{4.6b}$$

It is then easy to obtain from (4.1), with allowance for (3.7),

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial z}nu_\parallel=0, \tag{4.7}$$

$$\frac{\partial}{\partial t}nu_\parallel+\frac{\partial}{\partial z}n\left\{\langle v_\parallel^2\rangle-I\left\langle\frac{1}{v^2}\right\rangle\right\}=n\left(F_{01}+\frac{1}{m}\langle F\rangle\right). \tag{4.8}$$

Here $\langle\ldots\rangle$ denotes average quantities of the type (4.6b), and F is the force exerted by the RF wave on one particle. [14] It follows from linear plasma theory that

$$\left\langle\frac{1}{v^2}\right\rangle=\frac{m}{4\pi e^2}(1-\varepsilon), \tag{4.9}$$

where ε is the dielectric constant of the plasma. Equation (4.8) can then be rewritten in the form

$$\frac{\partial}{\partial t}mnu_\parallel+\frac{\partial}{\partial z}\left(mn\langle v_\parallel^2\rangle+\frac{\varepsilon-1}{4\pi}I_1\right)=n\left(F_{01}+\frac{\varepsilon-1}{8\pi n}\frac{\partial I_1}{\partial z}\right)$$
$$+\frac{k}{8\pi}\left\{\frac{\partial I_1}{\partial t}\frac{\partial\varepsilon}{\partial\omega}-\frac{\partial I_1}{\partial z}\frac{\partial\varepsilon}{\partial k}+I_1\left(\frac{\partial\omega}{\partial t}\frac{\partial^2\varepsilon}{\partial\omega^2}+\frac{\partial k}{\partial t}\frac{\partial^2\varepsilon}{\partial\omega\,\partial k}-\frac{\partial\omega}{\partial z}\frac{\partial^2\varepsilon}{\partial\omega\,\partial k}\right.\right.$$
$$\left.\left.-\frac{\partial k}{\partial z}\frac{\partial^2\varepsilon}{\partial k^2}-kF_{01}\frac{\partial^2\varepsilon}{\partial\omega^2}\right)\right\}. \tag{4.10}$$

If the frequency ν and the wave number k of the RF wave satisfy the dispersion equation $\varepsilon=0$, and if terms of order I_1 are neglected (as is done also in[15]), it follows that the total force exerted by the wave on the plasma is equal to the energy density gradient of the electric field, taken with a minus sign (cf. [14]).

The second term on the right-hand side of (4.10) is the averaged force acting on the plasma particles in the RF field. [17] At $k=0$ and $\varepsilon=1-\omega_p^2/\omega^2$, where ω_p is the plasma frequency, this force is potential with a quasipotential[16] $U=e^2E_1^2/4m\omega^2$.

5. We consider now the case of a quasi-stationary inhomogeneous field: $\nu=-\omega=\text{const}$, $\Omega=\text{const}$. The zeroth- and first-approximation drift equations coincide with (3.7). The second approximation equation takes, after rather laborious calculations, the form

$$D_0\bar{f}_2=\frac{\partial U}{\partial z}\frac{\partial f_0}{\partial v_\parallel}+\frac{I'}{2\omega^2}\frac{\partial^2 f_0}{\partial v_\parallel^2}-\frac{I}{\omega^2}\frac{\partial^2 f_0}{\partial v_\parallel\,\partial z}$$
$$-\frac{v_\perp}{2\Omega}\mathbf{e}_1\operatorname{rot}\mathbf{F}_0\frac{\partial f_0}{\partial\varepsilon v_\perp}+\frac{1}{\Omega}[\mathbf{F}_0\mathbf{e}_1]\frac{\partial f_0}{\partial\varepsilon\mathbf{r}}. \tag{5.1}$$

Where

$$U=\frac{I}{2\omega^2}+\frac{F_2^2+F_3^2-2(\Omega/\omega)F_2F_3\sin(\varphi_2-\varphi_3)}{4(\omega^2-\Omega^2)}$$

is the high-frequency quasipotential in the presence of a homogeneous magnetic field. [16] In the absence of a magnetic field ($\Omega=0$) and in the absence of the force F_0, Eq. (5.1) goes over into the kinetic equation, [10,11] but differs from the latter in that it has a second and third term with second derivatives. Under these conditions, Eq. (5.1) is analogous in the spatially-homogeneous case to the equation of the quasilinear theory[15] with a given external field.

