gy density and infinite curvature invariants. Then, by matter creation by the gravitational field, the matter density can increase. The universe can also be created at a singular point. In constrast to the singularities listed above, this singularity does not correspond to a finite proper time. Moreover, the energy density and the Hubble constant have from the very start nonzero positive values.

The process of expansion ends either with the Novikov regime, or with the Friedmann solution with constant energy density and Hubble constant corresponding to the node (ε_0, H_0). In this case, second viscosity has an important influence until $t=\infty$, creating entropy per particle the whole time.

Knowing the asymptotic dependence of second viscosity at high energy densities, we can also say something about the singular points. For $b_2=\frac{1}{2}$, $\beta>1$ there will be an odd number of singular points, i.e., there will certainly be one. They will alternate in accordance with the rule saddle–node–...–saddle. In all other cases there will be either none or an even number. The order of succession is: node–saddle–...–node. Compared with the Bianchi type I case investigated in[2], there are new possibilities for destruction (see (38)) and creation (see (23), (24)) of the universe. Moreover, this last does not correspond to a finite proper time.

The nature of the solution during the late stages of expansion and early stages of contraction is changed. However, the cosmological singularity remains, as before, an inescapable attribute of the evolution of the universe, both for contraction and expansion.

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Spontaneous production of positrons by a Coulomb center in a homogeneous magnetic field.

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It is shown that in a strong magnetic field $H \gg Z^2 \times {}^9$Oe the threshold for spontaneous production of positrons by a Coulomb field of a bare nucleus decreases, i.e., the critical charge Z_c becomes lower than $Z_c \approx 170$ in the absence of a field. In particular, for $H \approx 5\times10^{15}$ Oe the critical charge decreases to the charge of uranium ($Z_c \approx 92$). The threshold probability for positron production is calculated and is found to grow with increasing field and turns out to be larger than in the absence of the field. It is emphasized that the problem under consideration is quasi-one-dimensional as a result of smallness of the Coulomb interaction compared to the interaction of an electron with the magnetic field. This is confirmed by a calculation of the degree of compression of the critical atom in the direction perpendicular to the magnetic field. An estimate is made of the effect of vacuum polarization by strong Coulomb and magnetic fields on the magnitude of the critical charge.

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1. INTRODUCTION

The process of spontaneous production of positrons by the Coulomb field of bare nuclei ($Z>Z_c\approx170$) in the absence of external fields when the lowest electron level reaches the lower limit of the discrete spectrum: $\varepsilon=-mc^2$, was investigated in[1–3]. It is qualitatively clear that in a strong magnetic field for which the Larmor radius of the electron $l=(\hbar c/eH)^{1/2}$ is much smaller than the Bohr radius r_B the electron will experience a stronger attraction to the nucleus than in the absence of the field. Consequently, attainment of the lower limit of the discrete spectrum must occur for lighter nuclei with $Z<170$, and the threshold for the spontaneous production of positrons by the Coulomb field is lowered.[1)]

In the present paper we investigate the motion of a bound relativistic electron in the Coulomb field of a sta-

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tionary nucleus and a constant homogeneous strong magnetic field ($l/r_B \ll 1$). A discrete energy spectrum for the electron is obtained[2] and values have been obtained of the critical charge $Z_c(H)$ and of the degree of deformation of the K shell of the critical atom (Secs. 2, 3). Further, in Sec. 4 we have calculated the threshold probability for the production of positrons $W(H)$ and we have determined the behavior of the lowest electron level near the lower limit of the discrete spectrum. In Sec. 5 we have estimated the effect of vacuum polarization by strong Coulomb and magnetic fields on the value of the critical charge Z_c and we discuss the question of the spatial localization of the polarization charge in the transition through Z_c.

We shall use the system of units $\hbar = m = c = 1$, where m is the electron rest mass. With such a choice of the system of units, $r_B = \zeta^{-1}$, $l = (H_0/H)^{1/2}$, where $\zeta = Z/137$, while $H_0 = m^2c^3/e\hbar = 4.41 \times 10^{13}$ Oe is the quantizing magnetic field. We also take the magnetic field to be measured in units of H_0. We note that the quantity ζ^{-1} appearing in the formulas that follow is only approximately equal to the Bohr radius. The exact relationship of the average radius of the K shell of a relativistic atom to the quantity ζ has for $\zeta < 1$ the form $r_B = [1 + 2(1 - \zeta^2)^{1/2}]/2\zeta$, [1] but this is not of any importance for a qualitative interpretation of the results obtained.

2. THE ENERGY SPECTRUM

The Dirac equation for an electron in a Coulomb and a homogeneous magnetic field has the form

$$\mathscr{H}\psi = [\boldsymbol{\alpha}(\mathbf{p}+e\mathbf{A}) + V + \beta]\psi = \varepsilon\psi, \tag{1}$$

where

$$\mathbf{A} = (-{}^1/{}_2Hy, {}^1/{}_2Hx, 0), \quad V(r) = \begin{cases} -(\zeta/R)f(r/R), & r<R \\ -\zeta/r, & r>R \end{cases}, \tag{2}$$

the z axis is chosen along the magnetic field, R is the nuclear radius, while the cutoff function $f(x)$ depends on the distribution of the electric charge over the volume of the nucleus. In the presence of both a Coulomb and a magnetic field the component of the total angular momentum along the direction of the magnetic field is conserved ($[\hat{J}_z, \mathscr{H}] = 0$), and this enables us to separate, in a cylindrical system of coordinates $\{\rho, z, \theta\}$ the angular dependence in (1). Substituting in (1) the eigenfunction of $\hat{J}_z$:

$$\begin{gathered} \hat{J}_z\psi_M = M\psi_M, \quad M = \pm{}^1/{}_2, \pm{}^3/{}_2, \ldots, \\ \psi_M = \begin{pmatrix} \varphi_M \\ \chi_M \end{pmatrix}, \quad \varphi_M = \begin{pmatrix} c_1(\rho, z)e^{i(M-{}^1/{}_2)\theta} \\ c_2(\rho, z)e^{i(M+{}^1/{}_2)\theta} \end{pmatrix}, \quad \chi_M = \begin{pmatrix} b_1(\rho, z)e^{i(M-{}^1/{}_2)\theta} \\ b_2(\rho, z)e^{i(M+{}^1/{}_2)\theta} \end{pmatrix}, \end{gathered} \tag{3}$$

