Influence of viscosity on the evolution of a Bianchi type II universe

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Cosmological solutions for a homogeneous Bianchi type II model are investigated with allowance for dissipative viscous processes. A number of general conclusions drawn by Belinskiĭ and Khalatnikov [Sov. Phys. JETP **42**, 205 (1975)] from the study of a type I model is confirmed, and qualitatively new kinds of cosmological singularities, both initial and final, are found. The evolution of the universe for this case is described. It is shown how similar calculations can also be made for the Bianchi types VI, VII, VIII, and IX, but also why there is no point in doing this.

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1. INTRODUCTION

The evolution of the universe has been considered many times. But in the majority of cases no allowance has been made for processes that dissipate energy, the energy-momentum tensor being taken in the form corresponding to a perfect fluid. All the processes in the universe are then, of course, reversible and the two types of cosmological singularity—creation and destruction—differ only in the time direction. If viscous dissipative processes are included by using the energy-momentum tensor of a viscous fluid in Einstein's equations, the picture may be very different. For example, Murphy[1] has given an example of an exactly solvable flat Friedmann model with allowance for second viscosity, in which the singularity is in a certain sense eliminated. This effect is however characteristic only of isotropic models, as was shown by Belinskiĭ and Kalatnikov[2] in their calculation of anisotropic cosmological models of Bianchi type I. They found that viscosity does not eliminate the singularity, though it does allow other possibilities. For example, there is a solution for which the energy density and the Hubble constant are constant. In this case, according to[2], the standard development of the universe is as follows. Initially, the energy density is negligibly small and the metric corresponds to the Kasner solution. The energy density then increases and subsequently varies in accordance with the Friedmann laws, which govern the model during the later stages of expansion. Thus, in this case the gravitational field creates matter. The contraction begins and ends in the Friedmann solution, but in the middle the universe becomes anisotropic. Thus, viscosity has an isotropizing effect.

In this paper we consider Bianchi type II models. The results confirm the general conclusions of Belinskiĭ and Kalatnikov.[2] Additional possibilities are also found here.

Models of the Bianchi types VI, VII, VIII, and IX can also be considered by the method used here. However, the resulting systems of equations are cumbersome and require the introduction of multidimensional phase spaces. Furthermore, the picture they give hardly differs qualitatively from the investigated ones.

But how closely does this picture resemble the real universe? Our calculation shows that viscosity can significantly affect the evolution of the universe, but this by no means exhausts all dissipative processes. Therefore, our results are no more than a model of the true evolution. In addition, it is only justified to use just the two coefficients of viscosity when the terms with higher derivatives of the velocity are small. It would seem that this is the case near the initial cosmological singularity and, generally speaking, at not very high energy densities. With regard to the final stages of evolution, we may hope that the picture does not change qualitatively when the following dissipative terms are taken into account.

2. FROM EINSTEIN'S EQUATIONS TO A DYNAMICAL SYSTEM

Our metric is of the form

$$ds^2=dt^2-(a^2\mathbf{l}_\alpha\mathbf{l}_\beta+b^2\mathbf{m}_\alpha\mathbf{m}_\beta+c^2\mathbf{n}_\alpha\mathbf{n}_\beta)\,dx^\alpha dx^\beta, \tag{1}$$

where the Greek indices take the values 1, 2, 3, (We assume that the velocity of light and the Einstein gravitational contant are equal to unity.) The energy-momentum tensor of a viscous fluid has the form[3]

$$\begin{gathered}T_{ik}=(\varepsilon+p')u_iu_k-p'g_{ik}-\eta(u_{i;k}+u_{k;i}-u_ku^lu_{i;l}-u_iu^lu_{k;l}),\\ p'=p+(\xi-{}^2/{}_3\eta)u^k{}_{;k},\quad u^ku_k=1.\end{gathered} \tag{2}$$

Here, ε is the energy density, p is the pressure, η and ξ are the coefficients of first and second viscosity, and u_i are the components of the four velocity. The indices i, k, l take the values 0, 1, 2, 3.

