Mass of particles in a one-dimensional model with four-fermion interaction

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The infrared-asymptotic form of the one-particle Green function of a one-dimensional model with four-fermion interaction is investigated in the approximation of a large number of fermion fields. It is shown that the fermions become massive as a result of the interaction. The Green function has a branch point at $p^2 = m^2$. Spontaneous symmetry breaking does not occur and a mean field does not arise.

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INTRODUCTION

It is usual to associate the appearance of a fermion mass as a result of spontaneous breaking of γ_5-invariance with the formation of a Bose condensate of pairs of fermions.[1,2] However, the existence of a mean field of this type leads to unpleasant cosmological consequences.[3]

Below we consider a γ_5-invariant model in which a condensate does not arise but a fermion mass appears nevertheless. This is a one-dimensional model (one spatial coordinate) in which there are several Fermi fields. Such a model with two fields was proposed by Ansel'm[4] as an example of a model that has asymptotic freedom at short distances and does not have zero charge at large distances. In the paper[5] by Vaks and one of the authors this model was used as an example in which a fermion mass arises as a result of spontaneous breaking of γ_5-invariance. However, the proof given in this paper was not rigorous. The point is that at momenta of the order of the mass the interaction becomes strong and the parquet approximation, equivalent to the first order in the renormalization-group equations, is not applicable.

Gross and Neveu[6] considered an analogous model with a large number of fields ($N \gg 1$). In this case the interaction always remains weak (of order N^{-1}) and quantitative estimates are possible. In the leading approximation in N^{-1} there arises a Bose condensate (mean field), and as a consequence, a fermion mass. In the model in which only a discrete symmetry group exists (invariance under $\psi \to \gamma_5\psi$) the subsequent approximations in N^{-1} do not alter the qualitative result. However, in a model with a continuous symmetry group (invariance under $\psi \to \exp(i\alpha\gamma_5)\psi$), taking the next approximation in N^{-1} into account leads to the vanishing of the mean field.[7]

In the analogous nonrelativistic model[8] the pair correlation functions were calculated and decrease by a power law at large distances. An analogous situation is well known in the two-dimensional Bose gas.[9] This means that there is no long-range order in the system but massless excitations (the analog of Goldstone particles) exist. These excitations lead to infrared singularities and destroy the long-range order.

Below it is shown that, nevertheless, the fermion mass does not vanish. The infrared singularities, as in quantum electrodynamics, lead to the result that the one-particle Green function has a branch point at $p^2 = m^2$ instead of a pole.

Four-fermion interaction and the nonlinear σ-model

We consider a model with Lagrangian density

$$\mathscr{L}=i\bar{\psi}_k\hat{\partial}\psi_k+{}^1/{}_2 g^2[(\bar{\psi}_k\psi_k)^2-(\bar{\psi}_k\gamma_5\psi_k)^2], \tag{1}$$

where ψ_k is the fermion field, $k=1,\ldots,N$ is the isotopic index, and $\hat{\partial}=\gamma_\mu\partial^\mu$. Our choice of γ-matrices is the following:

$$\gamma_0=\sigma_x,\quad \gamma_1=-i\sigma_y,\quad \gamma_5=\gamma_0\gamma_1=\sigma_z.$$

This choice of γ_5-invariant interaction is convenient for the expansion in N^{-1} and permits isotopic $SU(N)$ symmetry. The interaction constant g is connected with the isoscalar and isovector $SU(N)$ interaction constants in

$$\mathscr{L}_{int}=g_1(\bar{\psi}_k\gamma_\mu\psi_k)^2+g_2(\bar{\psi}_k\gamma_\mu\tau_{kn}{}^a\psi_n)^2$$

by the relation $g^2=g_2=2Ng_1$. For $N=2$ the connection with the notation of the paper of Vaks and Larkin[5] is as follows:

$$g^2=\lambda_1=\lambda_2=\lambda_3=-2\lambda_4.$$

Introducing the intermediate boson fields σ and π, we can rewrite the Lagrangian (1) in the form

$$\mathscr{L}=\bar{\psi}_k[i\hat{\partial}-g(\sigma+i\pi\gamma_5)]\psi_k-{}^1/{}_2(\sigma^2+\pi^2) \tag{2}$$

or, denoting $\sigma+i\pi\gamma_5=\rho\exp(i\gamma_5\theta)$, in a form resembling the nonlinear σ-model:

$$\mathscr{L}=-\frac{1}{2}\rho^2+\sum_{k=1}^{N}\bar{\psi}_k[i\hat{\partial}-g\rho\exp(i\gamma_5\theta)]\psi_k=-\frac{1}{2}\rho^2+\sum_{k=1}^{N}\mathscr{L}_k. \tag{3}$$