we obtain

$$\begin{gathered} (\varepsilon - V - 1)c_1 + i[b_{1z} + b_{2\rho} + (M + {}^1/{}_2)b_2/\rho + \rho b_2/2l^2] = 0, \\ (\varepsilon - V - 1)c_2 + i[b_{1\rho} - (M - {}^1/{}_2)b_1/\rho - \rho b_1/2l^2 - b_{2z}] = 0, \\ (\varepsilon - V + 1)b_1 + i[c_{1z} + c_{2\rho} + (M + {}^1/{}_2)c_2/\rho + \rho c_2/2l^2] = 0, \\ (\varepsilon - V + 1)b_2 + i[c_{1\rho} - (M - {}^1/{}_2)c_1/\rho - \rho c_1/2l^2 - c_{2z}] = 0, \end{gathered} \tag{4}$$

where $(f)_z \equiv \partial f/\partial z$ and $(f)_\rho \equiv \partial f/\partial \rho$.

For a strong magnetic field ($l \ll \zeta^{-1}$) the distance between the Landau levels $\Delta\varepsilon_H \sim l^{-1}$ greatly exceeds the separation between the discrete values of the Coulomb spectrum $\Delta\varepsilon_\zeta \lesssim 2$, and therefore, just as in the nonrelativistic case, we can assume that the energy spectrum of the electron consists of Landau levels each of which has Coulomb sublevels. Therefore, in the zero order approximation with respect to the small parameter $l\zeta$ one can utilize the well-known solution of the Dirac equation for an electron in a magnetic field and separate the wave functions for the transverse motion. We then have for the lowest Landau level;

$$\begin{gathered} M = -{}^1/{}_2, \quad b_1 = c_1 = 0, \\ c_2 = g(z)\exp(-\rho^2/4l^2), \quad b_2 = if(z)\exp(-\rho^2/4l^2). \end{gathered} \tag{5}$$

Substituting the functions (5) into the original system of equations (4) and averaging the two remaining equations over the transverse motion we obtain

$$g_z - (\varepsilon + 1 - \overline{V})f = 0, \quad f_z + (\varepsilon - 1 - \overline{V})g = 0, \tag{6}$$

where

$$\overline{V}(z) = \frac{1}{l^2}\int_0^\infty V(\sqrt{\rho^2+z^2})\exp(-\rho^2/2l^2)\rho\, d\rho. \tag{7}$$

It follows from (7) that the effective potential $\overline{V}(z)$ is a function of z^2, and therefore the system (6) remains invariant under the simultaneous replacement

$$z \to -z, \quad g(-z) \to \pm g(z), \quad f(-z) \to \mp f(z), \tag{8}$$

i.e., all the solutions can be classified according to their parity. An even solution $g(z)$ corresponds to an odd $f(z)$ and conversely. Therefore, it is sufficient to obtain the solutions of (6) only in the region $z > 0$.

It can be easily shown[6] that for $z \gg \max(l, R)$ the effective potential (7) becomes a one dimensional Coulomb potential:

$$\overline{V}(z) \approx -(\zeta/z)[1 - O(l^2/z^2)], \tag{9}$$

and this on substitution into (6) yields

$$g_z - (\varepsilon + 1 + \zeta/z)f = 0, \quad f_z + (\varepsilon - 1 + \zeta/z)g = 0. \tag{10}$$

Solutions of the system (10) decaying as $z \to \infty$ are linear combinations of the Whittaker functions

$$g(z) = A(1+\varepsilon)^{1/2}z^{-1/2}[(\zeta/\lambda)W_{\varkappa_1, i\zeta}(2\lambda z) + W_{\varkappa_2, i\zeta}(2\lambda z)], \tag{11}$$

$$f(z) = A(1-\varepsilon)^{1/2}z^{-1/2}[(\zeta/\lambda)W_{\varkappa_1, i\zeta}(2\lambda z) - W_{\varkappa_2, i\zeta}(2\lambda z)], \tag{12}$$

where

$$\lambda = (1-\varepsilon^2)^{1/2}, \quad \varkappa_1 = \zeta\varepsilon/\lambda - {}^1/{}_2, \quad \varkappa_2 = \zeta\varepsilon/\lambda + {}^1/{}_2, \tag{13}$$

A is a normalization constant.

In order to obtain the solutions in the interior region we make an estimate as to when it is possible to neglect the terms $\varepsilon + 1$ and $\varepsilon - 1$ in comparison with the potential $\overline{V}(z)$ in the system (6). Since $|\varepsilon| < 1$ in the region of the discrete spectrum, it is sufficient to require that the inequality $|\overline{V}(z)| \gg 2$ should be satisfied. Taking into account the fact that in accordance with (2) and (7) the potential $|\overline{V}(z)|$ decreases monotonically with increasing z, we can only strengthen the inequality by replacing in it $\overline{V}(z)$ by the Coulomb asymptotic value (9), and from this we obtain $|\overline{V}(z)| \geq \zeta/z \gg 2$. This means that at distances $z \ll \zeta/2$ the inequality $|\bar{V}(z)| \gg \max\{\varepsilon-1, \varepsilon+1\}$ will certainly be satisfied. Consequently, in this region the system (6) takes on the approximate form

$$g_z+\bar{V}f=0, \quad f_z-\bar{V}g=0, \tag{14}$$

and from this we have

$$g(z)=B_1\cos w(z)+B_2\sin w(z), \tag{15}$$

$$f(z)=B_1\sin w(z)-B_2\cos w(z), \tag{16}$$

where

$$w(z)=\int_0^z \bar{V}(z')\,dz' \tag{17}$$

while B_1 and B_2 are normalization constants.

Equating the logarithmic derivatives of solutions (11) and (15) at some point z_0 which lies in the region where they overlap $l \ll z_0 \ll \zeta/2$ we obtain the equation for the determination of the energy spectrum;

a) in the case of even levels ($B_2=0$)

$$-\bar{V}(z_0)\,\mathrm{tg}\, w(z_0)=\frac{d}{dz}\ln g(z)\Big|_{z=z_0}; \tag{18}$$

b) in the case of odd levels ($B_1=0$) it is necessary to replace in the left hand side of (18) $\tan w$ by $-\cot w$.