If the Einstein equations based on the metric (1) are to be compatible, it is necessary to work in a comoving frame, in which $u^0=1$, $u^\alpha=0$. Then the energy-momentum tensor takes the form

$$\begin{gathered}T_0{}^0=\varepsilon,\quad T_\alpha^0=0,\quad T_\alpha{}^\beta=p'\delta_\alpha{}^\beta+\eta\varkappa_\alpha{}^\beta,\\ p'=p-(\xi-{}^2/{}_3\eta)\ (\ln\gamma^{1/2})^{\cdot}.\end{gathered} \tag{3}$$

Here, γ is the determinant of the spatial tensor

$$\gamma_{\alpha\beta}=-g_{\alpha\beta}\text{ and }\varkappa_{\alpha\beta}=\dot\gamma_{\alpha\beta}.$$

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We resolve the spatial components of four-vectors and four-tensors with respect to a triplet of frame vectors of a Bianchi type II space[4]:

$$A_{(a)(b)}=A_{\alpha\beta}e^{\alpha}{}_{(a)}e^{\beta}{}_{(b)};\quad A_{0(a)}=A_{0\alpha}e^{\alpha}{}_{(a)},\ u^{(a)}=u^{\alpha}e_{\alpha}{}^{(a)}. \tag{4}$$

We obtain the following expressions for the frame components of the Ricci four-tensor:

$$R_0{}^0=-{}^1/_2\dot{\varkappa}^{(a)}_{(a)}-{}^1/_4\varkappa^{(b)}_{(a)}\varkappa^{(a)}_{(b)},$$
$$R^{0}_{(a)}=0,\quad R^{(b)}_{(a)}=-\frac{1}{2d^{1/2}}(d^{1/2}\varkappa^{(b)}_{(a)})^{\cdot}-P^{(b)}_{(a)}, \tag{5}$$

where d is the determinant of the tensor $\gamma_{(a)(b)}$, and $P^{(b)}_{(a)}$ is the spatial curvature tensor, equal in our case to

$$P^{(1)}_{(1)}=\frac{1}{2}\frac{a^2}{b^2c^2},\quad P^{(2)}_{(2)}=P^{(3)}_{(3)}=-\frac{1}{2}\frac{a^2}{b^2c^2}. \tag{6}$$

Indices are raised and lowered by means of the tensor

$$\gamma_{(a)(b)}=\begin{pmatrix} a^2 & 0 & 0\\ 0 & b^2 & 0\\ 0 & 0 & c^2\end{pmatrix}. \tag{7}$$

For the components $T^{(b)}_{(a)}$ we obtain

$$T^{(b)}_{(a)}=-p'\delta^{(b)}_{(a)}+\eta\varkappa^{(b)}_{(a)},\quad p'=p-(\xi-{}^2/_3\eta)(\ln d^{1/2})^{\cdot}. \tag{8}$$

We denote

$$\dot a/a=n_1,\quad \dot b/b=n_2,\quad \dot c/c=n_3,\quad \dot d/d=6H.$$

It is clear that $n_1+n_2+n_3=3H$. We call H the Hubble constant. We form the system of Einstein equations. The components a and b of these equations give

$$-\frac{1}{d^{1/2}}\frac{\partial}{\partial t}(d^{1/2}n_1)-\frac{1}{2}\frac{a^2}{b^2c^2}=-p'+2\eta n_1-\frac{1}{2}T_i{}^i,$$
$$-\frac{1}{d^{1/2}}\frac{\partial}{\partial t}(d^{1/2}n_2)+\frac{1}{2}\frac{a^2}{b^2c^2}=-p'+2\eta n_2-\frac{1}{2}T_i{}^i, \tag{9}$$
$$-\frac{1}{d^{1/2}}\frac{\partial}{\partial t}(d^{1/2}n_3)+\frac{1}{2}\frac{a^2}{b^2c^2}=-p'+2\eta n_3-\frac{1}{2}T_i{}^i,$$

whence

$$-\frac{1}{d^{1/2}}\frac{\partial}{\partial t}[d^{1/2}(n_2-n_3)]=2\eta(n_2-n_3), \tag{10}$$
$$-\frac{1}{d^{1/2}}\frac{\partial}{\partial t}[d^{1/2}(n_3-n_1)]+\frac{a^2}{b^2c^2}=2\eta(n_3-n_1)$$

or

$$n_2-n_3=\mathrm{const}\cdot e^{\psi}/d^{1/2},\quad \dot\psi=-2\eta; \tag{11}$$
$$n_3-n_1=A=\frac{e^{\psi}}{d^{1/2}}\left(\mathrm{const}+\int e^{-\psi}d^{1/2}\frac{a^2}{b^2c^2}dt\right). \tag{12}$$

Hence

$$n_1=H-{}^2/_3A+\alpha_1e^{\psi}/d^{1/2},\quad n_2=H+{}^1/_3A+\alpha_2e^{\psi}/d^{1/2},$$
$$n_3=H+{}^1/_3A+\alpha_3e^{\psi}/d^{1/2}, \tag{13}$$

where α_1, α_2, α_3 are constant and $\sum\alpha_\beta=0$. Substituting (13) into Eqs. (9), we obtain

$$\dot H=-3H^2+\varepsilon-{}^1/_2W+{}^3/_2\xi H+{}^1/_6N, \tag{14}$$

where W is the enthalpy, $W=\varepsilon+p$, and N denotes the ratio a^2/b^2c^2.