Since in the two-dimensional model the continuous $SU(N)$ symmetry is not broken the Green function has a simple structure:

$$G_{ik}(x)=-i\left(\int\psi_i(x)\bar{\psi}_k(0)\exp\left(i\int\mathscr{L}\,d^2x\right)\prod_i D\bar{\psi}_i D\psi_i D\sigma D\pi\right)\times$$

$$\times\left(\int \exp\left(i\int \mathscr{L} d^2x\right)\prod_i D\overline{\psi}_i D\psi_i D\sigma D\pi\right)^{-1} = G(x)\delta_{ik}. \quad (4)$$

In formula (4), e.g., for G_{11}, we perform the averaging over $N-1$ components of the multiplet:

$$G(x) = -i\int \psi_1(x)\overline{\psi}_1(0)e^{i\Gamma}D\psi_1 D\overline{\psi}_1 D\sigma D\pi\left(\int e^{i\Gamma}D\psi_1 D\overline{\psi}_1 D\sigma D\pi\right)^{-1}. \quad (5)$$

Here,

$$\Gamma = \int \mathscr{L}_1 d^2x - \int V(\sigma,\pi)d^2x$$

plays the role of the effective action, and $V(\sigma, \pi)$ is the effective potential of the field ψ_1, produced by the $N-1$ fermion fields:

$$\int V(\sigma,\pi)d^2x = i\left[\int \ln\left(\prod_{k=1}^{N-1}\int \exp\left(i\int \mathscr{L}_k d^2x\right)D\psi_k D\overline{\psi}_k\right) - \frac{1}{2}\int \rho^2 d^2x\right]. \quad (6)$$

For large N, the functional integration over π and σ in formula (5) can be performed by the method of steepest descents. For this we represent the field in the form $\rho = \bar{\rho} + \rho_1$, where $\bar{\rho}$ is the Fourier component of ρ with zero momentum, and expand the effective potential to second order in ρ_1 and θ:

$$V = V(\bar{\rho}) + \frac{N}{2}\int d^2k[D_\rho(k)\rho_1^2 + D_\theta(k)\theta^2],$$

where, for $k^2 \ll (g\bar{\rho})^2$, we have

$$D_\rho(k) \approx \frac{g^2}{\pi}\frac{2g\bar{\rho}}{(-k^2)^{1/2}}\ln\left(1+\frac{(-k^2)^{1/2}}{2g\bar{\rho}}\right)$$
$$D_\theta(k) \approx -\frac{g^2}{\pi}\cdot 2g\bar{\rho}(-k^2)^{1/2}\ln\left(1+\frac{(-k^2)^{1/2}}{2g\bar{\rho}}\right). \quad (7)$$

In the integration over ρ_1 and θ, values $\rho_1 \sim (D_\rho N)^{-1/2}$ and $\theta \sim (D_\theta N)^{-1/2}$ are important. Therefore, ρ_1 is small in N^{-1} for all k and can be neglected. Inasmuch as $D_\theta \sim k^2$ for small k, in the infrared region the fluctuations of θ are not small and determine the character of the correlations at large distances. The integration over ρ by the method of steepest descents reduces to replacing $\bar{\rho}$ by its value at the point of the minimum of $V(\bar{\rho})$. Then for the effective action we have

$$\Gamma = \int d^2x[\overline{\psi}(i\hat{\partial} - g\bar{\rho}\exp(i\gamma_5\theta))\psi] + \frac{N}{2}\int D_\theta\theta^2 d^2k. \quad (8)$$

Denoting $g\bar{\rho} = m$ and replacing θ by $e\theta$ with $e = (\pi/g^2N)^{1/2}$, we arrive at the nonlinear σ-model[10]:

$$\mathscr{L}_\sigma = \overline{\psi}(i\hat{\partial} - m\exp 2ie\gamma_5\theta)\psi + \tfrac{1}{2}(\partial_\mu\theta)^2. \quad (9)$$

In this model the field θ varies in the interval $(0, \pi/e)$ and the γ_5 symmetry corresponds to the transformations

$$\psi \to \exp(i\alpha\gamma_5)\psi, \qquad \theta \to \theta - \alpha/e$$

with $\alpha = \text{const}$.