Since the ground energy level must be even due to the requirement that the wave function should have no nodes we shall below restrict ourselves to an investigation of Eq. (18). Since according to the condition on the choice of the point at which the solutions are joined $z_0 \gg l$ we can utilize for the potential $\bar{V}(z_0)$ the Coulomb asymptotic form (9) and, moreover, take into account the fact that

$$w(z_0)\approx-\zeta F\left(\frac{R}{l}\right)-\zeta\left[\ln\frac{z_0}{l}+\frac{1}{2}(\ln 2+C)+O\left(\frac{l^2}{z_0^2}\right)\right] \quad \text{as} \quad z_0/l\to\infty, \tag{19}$$

where $C=0.5772\ldots$ is the Euler constant, while

$$F(x)=\frac{1}{x}\int_0^x dy\, e^{y^2/2}\int_y^x\left[f\left(\frac{u}{x}\right)-\frac{x}{u}\right]e^{-u^2/2}u\,du \tag{20}$$

is a function which takes into account the finite nuclear size. On the other hand, $z_0 \ll \zeta/2 \lesssim 1$, and, therefore, one can replace the Whittaker functions appearing in the solution $g(z_0)$ by their expansion in the neighborhood of the origin

$$W_{\varkappa,i\mu}(x)\approx\sqrt{x}\left\{\frac{\Gamma(2i\mu)}{\Gamma(1/2+i\mu-\varkappa)}x^{-i\mu}+\frac{\Gamma(-2i\mu)}{\Gamma(1/2-i\mu-\varkappa)}x^{i\mu}\right\} \quad \text{as} \quad x\to 0. \tag{21}$$

When (9), (19), and (21) are taken into account, Eq. (18) is simplified and is transformed to the final form

$$\zeta\ln 2\lambda l+\mathrm{arctg}\frac{\lambda}{1-\varepsilon}+\arg\Gamma\left(-\frac{\zeta\varepsilon}{\lambda}+i\zeta\right)-\arg\Gamma(1+2i\zeta)$$
$$-\frac{\zeta}{2}(\ln 2+C)-\zeta F\left(\frac{R}{l}\right)=\frac{\pi}{2}+\pi n, \quad n=0,\pm1,\pm2,\ldots, \tag{22}$$

where the argument of the gamma-function is determined by the following expansion:

$$\arg\Gamma(x+iy)=-Cy+\sum_{k=1}^{\infty}\left[\frac{y}{k}-\mathrm{arctg}\frac{y}{x+k-1}\right]. \tag{23}$$

It follows from (22) that in the zero-order approximation with respect to the parameter $l\zeta$ the Coulomb sublevels of the lowest Landau level do not depend on the point at which the solutions are joined. By making the limiting transition to the nonrelativistic case ($\zeta \ll 1$, $1-\varepsilon \ll 1$), it can be easily verified that equation (22) yields the "one-dimensional" Coulomb spectrum obtained in the logarithmic approximation ($|\ln(l\zeta)| \gg 1$) in[6]:

$$\varepsilon_n=-\zeta^2/2n^2, \quad n=1,2,\ldots$$
$$\varepsilon_0=-\zeta^2/2n_0^2, \quad 1/n_0=|\ln(l\zeta)|-\ln\ln^2(l\zeta)+\ldots \tag{24}$$

It also follows from (22) that[3)]

$$\frac{\partial\varepsilon}{\partial(\ln H)}=-\frac{\lambda^3}{2[\operatorname{Im}\psi(-\zeta\varepsilon/\lambda+i\zeta)+\varepsilon\lambda-\lambda^2/2\zeta]}. \tag{25}$$

Substituting into the denominator of formula (25) the following representations for $\operatorname{Im}\psi(x+iy)$[11]:

$$\operatorname{Im}\psi(x+iy)=\sum_{k=0}^{\infty}\frac{y}{y^2+(x+k)^2}, \quad x\gtrless 0 \quad (\text{for } \varepsilon>0),$$
$$\operatorname{Im}\psi(x+iy)=\mathrm{arctg}\frac{y}{x}+\frac{y}{2(x^2+y^2)}$$
$$+4xy\int_0^{\infty}\frac{t\,dt}{[(t^2+x^2-y^2)^2+4x^2y^2](e^{2\pi t}-1)}, \quad x>0 \quad (\text{for } \varepsilon<0), \tag{26}$$

one can easily verify that the derivative (25) is negative for all values of $|\varepsilon| \leqslant 1$. Thus, the energy levels for a fixed ζ are lowered with increasing magnetic field, reaching the lower limit of the spectrum $\varepsilon=-1$ when the derivative has the value

$$\frac{\partial\varepsilon}{\partial\ln H}\Big|_{\varepsilon=-1}=-\frac{3\zeta_c^2}{1+4\zeta_c^2}<0. \tag{27}$$

Further in analogy with the derivation of (25) one can show that $\partial\varepsilon/\partial\zeta<0$ and attains for $\varepsilon=-1$ for the ground level the value

$$\frac{\partial\varepsilon}{\partial\zeta}\Big|_{\zeta=\zeta_c}=-\frac{3\pi}{1+4\zeta_c^2}\left\{1+\frac{4\zeta_c}{\pi}\left[\operatorname{Re}\psi(2i\zeta_c)-\frac{1}{2\zeta_c}\arg\Gamma(1+2i\zeta_c)-\frac{1}{2}\right]\right\}<0 \tag{28}$$

i.e., the ground level passes into the lower continuum for $\zeta>\zeta_c$.

3. THE CRITICAL CHARGE

We now consider the situation when the ground level of the electron falls on the limit of the lower continuum. Performing in this case the limiting transition $\varepsilon\to-1$ in formula (22) we obtain the transcendental equation implicitly determining the critical charge as a function of the magnetic field and of the nuclear radius:

$$\zeta_c\ln l\zeta_c+{}^1/_2\zeta_c(\ln 2-C)-\zeta_c F(R/l)-\arg\Gamma(1+2i\zeta_c)=\pi/2+\pi n. \tag{29}$$

As should have been expected, Eq. (29) contains the ratios of the three characteristic parameters of the problem; R, l, and r_B. (We recall that in the units adopted here $r_B=\zeta^{-1}$).