The 00 component of the Einstein equations gives

$$\varepsilon=3H^2-\frac{1}{4}N-\frac{1}{3}A^2-q^2\frac{e^{2\psi}}{d}+\alpha_1A\frac{e^{\psi}}{d^{1/2}}$$
$$=3H^2-\frac{1}{4}N-\frac{e^{2\psi}}{4d}(\alpha_2-\alpha_3)^2-\frac{3}{4}\left(\alpha_1\frac{e^{\psi}}{d^{1/2}}-\frac{2}{3}A\right)^2,\quad q^2=\frac{1}{2}\sum_{\beta=1}^{3}\alpha_\beta^2. \tag{15}$$

It can be seen from this that the state of the system must satisfy the inequality

$$\varepsilon\leqslant 3H^2-{}^1/_4N. \tag{16}$$

Differentiating Eq. (15) and then substituting the value $q^2e^{2\psi}/d$ from it, we obtain

$$\dot\varepsilon=9\xi H^2-3HW+4\eta(3H^2-\varepsilon-{}^1/_4N). \tag{17}$$

We have thus obtained the hydrodynamic equation $T^k_{i;k}=0$, which is already contained in Eqs. (15) and (16). As is shown in[2], this last equation is related to the law of increase of entropy. permitting here only evolutions corresponding to growth of the proper time t.

To complete the system, we need an equation for $\dot N$. It can be obtained by differentiating the relation

$$\dot N/N=4\alpha_1e^{\psi}/d^{1/2}-{}^8/_3A-2H.$$

We get

$$\frac{d}{dt}\left(\frac{\dot N}{N}\right)=-2\dot H-(2\eta+3H)\left(\frac{\dot N}{N}+2H\right)-\frac{8}{3}N. \tag{18}$$

Thus, the system has been obtained. To use the qualitative theory of dynamical systems, it is convenient to introduce the variable $p=\dot N/N$. Then

$$\dot p=-2\dot H-(2\eta+3H)(p+2H)-{}^8/_3N, \tag{19}$$
$$\dot N=pN. \tag{20}$$

One can obtain similarly complete systems of equations for the Bianchi types VI, VII, VIII, and IX. For this, it is necessary to obtain from the Einstein equations the expressions

$$n_2-n_1=A=\frac{e^{\psi}}{d^{1/2}}\int e^{-\psi}d^{1/2}(P^{(2)}_{(2)}-P^{(1)}_{(1)})dt,$$
$$n_3-n_2=B=\frac{e^{\psi}}{d^{1/2}}\int e^{-\psi}d^{1/2}(P^{(3)}_{(3)}-P^{(2)}_{(2)})dt,$$

where A and B are expressed in terms of integrals of the functions a, b, c. Thus, for Bianchi type VII

$$A=\frac{e^{\psi}}{d^{1/2}}\int e^{-\psi}d^{1/2}\frac{b^4-a^4}{a^2b^2c^2}dt,$$
$$B=\frac{e^{\psi}}{d^{1/2}}\int e^{-\psi}d^{1/2}\frac{b^2-a^2}{a^2c^2}dt.$$

Then writing

$$n_1 = H + \alpha_1 e^{\psi}/d^{1/3} - {}^2/_3 A - {}^1/_3 B,$$
$$n_2 = H + \alpha_2 e^{\psi}/d^{1/3} + {}^1/_3 A - {}^1/_3 B, \quad n_3 = H + \alpha_3 e^{\psi}/d^{1/3} + {}^1/_3 A + {}^2/_3 B,$$

we can readily obtain the expression

$$\dot{H} = -3H^2 + \varepsilon - {}^1/_2 W + {}^3/_2 \xi H - {}^1/_3 P_{(\alpha)}{}^{(\alpha)}.$$

However, in the expression for $\dot{\varepsilon}$ there is no cancellation of A, B, and α_β, in contrast to our case, in which the situation was made easier. Therefore, to obtain the system, we must also differentiate the resulting equation and then eliminate A, B, and α_β. Furthermore, the equations analogous to (18) become much more complicated. As a result, the dynamical system requires not four variables, as in our case, but at least eight. Therefore, despite the possibility of calculating exactly four more types, we do not do this, especially in view of the fact that the more complicated calculations give a result analogous to the one we have investigated.