In the infrared region the original model with four-fermion interaction is equivalent to the σ-model (9). In the region $k \gtrsim m$ these models are different. However, the interaction in this region leads only to a renormalization of the mass and Green function. These renormalizations are small in e^2. Below, being interested in the behavior of the one-particle Green function (4) at large distances, we consider the nonlinear σ-model in the entire range of momenta.

In zeroth order in e (the spherical model) we have free massive fermions:

$$G_0(p) = (\hat{p}+m)/(p^2-m^2). \quad (10)$$

In the coordinate representation,

$$G_0(x) = -i\frac{m}{2\pi}\left[K_0(m(-x^2)^{1/2}) + i\frac{\hat{x}}{(-x^2)^{1/2}}K_1(m(-x^2)^{1/2})\right] \quad (11)$$

where the K_n are cylindrical functions of imaginary argument. In this approximation the continuous γ_5 symmetry is broken, corresponding to the appearance of the nonzero vacuum average $\langle 0|\psi\overline{\psi}|0\rangle$ calculated by Gross and Neveu[6] in the leading approximation in N^{-1}. The subsequent approximations in e^2 substantially alter the form of the Green function.

First order of perturbation theory

The Green function of the σ-model (9) is related to the mass operator Σ by the Dyson equation

$$G^{-1}(p) = \hat{p} - m - \Sigma(p). \quad (12)$$

In first order in e^2 the mass operator has the form

$$\Sigma^{(1)} = 4im^2e^2\gamma_5\int G_0(p-k)\gamma_5 D_0(k)\frac{d^2k}{(2\pi)^2} + 2ime^2\int D_0(k)\frac{d^2k}{(2\pi)^2}, \quad (13)$$

$$D_0(k) = k^{-2}. \quad (14)$$

The integrals in (13) diverge at both large and small momenta. Introducing the ultraviolet and infrared cutoffs Λ and λ, we obtain

$$\Sigma^{(1)} = \frac{e^2}{\pi}\left\{m^2\hat{p}\frac{(\hat{p}-m)^2}{p^2(p^2-m^2)}\ln\frac{p^2-m^2}{m^2} - \frac{(\hat{p}-m)^2}{p^2-m^2}m\ln\frac{m}{\lambda} - m\ln\frac{\Lambda}{m}\right\}. \quad (15)$$

The last term in this expression does not depend on the momentum and designates the renormalization of the mass. Hereafter, by the symbol m we shall mean the renormalized mass. Thus, in first order in e^2, for the Green function we have

$$G^{(1)}(p) = \frac{\hat{p}}{p^2-m^2}\left(1-\frac{e^2}{\pi}\ln\frac{m^2}{m^2-p^2}\right)$$
$$+\frac{m}{p^2-m^2}\left(1-\frac{e^2}{\pi}\ln\frac{m}{\lambda}\right) + \frac{e^2}{\pi}\frac{\hat{p}}{p^2}\ln\frac{m^2}{m^2-p^2}. \quad (16)$$

The correction to the diagonal element of the Green function diverges logarithmically as $\lambda \to 0$. Therefore, it is necessary to sum the whole perturbation-theory in e^2. As will be seen below, the summation leads to the vanishing of the diagonal element of the Green function as $\lambda \to 0$. The Green function (16) has no singularities at $p^2 = 0$. This implies the absence of massless fermions in the model under consideration. The corrections to

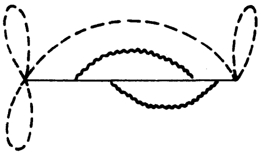

FIG. 1.

the Green function are large only in the region $|p^2 - m^2| \ll m^2$. As in quantum electrodynamics, summing the series of leading logarithms leads to the result that the pole at $p^2 = m^2$ in formula (10) for the Green function is replaced by a branch point:

$$G(p)=Z\frac{\hat{p}}{p^2-m^2}\left(\frac{p^2-m^2}{m^2}\right)^{e^2/\pi}. \tag{17}$$