We make an estimate of the effect of the finite nuclear size. In order to do this we expand the function $F(x)$ in a series (cf. (20)) and obtain:

a) in the case that the charge is distributed over a sphere ($f(x) \equiv 1$)

$$F(x) = -{}^{1}/_{6}x^2 + {}^{1}/_{60}x^4 + o(x^4); \tag{30}$$

b) for a homogeneous distribution ($f(x) = (3 - x^2)/2$)

$$F(x) = -{}^{1}/_{10}x^2 + {}^{1}/_{140}x^4 + o(x^4). \tag{31}$$

From (30) and (31) it follows that even for superstrong fields in which the Larmor radius for the electron becomes equal to the nuclear radius ($l \sim R$ for $H \sim 10^4$, or, in ordinary units, $H \sim 10^{18}$ Oe), the correction due to the finite nuclear size is negligibly small against the background of the large logarithm $\ln l\zeta_c$. Such a weak effect of nuclear size is associated with the fact that the strong magnetic field ($l \leqslant \zeta/2$) itself cuts off the one-dimensional Coulomb potential making it finite at the origin. Indeed, substitution of the point Coulomb potential in (7) yields $\overline{V}(0) = -\zeta(\pi/2)^{1/2}/l$.

From the above statement it follows that in a strong magnetic field $R \lesssim l \ll \zeta/2$ one can neglect the effect of finite nuclear size on ζ_c, i.e., one can set $R = 0$ in (29). Then, taking into account the fact that the minimum value of ζ_c corresponds to the choice $n = -1$ one can solve (29) with respect to the intensity of the magnetic field $H = H(\zeta_c)$;

$$H = 2\zeta_c^2 \exp\left\{-C + \frac{\pi - 2\arg\Gamma(1+2i\zeta_c)}{\zeta_c}\right\}. \tag{32}$$

The generalization of this formula to the case of excited states $n \leqslant -1$, $M \leqslant -\frac{1}{2}$ leads to the following result:

$$H = 2\zeta_c^2 \exp\left\{\psi\left(\left|M + \frac{1}{2}\right| + 1\right) - \frac{(2n+1)\pi + 2\arg\Gamma(1+2i\zeta_c)}{\zeta_c}\right\}. \tag{32'}$$

Equations (32) and (32′) were solved numerically. The results of the calculation are shown in Fig. 1. The condition $R \lesssim l \ll \zeta/2$ restricts the domain of applicability of these formulas to fields $1 \ll H \lesssim 10^4$. For the ground state $n = -1$, $M = -\frac{1}{2}$ the critical charge decreases monotonically with increasing field from the value $\zeta_c = 0.7$ ($Z_c = 96$) for $H = 10^2$ to $\zeta_c = 0.3$ ($Z_c = 41$) for $H = 2.4 \times 10^4$. For example, the uranium nucleus ($Z_c = 92$) becomes unstable with respect to the production of positrons in a field $H \approx 5.5 \times 10^{15}$ Oe. The next level for uranium, $n = -1$, $M = -\frac{3}{2}$, reaches the limit of the spectrum $\varepsilon = -1$ in a field which is larger than the indicated field by a factor of $e \approx 2.7$ (cf. curve 2).

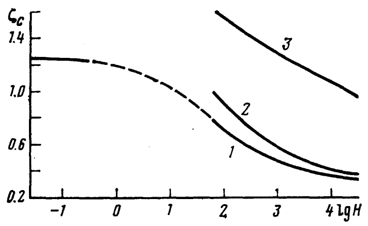

FIG. 1. Dependence of the critical charge on the value of the magnetic field: curve 1 corresponds to the ground level $n = -1$, $M = -\frac{1}{2}$; 2 corresponds to the level $n = -1$, $M = -\frac{3}{2}$; 3 corresponds to the level $n = -2$, $M = -\frac{1}{2}$.

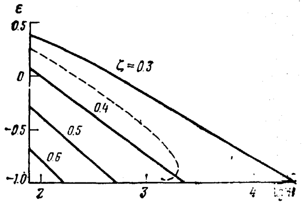

FIG. 2. Dependence of the ground energy level on the value of the magnetic field for different values of ζ; the dotted line is drawn according to Kraĭnov[10] for $\zeta = 0.3$.

On the other hand, in the limit of weak magnetic fields $H \ll 1$ one can estimate the change in the critical charge using the results of[3] (cf. formula (2.39)):

$$\varepsilon(\zeta) = -1 + 3\Lambda^3(\zeta_c - \zeta)/5\pi^2, \quad \varepsilon \to -1,$$

where $\Lambda = -\ln R$. Setting the small correction to the level $\varepsilon = -1$ equal to the Zeeman splitting: $|\Delta\varepsilon_H| = \mu g |M| H/2$, where in accordance with[12] $\mu = (8\zeta_c^2 - 3)/(6\zeta_c^2 + 9) \approx \frac{1}{3}$, we obtain for $Z_c \sim 170$ ($\Lambda \approx 4.2$)

$$g|M|H \approx 18(\zeta_c - \zeta). \tag{33}$$

In the ground state $|M| = \frac{1}{2}$, the Landé factor is $g = 2$, and therefore even for fields $H \sim 0.1$ the decrease in the critical charge is insignificant, $Z_c - Z \lesssim 1$. In Fig. 1 these fields correspond to the straight line in the initial segment of the dependence $\zeta_c(H)$ (curve 1). In the intermediate region $0.1 < H < 10^2$ which corresponds to the parameter $l\zeta \sim 1$ no calculations were carried out. The qualitative form of the dependence $\zeta_c(H)$ is this region is shown in the diagram by the dotted line.

From expression (22), fixing ζ and neglecting, as above, the dimensions of the nucleus, one can obtain the dependence of the energy of the ground level $n = -1$, $M = -\frac{1}{2}$ on the value of the magnetic field. Within the range of magnetic fields under consideration $10^2 \leqslant H \lesssim 10^4$ this dependence is shown graphically in Fig. 2 where for comparison we have also shown the analogous dependence $\varepsilon(H)$ for $\zeta = 0.3$ in accordance with the results of the paper by Kraĭnov[10] (a discussion of the differences is given in the Appendix).