3. INVESTIGATION OF THE SYSTEM

It remains to find the integral curves described by the equations (14), (17), (19), and (20). From physical considerations it is clear that N and ε cannot be negative. Therefore, the curves lie in a quarter of the ε, H, N space, on the outside of the parabolic cylinder $\varepsilon = 3H^2 - \frac{1}{4}N$. The plane $N=0$ is singular, since on it either $a=0$ or $bc=\infty$. If we consider formally the integral curves for $N=0$, they coincide completely with the curves constructed in[2]. It is easy to see that over a finite interval of time an integral curve that comes out of a point within the quarter space can reach neither the plane $N=0$ nor values $N=\infty$. Therefore, cosmological singularities can occur only at singular points of the system of equations.

Note that the trajectory of the system in the four-dimensional phase space depends on four constants, for example, the initial values of ε, H, N, and p. However, we do not know the initial value of p. Thus, we can obtain only a family of curves, which depend on one parameter. However, near the singular points this dependence frequently disappears—the difference between the trajectories of the family becomes infinitesimally small.

Let us consider the singular points for finite values of ε and H. Equating $\dot{\varepsilon}$, $\dot{H}$, and $\dot{N}$ to zero, we obtain two groups of solutions. One of them has $N<0$, i.e., it is unphysical. The other coincides completely with the singular points in[2]. These points are situated where $H \geqslant 0$, $N=0$, $\varepsilon = 3H^2$ and at them $\xi(\varepsilon) = W(\varepsilon)(3\varepsilon)^{-1/2}$. This last condition is certainly satisfied for $\varepsilon = H = 0$. Whether there are other singular points is determined by the dependence $\xi(\varepsilon)$.

Two separatrices lie in the plane $N=0$. One of them is the parabola $\varepsilon = 3H^2$. Its characteristic number is

$$\lambda_1 = 3\varepsilon_0 \left[\frac{d}{d\varepsilon}\left(\xi - \frac{W}{(3\varepsilon)^{1/2}}\right)\right]_0,$$

where the subscript zero is appended to quantities taken at a singular point. It can have either sign. At two neighboring singular points, the signs of λ_1 are necessarily opposite. If for large ε

$$\xi(\varepsilon) = W(\varepsilon)/(3\varepsilon)^{1/2},$$

then $\lambda_1 = 0$, and we obtain the singular line $\varepsilon = 3H^2$. However, this case does not lead to any interesting effects. The characteristic number of the second separatrix, which leads to the point $\varepsilon = N = 0$, $H = \infty$, $\lambda_2 = -6H_0 - 4\eta_0$, is definitely negative.

If we consider the system of equations near a singular point in the four-dimensional phase space, we find that to the given values of N, ε, and H there correspond two singular points with values $p_0 = -2H_0$ and $p_0 = \infty$. They have identical separatrices in the plane $N=0$. We have already given them. It would be meaningless to speak of a third separatrix in the N, ε, H space, or of any other phase trajectory. We have already mentioned this above. However, one could speak of its behavior near a singular point, since all lines of the family of trajectories have here the same asymptotic behavior. For the point with $p_0 = -2H_0$ this is

$$\frac{d\varepsilon}{dN} = -\frac{3}{2}\frac{\xi_0}{2H_0+\lambda_1}, \quad \frac{dH}{dN} = \frac{1}{24H_0}\frac{\lambda_1+2H_0-6\xi_0}{2H_0+\lambda_1}.$$

This straight line is tangent to the parabolic cylinder $\varepsilon = 3H - \frac{1}{4}N$, the boundary of the admissible region. The characteristic number corresponding to it, $\lambda_3 = -2H_0$, is also always negative. Therefore, depending on the sign of λ_1, the point is either an attracting node, or a saddle. One can show that these kinds of points must alternate, i.e., if we move along the parabola $N=0$, $\varepsilon = 3H^2$, a saddle must follow a node, and vice versa.