The Green function in the infrared region

The derivation of formula (17) is conveniently carried out in the coordinate representation. For this we transform the fermion operators:

$$\begin{aligned}\psi(x)&\to\exp(ie\gamma_5\theta(x))\psi(x),\\ \bar{\psi}(x)&\to\bar{\psi}(x)\exp(ie\gamma_5\theta(x)).\end{aligned} \tag{18}$$

In the new variables,

$$G(x-x')=-i\langle\exp(ie\gamma_5\theta(x))\psi(x)\bar{\psi}(x')\exp(ie\gamma_5\theta(x'))\rangle, \tag{19}$$

where $\langle\ldots\rangle$ denotes averaging with the Lagrangian

$$L=\bar{\psi}(x)[i\hat{\partial}-m-e\gamma_5\hat{\partial}\theta(x)]\psi(x)+{}^1/_2(\partial_\mu\theta)^2, \tag{20}$$

$$\langle A\rangle=\left(\int A\exp\left(i\int L\,d^2x\right)D\psi\,d\bar{\psi}\,D\theta\right)\left(\int\exp\left(i\int L\,d^2x\right)D\psi\,D\bar{\psi}\,d\theta\right)^{-1}. \tag{21}$$

Formula (19) is conveniently rewritten in the form

$$\begin{gathered}G=G_1+G_2,\\ G_1(x-x')=-i\langle\exp(ie\gamma_5\theta(x))\langle\psi(x)\bar{\psi}(x')\rangle\cdot\exp(ie\gamma_5\theta(x'))\rangle.\end{gathered} \tag{22}$$

The first term is a product of averages. Examples of the diagrams appearing in it are shown in Fig. 1. The second term represents the set of all diagrams in which at least one boson line links an end of a fermion line with a point in the middle of it (Fig. 2).

In the first term the average $g(x-x') = -i\langle\psi(x)\bar{\psi}(x')\rangle$ can be expressed in terms of the Green function of the massive Thirring (MT) model:

$$\mathscr{L}_{MT}=\bar{\psi}(i\hat{\partial}-m)\psi+{}^1/_2e^2(\bar{\psi}\gamma_\mu\psi)^2. \tag{23}$$

For this we replace the four-fermion interaction of the Thirring model by an interaction of the fermions with intermediate boson fields φ and θ:

$$\begin{gathered}\mathscr{L}_{MT}\to\bar{\psi}(i\hat{\partial}-m)\psi-ej_\mu\partial^\mu\varphi+ej_\mu{}^5\partial^\mu\theta-{}^1/_2(\partial_\mu\varphi)^2+{}^1/_2(\partial_\mu\theta)^2,\\ j_\mu=\bar{\psi}\gamma_\mu\psi,\quad j_\mu{}^5=\bar{\psi}\gamma_\mu\gamma_5\psi.\end{gathered} \tag{24}$$

It is easy to convince oneself that such a replacement is correct: averaging over φ and θ in formula (23) leads us to a four-fermion interaction, since, in two-dimensional space,

$$(j_\mu{}^5j_\nu{}^5-j_\mu j_\nu)k_\mu k_\nu/k^2=j_\alpha j^\alpha.$$

The averaging over φ can be performed making the gauge transformation $\psi\to\psi e^{-ie\varphi}$. As a result we obtain

$$G_{MT}(x)=g(x)\langle\exp\{-ie[\varphi(x)-\varphi(0)]\}\rangle=g(x)(-x^2)^{e^2/4\pi}. \tag{25}$$

Perturbation theory for the Green function of the massive Thirring model has no infrared singularities, and a small interaction leads only to a mass renormalization, equal to the last term in formula (15). Therefore,

$$g(x)=-\frac{i}{2\pi}m(-x^2\Lambda^2)^{-e^2/4\pi}\left[K_0(m(-x^2)^{1/2})+i\frac{\hat{x}}{(-x^2)^{1/2}}K_1(m(-x^2)^{1/2})\right]. \tag{26}$$

Now we can calculate

$$\begin{aligned}G_1=-\frac{im}{2\pi}(-x^2\Lambda^2)^{-e^2/4\pi}\Big\{&K_0(m(-x^2)^{1/2})\langle\exp[ie\gamma_5(\theta(x)+\theta(0))]\rangle\\ &+i\frac{\hat{x}}{(-x^2)^{1/2}}K_1(m(-x^2)^{1/2})\langle\exp[ie\gamma_5(\theta(x)-\theta(0))]\rangle\Big\}.\end{aligned} \tag{27}$$