It is also of interest to compare the transverse dimension of the atom l with its linear dimension along the z axis in the ground state $\varepsilon = -1$. The normalized one-dimensional wave functions in this case are equal to

$$g(z) = A[\theta(z_0 - z)\cos w(z) + B\theta(z - z_0)K_{2i\zeta_c}(\tau)], \tag{34}$$

$$f(z) = A\left[\theta(z_0 - z)\sin w(z) + \frac{B}{2\zeta_c}\theta(z - z_0)\tau\frac{d}{d\tau}K_{2i\zeta_c}(\tau)\right], \tag{35}$$

where

$$\tau = (8\zeta_c z)^{1/2}, \quad A = \left(\frac{3\zeta_c}{1+4\zeta_c^2}\right)^{1/2}, \quad B = \left(\frac{2\zeta_c \operatorname{sh} 2\pi\zeta_c}{\pi}\right)^{1/2}.$$

Here just as in the three-dimensional case in the absence of a magnetic field[3] the transition to the level $\mathcal{E} = -1$ corresponds to the degeneration of the Whittaker functions (cf. (11), (12)) into the Macdonald functions $K_\mu(x)$.

A calculation of the root-mean-square longitudinal size of the atom $\langle z^2\rangle^{1/2}/2$ yields

$$\langle z^2\rangle \approx 2(AB)^2\int_0^\infty z^2\,dz\left[K^2_{2i\zeta_c}(\sqrt{8\zeta_c z})+\frac{2z}{\zeta_c}K'^2_{2i\zeta_c}(\sqrt{8\zeta_c z})\right]=\frac{(9+4\zeta_c^2)(1+\zeta_c^2)}{35\zeta_c^2},$$

from which we obtain

$$\frac{\langle z^2\rangle^{1/2}}{2}=\frac{1}{2\zeta_c}\left[\frac{(9+4\zeta_c^2)(1+\zeta_c^2)}{35}\right]^{1/2} \tag{36}$$

Calculations according to this formula show that as the field is varied in the range $10^2 < H < 10^4$ the longitudinal dimension grows insignificantly from 0.5 to 0.9 while the degree of deformation of the K-shell of the atom, $\langle z^2\rangle^{1/2}/2l$ grows sharply from 5 to 140. Therefore it is reasonable to speak not of an elongation of the K shell along the magnetic field but of its compression in the transverse plane. An analogous result is apparently valid also for excited states.

4. THE THRESHOLD PROBABILITY OF POSITRON PRODUCTION.

As is well known, [2] in order to determine the threshold probability for the emergence of a positron it is necessary to calculate the width Γ of the quasistationary level $\varepsilon=-1+i\Gamma$ obtained by an analytic continuation of the function $\varepsilon(\zeta)$ into the region $\zeta > \zeta_c$. In order to carry this out we divide equation (22) by ζ, (29) by ζ_c and subtract the second from the first. As a result we obtain

$$f(\varepsilon,\zeta)=A(\zeta,\zeta_c), \tag{37}$$

where

$$f(\varepsilon,\zeta)=-\frac{1}{\pi\zeta}\left\{\zeta\ln\frac{\lambda}{\zeta}+\operatorname{arctg}\frac{\lambda}{1-\varepsilon}+\arg\Gamma\left(-\frac{\zeta\varepsilon}{\lambda}+i\zeta\right)\right\}, \tag{38}$$

$$A(\zeta,\zeta_c)=\frac{1}{2}\left(\frac{1}{\zeta}-\frac{1}{\zeta_c}\right)+\frac{1}{\pi}\left[\ln\frac{\zeta}{\zeta_c}+\frac{1}{\zeta_c}\arg\Gamma(1+2i\zeta_c)-\frac{1}{\zeta}\arg\Gamma(1+2i\zeta)\right]. \tag{39}$$

We consider the function $f(\varepsilon,\zeta)$. It follows from (38) that for $\varepsilon<-1$ it acquires an imaginary part

$$\operatorname{Im} f(\varepsilon,\zeta)=-\frac{1}{4\pi\zeta}\ln\left(\frac{1-\exp\{-2\pi\zeta-2\pi\zeta|\varepsilon|/(\varepsilon^2-1)^{1/2}\}}{1-\exp\{2\pi\zeta-2\pi\zeta|\varepsilon|/(\varepsilon^2-1)^{1/2}\}}\right), \tag{40}$$

which near $\varepsilon=-1$ is exponentially small:

$$\operatorname{Im} f(\varepsilon,\zeta)\approx-\frac{\operatorname{sh}2\pi\zeta_c}{2\pi\zeta_c}\exp\left(-\frac{2\pi\zeta_c|\varepsilon|}{(\varepsilon^2-1)^{1/2}}\right)\quad\text{as}\quad \varepsilon\to-1. \tag{41}$$

As $\varepsilon\to-1$, Eq. (37) assumes the form

$$\frac{(1+4\zeta_c^2)(1+\varepsilon)}{6\pi\zeta_c^2}-i\frac{\operatorname{sh}2\pi\zeta_c}{2\pi\zeta_c}\exp\left(-\frac{2\pi\zeta_c|\varepsilon|}{(\varepsilon^2-1)^{1/2}}\right)\theta(-1-\varepsilon)=A(\zeta,\zeta_c). \tag{37'}$$

We seek its solution in the form $\varepsilon=\varepsilon'+i\varepsilon''$ with a small imaginary part $\varepsilon''\ll\varepsilon'$. Then we have

$$\varepsilon=-\left(1+\frac{k^2}{2}\right)+i\frac{3\operatorname{sh}2\pi\zeta_c}{1+4\zeta_c^2}\exp\left(-\frac{2\pi\zeta_c}{k}\right). \tag{42}$$

where

$$k=2\zeta_c\left(\frac{-3\pi A(\zeta,\zeta_c)}{1+4\zeta_c^2}\right)^{1/2}. \tag{43}$$