The solution of the equation near the singular point has the form $(C_1 > 0)$

$$\begin{gathered} N = C_1 e^{-2H_0 t}, \\ \varepsilon = \varepsilon_0 + C_2 e^{\lambda_1 t} + C_3 e^{\lambda_2 t} - \frac{3}{2}\frac{\xi_0}{2H_0+\lambda_1} C_1 e^{-2H_0 t}, \\ H = H_0 + \frac{C_2}{6H_0} e^{\lambda_1 t} - \frac{2\lambda_1 + 12H_0 - 3\xi_0}{6H_0(8\eta_0 + 3\xi_0)} C_3 e^{\lambda_2 t} + \frac{1}{24H_0}\frac{\lambda_1 + 2H_0 - 6\xi_0}{2H_0+\lambda_1} C_1 e^{-2H_0} \end{gathered} \tag{21}$$

Moreover, for a node,

$$d \sim e^{6H_0 t}, \quad a^2 \sim b^2 \sim c^2 \sim e^{2H_0 t} \tag{22}$$

as $t \to \infty$. This corresponds to the late stages of infinite isotropic expansion of the universe. This takes place in accordance with Friedmann, but the energy density does not tend to zero but some finite value. This is due to the influence of second viscosity. The energy of the motion of the expanding matter is dissipated and goes over into internal energy of the matter. The entropy density satisfies

$$\sigma \sim \exp\left(\int \frac{d\varepsilon}{W}\right) \to \text{const},$$

and the energy-momentum tensor takes the asymptotic form $T_0^0 = -\varepsilon_0$, $T_\alpha^\beta = -\varepsilon_0 \delta_\alpha^\beta$. This is similar to a perfect fluid with effective pressure $\bar{p} = -\varepsilon$.

The entropy per particle in a distinguished volume of space is proportional to $\int \sigma d^{1/2} d^3x$, i.e., it increases with the time like $d^{1/2}\sigma \sim e^{3H_0 t}$ as $t\to\infty$. Thus, the entropy per particle accumulates the whole time.

In the case $\lambda_1>0$, i.e., for a saddle, all this is true for $C_2=0$. The lines also end at a singular point. Here, the outgoing separatrix has no physical meaning, since it lies in the plane $N=0$. However, for Bianchi type I it is this separatrix that gives the solutions found by Murphy.[1]

Points with $p_0=\infty$ are completely different. For them $\lambda_3=+\infty$, i.e., they correspond to a saddle in all cases. However, since $|\lambda_3|\gg|\lambda_1|$, $|\lambda_3|\gg|\lambda_2|$, and the case $C_1=0$ corresponds to unphysical solutions, one of the separatrices suppresses the others. This is manifested in the fact that a system which for $t=-\infty$ is at the singular point leaves it parallel to the N axis. Moreover, for $t\to-\infty$

$$H=H_0,\quad \varepsilon=\varepsilon_0,\quad N\sim\exp[C\exp(-2\eta_0-3H_0)t],\tag{23}$$

where C is a negative constant. Having reached the maximal value of N, it returns, and either falls into the same singular point (corresponding to the same ε, H, N but with $p_0=-2H_0$), or into some other one, including infinity. As $t\to-\infty$,

$$\begin{gathered}d\sim\exp(6H_0t),\ a^2\sim\exp[3H_0t+{}^1/_2C\exp(-2\eta_0-3H_0)t],\\ b^2c^2\sim\exp[3H_0t-{}^1/_2C\exp(-2\eta_0-3H_0)t],\end{gathered}\tag{24}$$

i.e., $a^2\to0$, $b^2c^2\to\infty$.

As we see, this singularity corresponds to $t=-\infty$, which differs strongly from the singularities for Bianchi type I and the remaining singularities in our case. Physically, this is rapid isotropization of the universe, due basically to second viscosity, and is characteristic of type II. It is logical to assume that for types VI, VII, VIII, and IX we must also obtain analogous solutions.

Thus, we have considered singular points at finite values of ε. They all lie on the parabola $\varepsilon=3H^2$, $N=0$. It is clear from the inequality (16) that one cannot have singular points with $N=\infty$ and finite H.

We now consider the case of low and high energy densities. For this, we must particularize the equation of state of the matter and the asymptotic dependence of the coefficients of viscosity on the energy density. We set

$$W=\gamma\varepsilon,\quad 1\leqslant\gamma=\text{const}\leqslant2.\tag{25}$$

The analysis of the relativistic kinetic equations for simplified gas models mentioned in[2] shows that it is reasonable to assume at small ε

$$\begin{gathered}\eta=\eta_1(\varepsilon/\varepsilon_1)^{a_1},\ \xi=\xi_2(\varepsilon/\varepsilon_2)^{a_2};\\ a_1\geqslant1,\ a_2\geqslant1,\ \varepsilon\ll\varepsilon_1,\ \varepsilon_2.\end{gathered}\tag{26}$$