6. In the presence of an RF field, the distribution function depends on the phase of the field θ_2. If the frequency of the RF field is close to the frequency of the cyclotron rotation of the particle in the field $\mathbf{B}_0$ (the resonance condition), then in accordance with the Bogolyubov–Mitropol'skiĭ general ideas[18] it is necessary to take into account also the dependence of the distribution frequency on the "resonant" phase difference. Thus, under resonance conditions, the distribution function takes the form

$$f=f(t, \mathbf{r}, \mathbf{v}, \theta_2, \psi), \tag{6.1}$$

where

$$\psi=s_1\theta_1+s_2\theta_2 \tag{6.2}$$

is the resonant phase difference; s_1 and s_2 are certain prime numbers, then the operators $\partial/\partial t$ and ∇ in the initial Vlasov kinetic equation must be defined not by (2.8) and (2.9) but in the form

$$\begin{aligned} &\frac{\partial}{\partial t}\to\frac{\partial}{\partial t}-\omega\frac{\partial}{\partial\theta_2}+\frac{\partial\psi}{\partial t}\frac{\partial}{\partial\psi}, \\ &\nabla\to\nabla+\mathbf{k}\frac{\partial}{\partial\theta_2}+\nabla\psi\frac{\partial}{\partial\psi}. \end{aligned} \tag{6.3}$$

We shall assume conditions (1.1), (1.2), and (1.4) to be satisfied. In the resonance region we have

$$s_1\Omega+s_2\nu=\varepsilon\Delta. \tag{6.4}$$

The resonance phase difference ψ must therefore be included among the slow variables. Then Eq. (2.1) can be written in the form

$$\nu\left(\frac{s_2}{s_1}\frac{\partial}{\partial\theta_1}-\frac{\partial}{\partial\theta_2}\right)f=\varepsilon\left\{\frac{\Delta}{s_1}\frac{\partial f}{\partial\theta_1}+Lf+M\frac{\partial f}{\partial\psi}\right\}, \tag{6.5}$$

where

$$\begin{aligned} M=&\Delta+s_1A_0+G_1\cos\theta_1+G_2\sin\theta_1+s_1(A_3\cos 2\theta_1\\ &+A_4\sin 2\theta_1+A_5\cos\theta_2+A_6\sin\theta_2+A_7\cos\theta_+\\ &+A_8\sin\theta_++A_9\cos\theta_-+A_{10}\sin\theta_-),\\ &G_{1,2}=s_1A_{1,2}+s_2v_\perp k_{2,3}, \end{aligned} \tag{6.6}$$

the operator L is defined by formula (2.11). The distribution function is sought in the form of the expansion (3.1). In the zeroth approximation we get from (6.5)

$$\left(\frac{s_2}{s_1}\frac{\partial}{\partial\theta_1}-\frac{\partial}{\partial\theta_2}\right)f_0=0, \tag{6.7}$$

from which it is seen that f_0 does not depend on the fast phases θ_1 and θ_2:

$$f_0=f_0(t, \mathbf{r}, v_\parallel, v_\perp, \psi).$$

In the first-order approximation

$$\nu\left(\frac{s_2}{s_1}\frac{\partial}{\partial\theta_1}-\frac{\partial}{\partial\theta_2}\right)f_1=L_0f_0+M_0\frac{\partial f_0}{\partial\psi}. \tag{6.8}$$

The functions f_i can be represented as sums of "constant" parts that depend on the resonant phase difference, and periodic parts that depend on the fast phases:

$$f_i=\langle f_i(\psi)\rangle+\tilde{f}_i. \tag{6.9}$$

Then, averaging (6.8) over the fast phases, we can obtain an equation for the function f_0:

$$\langle L_0f_0\rangle+\left\langle M_0\frac{\partial f_0}{\partial\psi}\right\rangle=0. \tag{6.10}$$

This is the zeroth-approximation drift-kinetic equation. Its actual form depends on the resonance under consideration.

For the alternating component $\tilde{f}_1$ we get from (6.8) and (6.10) the equation

$$\nu\left(\frac{s_2}{s_1}\frac{\partial}{\partial\theta_1}-\frac{\partial}{\partial\theta_2}\right)\tilde{f}_1=\widetilde{L_0f_0}+\widetilde{M_0\frac{\partial f_0}{\partial\psi}}. \tag{6.11}$$

The alternating parts of the functions can be represented in the form

$$\tilde{f}=\sum_m f^{(m)}(\psi)e^{i(m,\theta)}, \tag{6.12}$$

where

$$m\to(m_1, m_2), \quad (m, \theta)=m_1\theta_1+m_2\theta_2.$$

It is then easy to obtain from (6.11)

$$\tilde{f}_1=\sum_m\frac{\alpha_m+\beta_m}{i\nu(s_2m_1/s_1-m_2)}e^{i(m,\theta)}, \quad s_1m_2-s_2m_1\neq 0, \tag{6.13}$$

where α_m and β_m are the coefficients in the expansions

$$\widetilde{L_0f_0}=\sum_m\alpha_m e^{i(m,\theta)}, \quad \widetilde{M_0\frac{\partial f_0}{\partial\psi}}=\sum_m\beta_m e^{i(m,\theta)}.$$