From this the desired threshold probability for the emergence of the positron[4] is equal to

$$W(\zeta,\zeta_c)=\frac{6\zeta_c\operatorname{sh}2\pi\zeta_c}{1+4\zeta_c^2}\exp\left(-\frac{2\pi\zeta_c}{k}\right). \tag{44}$$

The dependence of the probability of creation of e^+ on the magnetic field calculated according to this formula is shown in Fig. 3. Greater values of the threshold probability compared with the analogous effect in the absence of a field[2,5] can be easily explained by calculating the transparency of the Coulomb barrier for positrons. Indeed, by combining the basic one-dimensional equations (6) we obtain the equation

$$\left\{\frac{d^2}{dz^2}+(\varepsilon+1-\bar V)^{-1}\frac{d\bar V}{dz}\frac{d}{dz}+[(\varepsilon-\bar V)^2-1]\right\}g(z)=0,$$

which by the replacement $g(z)=(\varepsilon+1-\bar V)^{1/2}\chi(z)$ reduces to an equation of the Schrödinger type;

$$\chi_{zz}+\varkappa^2\chi=0, \tag{45}$$

where $\varkappa^2=2(E-U)$, $E=(\varepsilon^2-1)/2$, while the effective potential is

$$U=-\frac{1}{2}\bar V^2+\varepsilon\bar V+\frac{\bar V_{zz}}{4(\varepsilon+1-\bar V)}+\frac{3}{8}\left(\frac{\bar V_z}{\varepsilon+1-\bar V}\right)^2,\quad \bar V_z=\frac{d\bar V}{dz}. \tag{46}$$

Setting in (46) $\varepsilon\approx-1$ we find that in the region $z\gg l$

$$U\approx-\frac{\zeta\varepsilon}{z}-\frac{\zeta^2+1/4}{z^2}. \tag{46'}$$

A graph of the function (46') is shown in Fig. 4. For comparison we also show the value of the analogous potential from the review of Zel'dovich and Popov. [3] It can be easily seen that with increasing magnetic field the height of the barrier is lowered since the value of the critical charge is reduced. Therefore the transparency of the one-dimensional barrier

$$D\sim\exp\left\{-2\int_{z_1}^{z_2}[2(U-E)]^{1/2}dz\right\}=\exp\left\{-\frac{2\pi\zeta_c|\varepsilon|}{k}\right\},$$

which with an accuracy up to the expression in front of the exponential coincides with the probability (44), turns out to be greater when the field is switched on than in the well-known case[3] of free atoms.

It is obvious that the state $\varepsilon=-1$ is a localized one ($D=0$). This can also be seen from the expression for the wave function for the ground state (cf. (34))

$$c_2(r)\sim K_{2i\zeta_c}(\sqrt{8\zeta_c z})\exp\left(-\frac{\rho^2}{4l^2}\right)\sim\frac{1}{z^{1/4}}\exp\left\{-\sqrt{8\zeta_c z}-\frac{\rho^2}{4l^2}\right\},\quad z\to\infty.$$

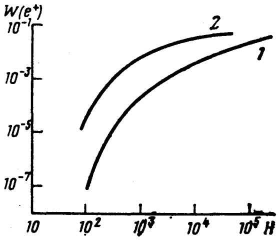

FIG. 3. Dependence of the probability of positron production on the intensity of the magnetic field (in units of $mc^2/\hbar$): curve 1 represents the threshold probability for $Z=Z_c+1$, 2—for $Z=Z_c+2$.

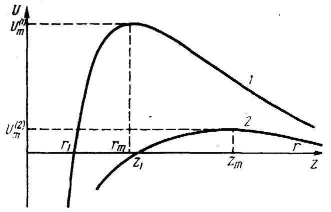

FIG. 4. The effective potential U: 1—in the absence of a field ($\zeta_0 = 1.25$) and 2—for the field $H = 2.4\times10^{16}$ Oe ($\zeta_c = 0.5$). The parameters are:

$$r_1 = \zeta_0/2 - 1/2\zeta_0, \quad r_m = 2r_1,$$
$$r_2 = 2\zeta_0/(\varepsilon^2-1) \to +\infty, \quad U_m^{(1)} = {}^1\!/_2 + 1/2(\zeta_0^2-1);$$
$$z_1 = \zeta_c/2 + 1/8\zeta_c, \quad z_m = 2z_1,$$
$$z_2 = 2\zeta_c/(\varepsilon^2-1) \to +\infty, \quad U_m^{(2)} = {}^1\!/_2 - 1/8(\zeta_c^2 + {}^1\!/_4)$$

5. VACUUM POLARIZATION, RADIATION CORRECTIONS AND THE CRITICAL CHARGE.

The change in the Coulomb potential (2) associated with the deviation of the permittivity of vacuum from unity in strong fields will have only a small effect on the spectrum $\varepsilon(\zeta)$ and on the value of the critical charge. In order to confirm this it is sufficient to examine the Heisenberg–Euler Lagrangian for constant and homogeneous electric and magnetic fields. [13] It was shown by Migdal[14] that a sufficiently strong field can indeed be regarded as homogeneous even at small distances from the Coulomb center and this enables us in our case to obtain from the Lagrangian indicated above the permittivity of vacuum near an external charge of arbitrarily small radius.

Along the z axis where the E and H fields are parallel we have with logarithmic accuracy for the induction $Ze/4\pi r^2 = \partial\mathscr{L}/\partial E$ and correspondingly for the intensity[13]

$$E = \frac{Ze}{4\pi r^2}\left(1 + \frac{\alpha}{3\pi}\ln\frac{H}{H_0}\right), \quad \text{if} \quad H \gg H_0, E, \tag{47}$$

and the well known result due to Uehling

$$E = \frac{Ze}{4\pi r^2}\left(1 + \frac{\alpha}{3\pi}\ln\frac{E}{E_0}\right), \quad \text{if} \quad E \gg H \gg H_0.$$

In the plane perpendicular to the magnetic field and passing through the center of the nucleus we obtain with the same logarithmic accuracy

$$E = \frac{Ze}{4\pi r^2}\left[1 + \frac{\alpha}{3\pi}\ln\frac{(E^2-H^2)^{1/2}}{H_0}\right], \quad \text{if} \quad E \gtrsim H \gg H_0. \tag{48}$$

From (47) and (48) it follows that the depth of potential (2) increases insignificantly by $|\Delta V| \sim \alpha\zeta/r$, and this leads to a small negative shift of the energy levels $\sim \alpha mc^2$ and correspondingly to a small decrease in the critical charge ($\sim \alpha Z_c < 1$) in agreement with the assertion made at the beginning of this section. We note that the accuracy of these estimates which are valid, generally speaking, in the Born approximation $\zeta \ll 1$ improves with increasing field H due to a decrease in ζ_c ($\zeta_c \sim 0.3$ for $H \sim 10^4$).