For high energy densities, we have

$$\begin{gathered}\eta=\eta_1(\varepsilon/\varepsilon_1)^{b_1},\quad \xi=\xi_2(\varepsilon/\varepsilon_2)^{b_2};\\ 0\leqslant b_2\leqslant{}^1/_2,\ b_1\geqslant b_2+{}^1/_2,\ \varepsilon\gg\varepsilon_1,\ \varepsilon_2.\end{gathered}\tag{27}$$

For small ε, we have singular points at $\varepsilon=H=N=0$ and at $\varepsilon=0$, $H=\pm\infty$. The straight line $\varepsilon=0$ passing through them is also a solution of the system, which has the form

$$\begin{gathered}\varepsilon=0,\quad H=\frac{1}{3t},\quad N=-\frac{2}{3}a^2C\frac{t^{a-2}}{(C-t^a)^2},\\ a=\text{const},\quad C=\text{const}.\end{gathered}\tag{28}$$

Near the singular points, it has the asymptotic behavior

$$\varepsilon=0,\quad H=1/3t,\quad N\sim t^{-2-2\Delta}\ (\Delta=\text{const}>0,\ t\to\pm\infty).\tag{29}$$

For the point $\varepsilon=H=N=0$ and

$$\varepsilon=0,\quad H=1/3t,\quad N\sim t^{-2+2\Delta}\ (\Delta=\text{const}>0,\ t\to\pm0)\tag{30}$$

for the points $\varepsilon=0$, $H=\pm\infty$. For $\varepsilon=0$, $H=-\infty$ this solution $(t\to-0)$ is unique since this point is a saddle and (30) describes the outgoing separatrix.

We here see clearly how the general solution (28) ceases to depend on the one constant near the singularities and takes on the Kasner form. Near the singular points, the coefficients a, b, and c have the form

$$a^2\sim t^{-\Delta},\quad b^2c^2\sim t^{2+\Delta},\quad d\sim t^2\tag{31}$$

for the coordinate origin and

$$a^2\sim t^{\Delta},\quad b^2c^2\sim t^{2-\Delta},\quad d\sim t^2\tag{32}$$

for the points $\varepsilon=0$, $H=\pm\infty$.

Besides this general solution, the Novikov solution also passes through the origin[5-7]:

$$H=\frac{2}{3\gamma t},\quad \varepsilon=\frac{6-\gamma}{4\gamma^2t^2},\quad N=\frac{(3\gamma-2)(2-\gamma)}{4\gamma^2t^2},\tag{33}$$

for which

$$a^2\sim t^{2/\gamma-1},\quad b^2c^2\sim t^{2/\gamma+1},\quad d\sim t^{4/\gamma}.\tag{34}$$

They both enter the node from the side of positive H and leave it in the direction of negative H. The ingoing curves $(t\to\infty)$ correspond to late stages of expansion; the outgoing curves $(t\to-\infty)$, to early stages of contraction of the universe.

Besides (30) the point $\varepsilon=0$, $H=\infty$ has an additional family of solutions. This point is a node, and integral curves begin at it. The coefficients of their asymptotic behaviors depend on which of the viscosities is dominant at low energy densities. Suppose, for example, $\eta>\xi$ (i.e., $a_1<a_2$) and in the limiting form of the system of equations

$$\begin{gathered}\dot H=-3H^2,\quad \dot\varepsilon=-3\gamma H\varepsilon+12\eta H^2+9\xi H^2,\\ \dot p=(\gamma-2)\varepsilon-3N-3Hp-3\xi H,\quad \dot N=pN\end{gathered}$$

we can ignore the last term on the right-hand side of the second equation. We then obtain the solution

$$H=\frac{1}{3t},\ N\sim t^{-2+2\Delta},\quad \varepsilon=\varepsilon_1\left[\frac{1+\gamma(a_1-1)}{a_1-1}\frac{3\varepsilon_1}{4\eta_1}t\right]^{1/(a_1-1)}\quad (t\to0)\tag{35}$$

with the same metric as in (32). But if $a_1 = 1$, then the expression (35) no longer holds. In it, we must set

$$\varepsilon \sim t^{-\gamma} \exp(-4\eta_1/3\varepsilon_1 t).$$

If second viscosity is dominant, only the numerical coefficients are changed in the equations.

Thus, the point $\varepsilon = 0$, $H = \infty$ corresponds to the start of cosmological expansion. And, because of the two viscosities, the singularity of the gravitational field creates matter.