To find the equation for the "dc" component of the function f_1, we must use (6.5) in the second approximation

$$\nu\left(\frac{s_2}{s_1}\frac{\partial}{\partial\theta_1}-\frac{\partial}{\partial\theta_2}\right)f_2=\frac{\Delta}{s_1}\frac{\partial f_1}{\partial\theta_1}+L_0f_1+L_1f_0+M_1\frac{\partial f_0}{\partial\psi}+M_0\frac{\partial f_1}{\partial\psi}. \tag{6.14}$$

After averaging over the fast phases, this yields the first-approximation drift-kinetic equation in the resonance region

$$\langle L_0f_1\rangle+\left\langle M_0\frac{\partial f_1}{\partial\psi}\right\rangle=-\langle L_1\rangle f_0-\langle M_1\rangle\frac{\partial f_0}{\partial\psi}. \tag{6.15}$$

The described procedure can be continued. The corresponding calculations, however, become exceedingly cumbersome.

7. We consider now the region of electron-cyclotron resonance (ECR). The zeroth-approximation drift-kinetic equation takes in this case the form

$$Kf_0\equiv(D_0+D_{70}\cos\psi+D_{80}\sin\psi)f_0+(\Delta+A_0+A_{70}\cos\psi+A_{80}\sin\psi)\frac{\partial f_0}{\partial\psi}=0. \tag{7.1}$$

Here

$$\psi=\theta_1+\theta_2, \quad \Delta=\Omega+\nu, \quad \Omega=eB_0/mc.$$

The characteristics of Eq. (7.1) coincide with the averaged equations of motion under the ECR conditions, which generalize the known Canobbio equations.[19,20]

The alternating part of the function f_1 is given, in accordance with (6.13), by the formula

$$\bar{f}_1=\alpha_1\cos\theta_1+\alpha_2\sin\theta_1+\alpha_3\cos 2\theta_1+\alpha_4\sin 2\theta_1$$
$$+\alpha_5\cos\theta_2+\alpha_6\sin\theta_2+\alpha_9\cos\theta_-+\alpha_{10}\sin\theta_-, \qquad (7.2)$$

where

$$\alpha_1=-\frac{1}{v}\left\{D_2+(A_2-k_3v_\perp)\frac{\partial}{\partial\psi}\right\}f_0,\quad \alpha_2=\frac{1}{v}\left\{D_1+(A_1-k_2v_\perp)\frac{\partial}{\partial\psi}\right\}f_0,$$
$$\alpha_3=-\frac{1}{2v}\left(D_4+A_4\frac{\partial}{\partial\psi}\right)f_0,\quad \alpha_4=\frac{1}{2v}\left(D_3+A_3\frac{\partial}{\partial\psi}\right)f_0,$$
$$\alpha_5=\frac{1}{v}\left(D_6+A_6\frac{\partial}{\partial\psi}\right)f_0,\quad \alpha_6=-\frac{1}{v}\left(D_5+A_5\frac{\partial}{\partial\psi}\right)f_0, \qquad (7.3)$$
$$\alpha_9=-\frac{1}{2v}\left(D_{10}^{(0)}+A_{10}^{(0)}\frac{\partial}{\partial\psi}\right)f_0,\quad \alpha_{10}=\frac{1}{2v}\left(D_9^{(0)}+A_9^{(0)}\frac{\partial}{\partial\psi}\right)f_0.$$

The first-approximation drift-kinetic equation takes in the ECR region the form

$$K\langle f_1\rangle=-{}^1/_2(D_1\alpha_1+D_2\alpha_2+A_1\alpha_2-A_2\alpha_1)$$
$$-\frac{1}{2}\sum_{1\leq i\leq 10}G_i\frac{\partial\alpha_i}{\partial\psi}+Q_1\cos\psi+Q_2\sin\psi. \qquad (7.4)$$

The formulas for the coefficients Q_i and G_i are quite complicated. In the absence of cyclotron-resonance conditions, the function f_0 does not depend on the phases at all. Then, after additional averaging over the phases, Eq. (7.4) goes over into an equation whose characteristics are the known equations of motion of a charged particle in the drift approximation.[1,2]