The interaction of an electron with the photon vacuum which leads to positive additions to the potential (2) due to the self-energy and vertex parts of the Lamb shift has not been investigated for strong fields ($\zeta > 1$) even in the absence of a magnetic field. In our case one can expect that the well-known estimates which are valid in the Born approximation will describe correctly the small increase in the critical charge by an amount $\alpha Z_c \lesssim 1$ corresponding to this interaction.

The interaction of an electron with the photon vacuum in a superstrong magnetic field $H \gg 1$ leads to a considerable change in the anomalous magnetic moment of the electron[15]:

$$\frac{\Delta\mu}{\mu_B} \approx \frac{\alpha}{4\pi H}[\ln^2 2H - C\ln 2H], \tag{49}$$

and this enables us to remove the well-known limitation on the value of the magnetic field $H \lesssim \alpha^{-1}$ ($\sim 10^{16}$ Oe) which follows from the solution of the Foldy equation if the standard value of the anomalous magnetic moment $\Delta\mu/\mu_B = -\alpha/2\pi$ is utilized. Indeed, from formula (49) it follows that in the range of fields under investigation $1 \ll H \lesssim 10^4$ the contribution to the energy is $H\Delta\mu \ll 1$, i.e., it has only a small effect on the results obtained above.

Finally, a few words on the distribution of the polarization charge and on its magnitude. Here the situation must be analogous to that discussed in[3]. As in the case of the absence of a field, the localization of the polarization charge follows the density inside the barrier of the quasistationary state characterized by the energy (42). Therefore the cloud of the vacuum charge is compressed in the transverse direction ($\perp \mathbf{H}$). In passing through Z_c the density of the vacuum charge $\rho_{vac}(r)$ undergoes a discontinuity associated with the distortion of the wave functions of the lower continuum by the discrete level approaching it:

$$\Delta\rho_{vac}(r) = -e\rho_c(r), \quad \rho_c(r) = b_2^2(r) + c_2^2(r)$$

(cf. (5), (34), (35)). The total vacuum charge for $\zeta > \zeta_c$ is equal to $-e$, i.e., it is smaller by a factor of two than in the case considered in[3] due to the lifting by the magnetic field of the degeneracy with respect to spin.

6. CONCLUSION

Thus, in a strong magnetic field $H \gg \zeta^2$ the threshold for the spontaneous production of positrons by a Coulomb center is appreciably lowered, and the threshold probability of production turns out to be higher than in the absence of the field. This effect is possible because of the exact compensation of the diamagnetic and paramagnetic contributions to the ground state for particles of spin $s = \frac{1}{2}$, since only for them is the Landau ground level $\varepsilon = (m^2 + p_z^2)^{1/2}$ independent of the field. For scalar particles ($s = 0$) in a strong magnetic field the Landau ground level rises as the intensity H increases, $\varepsilon = (m^2 + p_z^2 + 2/l^2)^{1/2}$, and this in the problem under consideration involving a Coulomb center would lead to an increase in the critical charge.

Another possible electrodynamic process for a Coulomb center in a strong constant electromagnetic field—the production of free relativistic electron-positron pairs—was discussed in the Born approximation ($\zeta \ll 1$) by Narozhnyĭ and Nikishov.[16] As is shown in[16], up to fields $H \sim 10^2$ (in units of H_0) the approximation of a stationary center for such light nuclei turns out to be justified. In our problem in the same field $H \sim 10^2$ the critical charge is $Z_c \sim 90$, so that the approximation of a stationary center can be violated only for very large values of the field which would displace Z_c in the direction of light elements. In the latter case there arises not only the problem of finding bound states in a relativistic two-body problem in an external field, but also the necessity to take into account the change in the electromagnetic structure of the nucleus.

We note possible applications of the problem considered in this paper. According to the available calculations of the equilibrium nuclear composition of the crust of a pulsar one can expect an increase in the charge Z and the mass number A as the center of the star is approached, where the neutron-rich bare nuclei decay adding to the free neutron liquid. On the other hand, the frozen-in magnetic field grows as the density of matter increases according to the law $H = H_p(\rho/\rho_p)^{2/3}$, where $\rho_p \sim 10^6$ g/cm^3 and $H_p \sim 10^{12}$ Oe are the density and the magnetic field at the surface of the pulsar. Therefore, for a certain heavy nucleus Z one can expect that the magnetic field will exceed the critical field H_c (cf. formula (32)) needed for the switching on of the process of spontaneous production of positrons considered in this paper. Indeed, according to the estimates of Kirzhnits[17] without taking the magnetic field into account we have within the inner crust of the pulsar $\rho \sim 4 \times 10^{13}$ g/cm^3, which corresponds to a field $H \sim 10^{17}$ Oe, while the average charge is $Z \sim 74$, i.e., greater than $Z_c(H)$. Moreover, in the above magnetic field the interaction $\mu_n H \sim 1$ MeV, i.e., it is necessary to modify the Bethe–Weizsäcker formula, and it is desirable to take this as well as the spontaneous production of positrons into account in calculating the equilibrium nuclear composition of a pulsar crust.

Another possible application of the asymptotic solution of the Dirac equation for a Coulomb center in a strong magnetic field obtained here can be in the investigation of the problem of magnetic freezing-out in semiconductors with a narrow forbidden gap[18] 6) where the characteristic strong magnetic fields are considerably lower than astrophysical ones.

In conclusion the authors express their gratitude to L. A. Klebanov for carrying out the numerical calculations, to D. A. Kirzhnits, V. S. Popov and Ya. A. Smorodinskiĭ for useful discussions of the results of this paper.