Let us now consider the behavior of the solutions at high densities. The asymptotic behavior of the coefficients of viscosity (27) leads to three qualitatively different cases. Let us consider them successively.

1) The case $0 \leq b_2 < \frac{1}{2}$, $b_1 > \frac{1}{2}$. It is easy to show that the integral curves cannot go off to infinity with respect to ε for $H > 0$. This is possible only in the lower half-plane $H < 0$. Here, one can have two solutions, which lie on the surface of the parabolic cylinder $\varepsilon = 3H^2 - \frac{1}{4}N$. One of them is the Friedmann solution;

$$\begin{gathered} \varepsilon = \frac{4}{3\gamma^2\tau^2}, \quad H = \frac{2}{3\gamma\tau}, \quad N \sim \tau^{-4/3\gamma}; \\ a^2 \sim b^2 \sim c^2 \sim \tau^{4/3\gamma}, \quad d \sim \tau^{4/\gamma}, \quad \tau = t - t_1 < 0, \quad \tau \to 0. \end{gathered} \tag{36}$$

Here, t_1 is the finite instant of time corresponding to the cosmological singularity. The trajectories approach the Friedmann trajectory in accordance with the law

$$3H^2 - {}^1/_4 N - \varepsilon \sim \exp(-\rho^2 |\tau|^{1-2b_1}), \tag{37}$$

where ρ is a function of η_1, b_1, γ, and ε_1. It can be seen from this that the system's tending to the isotropic state in the final stages of contraction of the universe is a result of first viscosity.

There exists one further solution;

$$\begin{gathered} H = \delta/3\tau, \quad N \sim \tau^{-2}, \quad \varepsilon = 3H^2 - {}^1/_4 N \sim \tau^{-2}; \\ a^2 \sim \tau^{\delta-1}, \quad b^2c^2 \sim \tau^{\delta+1}, \quad d \sim \tau^{2\delta}. \end{gathered} \tag{38}$$

Here, δ is a quantity that depends irrationally on γ. For all permitted values of the argument we have $\delta \geq 1$, with equality achieved at $\gamma = 2$. This solution is asymptotic. Trajectories can separate from it, and we have

$$3H^2 - \varepsilon - {}^1/_4 N \sim |\tau|^{2b_1-3}. \tag{39}$$

However, this correction to the expression for the energy density increases more slowly than the energy density itself, and actually decreases for $b_1 > \frac{3}{2}$. This solution owes its existence to first viscosity and is peculiar to type II. It gives one further possibility for destruction of the universe in a non-Friedmann manner. Since allowance for the higher dissipative terms can significantly affect the solution in this region, we shall not consider it in detail.

2) For $b_2 = \frac{1}{2}$, $b_1 \geq 1$ we cannot ignore the terms with second viscosity in Eqs. (14) and (17). However, for $H < 0$ we obtain the same solutions, in which γ is replaced by $\gamma' = \gamma(1 + \beta)$, where

$$\beta = 3^{1/2}\xi_2/\gamma\varepsilon_2^{1/2}. \tag{40}$$

We note only that γ' cannot be greater than 2. The solution (38) exists for $\gamma' < \gamma_0 \approx 2.5$.

For $\beta > 1$, the integral curves can go off to infinity with respect to ε when $H > 0$ as well. The solutions corresponding to this process are obtained from (36) and (38) by the formal replacement of γ by $\gamma(1 - \beta) < 0$. They correspond to a second viscosity which is so large that the energy of the motion of the matter dissipated through through it suffices to increase the internal energy continuously, i.e., the supply of energy through dissipation exceeds its loss through expansion of the universe.

The case $\beta = 1$ corresponds to an infinite number of singular points on the parabola $\varepsilon = 3H^2$, $N = 0$. We have already described them above.

These solutions can however be changed significantly when allowance is made for higher viscosities. It may even happen that there are no solutions at all with $\varepsilon \to \infty$, $H \to +\infty$.