The obtained equations become much simpler in the case of a homogeneous magnetic field, and also at ω = const and $\mathbf{k}=0$. Under these conditions, (7.1) takes the form

$$\left\{\frac{\partial}{\partial t}+v_\parallel\frac{\partial}{\partial z}+F_{01}\frac{\partial}{\partial v_\parallel}+\frac{1}{2}\left[F_2\cos(\psi+\varphi_2)+F_3\sin(\psi+\varphi_3)\right]\frac{\partial}{\partial v_\perp}\right\}f_0$$
$$=-\left\{\Delta+\frac{1}{2v_\perp}\left[-F_2\sin(\psi+\varphi_2)+F_3\cos(\psi+\varphi_3)\right]\right\}\frac{\partial f_0}{\partial\psi}. \qquad (7.5)$$

Here $\Delta=\Omega-\omega$ and $F_i=-eE_i/m$. Outside the resonance region, Eq. (7.5) goes over into the drift-kinetic equation that describes the distribution of the Larmor-circle centers as $B_0\to\infty$.[5,6,21]

[1]A. I. Morozov and L. S. Solov'ev, in: Voprosy teorii plazmy (Problems of Plasma Theory), Vo. 2, Atomizdat, 1963, p. 194.
[2]D. V. Sivukhin, *ibid.*, Vol. 1, p. 7.
[3]R. D. Hazeltine, Plasma Physics **15**, 77 (1973).
[4]G. Chew, M. L. Goldberger, and F. Low. Proc. R. Soc. Lond. **A236**, 112 (1956).
[5]R. Z. Sagdeev, B. B. Kadomtsev, L. I. Rudakov, and A. A. Vedenov, Proc. Second Intern. Conf. on Peaceful Uses of Atomic Energy, Geneva, 1958.
[6]T. F. Volkov, in: Voprosy teorii plazmy (Problems of Plasma Theory), Vol. 4, Atomizdat, 1964, p. 3.
[7]V. P. Milant'ev, Zh. Prikl. Mekh. Tekh. Fiz. **1**, 3 (1971).
[8]R. A. Demirkhanov, Second Colloque International, Saclay, 1968; A. A. Ivanov, in: Vaprosy teorii plazmy (Problems of Plasma Theory), Vol. 6, Atomizdat, 1972, p. 139.
[9]N. E. Andreev, A. D. Kiriĭ, and V. P. Silin, Zh. Eksp. Teor. Fiz. **57**, 1024 (1969) [Sov. Phys. JETP **30**, 559 (1970)].
[10]A. G. Litvak, M. I. Petelin, and E. I. Yakubovich, Zh. Tekh. Fiz. **35**, 108 (1965) [Sov. Phys. Tech. Phys. **10**, 81 (1965)].
[11]A. G. Litvak and V. Yu. Trakhtengerts, Zh. Eksp. Teor. Fiz. **60**, 1702 (1971) [Sov. Phys. JETP **33**, 921 (1971)].
[12]A. A. Vedenov and L. I. Rudakov, in: Fizika plazmy i problema upravlyaemykh termoyadernykh reaktsii (Plasma Physics and the Problems of Controlled Thermonuclear Reactions), Vo. 4, Akad. Nauk SSSR, 1958, p. 42.
[13]V. I. Kariman, Nelineĭnye volny v dispergiruyushchikh sredakh (Nonlinear Waves in Dispersive Media), Nauka, 1973.
[14]R. Klima, Plasma Phys. **7**, 329 (1972).
[15]B. B. Kadomtsev, in: Voprosy teorii plazmy (Problems of Plasma Theory), Vol. 4, Atomizdat, 1964, p. 188.
[16]A. B. Gaponov and M. A. Miller, Zh. Eksp. Teor. Fiz. **34**, 242, 1958 [Sov. Phys. JETP **7**, 168 (1958)]; M. A. Miller, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **1**, 110 (1958).
[17]V. L. Ginzburg, Rasprostranenie elektromagnitnykh voln v plazme (Propagation of Electromagnetic Waves in a Plasma), Nauka, 1967, p. 331.
[18]N. N. Bogolyubov and Yu. A. Mitropol'skiĭ, Asimptoticheskie metody v teorii nelineĭnykh kolebaniĭ (Asymptotic Methods in the Theory of Nonlinear Oscillations), Nauka, 1974, p. 202.
[19]E. Canobbio, Nucl. Fusion **9**, 27 (1969).
[20]V. P. Milant'ev, Zh. Tekh. Fiz. **43**, 1405 (1973) [Sov. Phys. Tech. Phys. **18**, 893 (1974)].
[21]V. N. Tsytovich, Teoriya turbulentnoĭ plazmy (Theory of Turbulent Plasma), Atomizdat, 1971, p. 375.

Translated by J. G. Adashko