APPENDIX

Here we shall carry out a comparison of the results obtained above with the conclusions of the paper by Kraĭnov[10] in which he also considered a relativistic atom in a strong magnetic field. It was asserted in[10] that within the framework of a single-particle problem the level $\varepsilon = -1$ cannot be reached as a result of the appearance of a characteristic bending of the curve $\varepsilon(H)$ at a certain "critical" field H_c similar to the manner in which this happens, for example, for scalar particles in a short range potential.[3, 14] This result is not valid for a Coulomb field due to the presence of a barrier in the one-dimensional potential (46′) which prevents the delocalization of the single-particle wave function of the ground state (Cf. Sec. 4). Let us examine this question in greater detail.

1. In solving the one-dimensional equations (6) Kraĭnov[10] utilized the phenomenological Coulomb potential

$$\tilde{V}(z) = -\zeta/(|z|+l), \qquad (A.1)$$

which differs at small distances from the effective potential (7). Although such a cutting-off of the potential at small distances should not qualitatively affect the principal results, it is more difficult in this scheme to give a correct estimate of the effect of finite nuclear size.

2. Because the potential of (A.1) is finite at the origen the solutions of the system (6) in the region $z > 0$ have the form (11), (12) with the shifted argument $z \to z + l$. The joining of the even solutions at the origin leads in this case to the quantization condition (cf. [10])

$$W_{\varkappa_1, i\zeta}(2\lambda l) = \lambda\zeta^{-1} W_{\varkappa_2, i\zeta}(2\lambda l), \qquad (A.2)$$

which for fields $H \gg 1$ reduces by expanding the Whittaker functions in series in terms of $\lambda l \ll 1$ to

$$\zeta \ln 2\lambda l + \operatorname{arctg}\frac{\lambda}{1-\varepsilon} + \arg\Gamma\left(-\frac{\zeta\varepsilon}{\lambda} + i\zeta\right) - \arg\Gamma(1+2i\zeta) = \frac{\pi}{2} + \pi n. \qquad (A.3)$$

This expression differs from (22) only by the absence of the last two terms in the left hand side which originate in different definitions of the one-dimensional potential but which are not significant for the differentiation with respect to $\ln H$ (cf. the footnote following formula (25)). The result of such a differentiation of (A.3) coincides with (25), and consequently also in the model considered in[10] the derivative $\partial\varepsilon/\partial(\ln H)$ is everywhere negative and finite, i.e., the energy levels must be monotonically lowered in value with increasing magnetic field.

3. For the ground state ($n = -1$) equation (A.3) can be brought to the form

$$\varepsilon = \cos\left[\zeta \ln\frac{H}{4\lambda^2} + 2\arg\Gamma(1+2i\zeta) - 2\arg\Gamma\left(-\frac{\zeta\varepsilon}{\lambda} + i\zeta\right)\right]. \qquad (A.4)$$

In the Born approximation $\zeta \ll 1$ this expression reduces to formula (1) from[10]:

$$\varepsilon \approx \cos(\zeta \ln(H/\lambda^2)). \qquad (A.5)$$

From (A.5) it follows that the derivative $\partial\varepsilon/\partial \ln H$ becomes infinite for a value of the energy $\varepsilon \approx -1 + 2\zeta^2$ in a magnetic field $H_c \approx \exp\{-2 + \pi/\zeta + 2\ln 2\zeta\}$ in contradiction with what was proved above. Such a discrepancy is explained by the use of the Born result (A.5) instead of the formula (A.4) which is asymptotically exact for $H \gg 1$ and valid for any $\zeta > 0.3$. Indeed, substitution into (A.4) of the indicated values of ε and H for the minimum value of $\zeta \sim 0.3$ considered by us shows that the second and the third terms in the argument of the cosine in (A.4) are comparable in order of magnitude with the first one, i.e., the Born result (A.5) would have been valid for considerably smaller ζ. But in this case the approximation of a point nucleus no longer has sense, since in the field $H_c \sim e^{\pi/\zeta}$, $\zeta \ll 0.3$, the Larmor radius is much smaller than the nuclear radius.

Thus, the assertion of Kraĭnov[10] that the level $\varepsilon = -1$ is unattainable within the framework of a single-particle problem is erroneous. Graphically, the discrepancy is illustrated in Fig. 2, where the dotted line shows the dependence $\varepsilon(H)$ obtained in accordance with the formula (A.5) for $\zeta = 0.3$. We also note that use of the approximate potential (A.1) would lead to a value of the magnetic field, at which the lower limit of the discrete spectrum is attained, which is by a factor of $2e^c \approx 3.5$ greater compared to the value obtained above (cf. Fig. 1).

1)The question of the existence of such strong magnetic fields ($l \sim \hbar/mc$ for $H \sim 10^{13}$ Oe) has been recently discussed in connection with the theory of pulsars.[4] In some of the estimates[5] a possibility is indicated that fields up to $H \sim 10^{18}$ Oe exist inside a pulsar.

2)In the nonrelativistic case the analogous problem for a hydrogen-like atom or an exciton was solved in the paper by Hasegawa and Howard.[6] Gor'kov and Dzyaloshinskiĭ[7] took into account the motion of the exciton in this case. Kadomtsev and Kudryavtsev[8,9] have made a nonrelativistic investigation of a many-electron atom in a strong magnetic field. In the paper by Kraĭnov[10] devoted to a relativistic investigation of a hydrogen-like atom in a strong field the spontaneous production of positrons was not investigated as a result of an erroneous assertion that within the framework of a single-particle problem the level $\varepsilon = -mc^2$ could not be reached (for details see the Appendix).

3)More accurately, the numerator of (25) must in place of λ^3 contain $\lambda^3(1 + xF_x'|_{x=R/l})$. But the contribution due to finite nuclear size is small compared to unity within the range of magnetic fields under discussion (cf. below (30) and (31)).

4)In units of $mc^2/\hbar = 9 \times 10^{20}$ Hz.

5)With the same displacement from threshold $Z - Z_c = 1$ which enables us to utilize the single particle approximation.

6)We are grateful to V. M. Édelshteĭn for bringing to our attention this work of Fal'kovskiĭ.

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