3) The case $b_2 = 0$, $b_1 = \frac{1}{2}$ corresponds to the minimal possible viscosities. In this case, the trajectories in the half-space $H > 0$ cannot of course go off to infinity. In the lower half-space, because of the insufficient isotropizing action of the first viscosity, new solutions occur in this case, and they depend on the constant ω:

$$\omega = \frac{4\eta_1}{3^{1/2}(2-\gamma)\varepsilon_1^{1/2}}. \tag{41}$$

For $\omega > 1$, the trajectories tend as before to (36), but more slowly:

$$3H^2 - \varepsilon - {}^1/_4 N \sim |\tau|^{2(2-\gamma)(\omega-1)/\gamma}. \tag{42}$$

But if $\omega < 1$, the integral curves tend to

$$\begin{gathered} H = \{[3 - 3\omega^2(1-\gamma/2)]\tau\}^{-1}; \quad \varepsilon = 3\omega^2 H^2, \quad N \sim \tau^{2/[3\omega^2(1-\gamma/2)-3]}, \\ a^2 \sim \tau^{2/[1-(1-\gamma/2)\omega^2]}, \quad b^2c^2 \sim a^4, \quad d \sim a^6. \end{gathered} \tag{43}$$

At the same time

$$3\omega^2 H^2 - \varepsilon - {}^1/_4 N \sim \tau^{\gamma/[\omega^2(1-\gamma/2)-1]}.$$

With regard to the second solution, it takes the form

$$H \sim \tau^{-1}, \quad \varepsilon \sim \tau^{-2}, \quad N \sim \tau^{-2}$$

with coefficients that cannot be calculated explicitly.

4. CONCLUSIONS

We can now consider the complete evolution of the universe. Its contraction begins with the Novikov regime and ends either with the isotropic Friedmann singularity, or the nonisotropic solution (38). These both correspond to the case $\varepsilon = \infty$, $H = -\infty$. Possibilities for expansion of the universe are much greater. Expansion can start from a Kasner singularity with zero energy density and infinite curvature invariants. Then, by matter creation by the gravitational field, the matter density can increase. The universe can also be created at a singular point. In constrast to the singularities listed above, this singularity does not correspond to a finite proper time. Moreover, the energy density and the Hubble constant have from the very start nonzero positive values.

The process of expansion ends either with the Novikov regime, or with the Friedmann solution with constant energy density and Hubble constant corresponding to the node (ε_0, H_0). In this case, second viscosity has an important influence until $t=\infty$, creating entropy per particle the whole time.

Knowing the asymptotic dependence of second viscosity at high energy densities, we can also say something about the singular points. For $b_2=\frac{1}{2}$, $\beta>1$ there will be an odd number of singular points, i.e., there will certainly be one. They will alternate in accordance with the rule saddle–node–...–saddle. In all other cases there will be either none or an even number. The order of succession is: node–saddle–...–node. Compared with the Bianchi type I case investigated in[2], there are new possibilities for destruction (see (38)) and creation (see (23), (24)) of the universe. Moreover, this last does not correspond to a finite proper time.

The nature of the solution during the late stages of expansion and early stages of contraction is changed. However, the cosmological singularity remains, as before, an inescapable attribute of the evolution of the universe, both for contraction and expansion.

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Spontaneous production of positrons by a Coulomb center in a homogeneous magnetic field.

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It is shown that in a strong magnetic field $H > Z^2 \times {}^9$Oe the threshold for spontaneous production of positrons by a Coulomb field of a bare nucleus decreases, i.e., the critical charge Z_c becomes lower than $Z_c \approx 170$ in the absence of a field. In particular, for $H \approx 5\times10^{15}$ Oe the critical charge decreases to the charge of uranium ($Z_c \approx 92$). The threshold probability for positron production is calculated and is found to grow with increasing field and turns out to be larger than in the absence of the field. It is emphasized that the problem under consideration is quasi-one-dimensional as a result of smallness of the Coulomb interaction compared to the interaction of an electron with the magnetic field. This is confirmed by a calculation of the degree of compression of the critical atom in the direction perpendicular to the magnetic field. An estimate is made of the effect of vacuum polarization by strong Coulomb and magnetic fields on the magnitude of the critical charge.

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1. INTRODUCTION

The process of spontaneous production of positrons by the Coulomb field of bare nuclei ($Z > Z_c \approx 170$) in the absence of external fields when the lowest electron level reaches the lower limit of the discrete spectrum: $\varepsilon = -mc^2$, was investigated in[1–3]. It is qualitatively clear that in a strong magnetic field for which the Larmor radius of the electron $l=(\hbar c/eH)^{1/2}$ is much smaller than the Bohr radius r_B the electron will experience a stronger attraction to the nucleus than in the absence of the field. Consequently, attainment of the lower limit of the discrete spectrum must occur for lighter nuclei with $Z < 170$, and the threshold for the spontaneous production of positrons by the Coulomb field is lowered.[1)]

In the present paper we investigate the motion of a bound relativistic electron in the Coulomb field of a sta-