The latter averaging is performed with the Lagrangian (20). The interaction of the θ-bosons through virtual fermions is small at small momenta. It is proportional to the momenta of the bosons taking part in the interaction. Therefore, in formula (27) the averaging can be performed over the free θ-field with the Green function

$$D^{-1}(k)=k^2-e^2\Pi(k), \tag{28}$$

where $\Pi(k)$ is the polarization operator, which is proportional to k^2.

The renormalization of the Green function reduces to a renormalization of the charge in the σ-model. In the original model with four-fermion interaction allowance for the vacuum polarization in formula (28) would imply an excess of accuracy in N^{-1}. Thus, in the expression (27) the averaging can be performed over the free fields:

$$\langle\exp\{ie\gamma_5[\theta(x)-\theta(0)]\}\rangle=\exp\{ie^2[D(0)-D(x)]\}=(-x^2\Lambda^2)^{-e^2/4\pi}, \tag{29}$$

$$\langle\exp\{ie\gamma_5[\theta(x)+\theta(0)]\}\rangle=\exp\{ie^2[D(0)+D(x)]\}=(-x^2\Lambda^{-2}\lambda^4)^{e^2/4\pi}, \tag{30}$$

The diagonal part of G_1 tends to zero as a power as $\lambda\to 0$. The vanishing of the diagonal elements has occurred in conformity with the general theorem on the impossibility of spontaneous breaking of a continuous symmetry in a two-dimensional theory.

Thus,

$$G_1=\frac{m}{2\pi}\frac{\hat{x}}{(-x^2)^{1/2}}(-x^2\Lambda^2)^{-e^2/2\pi}K_1(m(-x^2)^{1/2}). \tag{31}$$

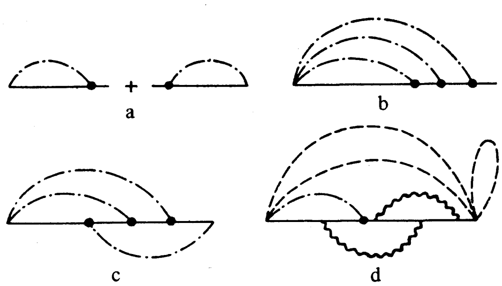

FIG. 2.

The Fourier transform of $G_1(x)$ has the form

$$G_1(p) \sim pF(2-e^2/2\pi,\ 1-e^2/2\pi;\ 2;\ -p^2/m^2), \qquad (32)$$

where F is the hypergeometric function. In the region $|p^2 - m^2| \ll m^2$ this expression goes over into formula (17).

We now estimate G_2. In first order of perturbation theory in e^2 (Fig. 2a),

$$G_2^{(1)} = \frac{e^2}{\pi}\left[\frac{p}{2p^2}\ln\frac{m^2}{m^2-p^2} - \frac{1}{p-m}\ln\frac{m}{\Lambda}\right]. \qquad (33)$$

The second term in this expression leads to a renormalization of the Green function, so that in formula (31) Λ is replaced by m. As regards the first term, at $p^2 = m^2$ it has only a logarithmic singularity and at large distances it falls off faster than $G_1(x)$. Diagrams of higher order, e.g., (b) and (c) in Fig. 2, lead either to a renormalization of the Green function or to terms having a weaker singularity at $p^2 = m^2$ than that of G_1. Figure 2d depicts the diagram (a) dressed by the boson lines shown in Fig. 1. As a result of this dressing, as in the calculation of G_1 the diagonal elements of G_2 vanish, and the nondiagonal elements acquire a factor $(x^2\Lambda^2)^{-e^2/2\pi}$. The same applies to the diagrams (b) and (c). The most singular part in G_2 is therefore proportional to G_1. The coefficient of proportionality is small for small e^2. Thus, taking G_2 into account leads to a change of the coefficient of G_1 and the Green function has the form (17). An essential point is that all the diagrams that appear in both G_1 and G_2 contain a fermion line joining the points 0 and x, and, therefore, at large space-like intervals they fall off exponentially with increase of x.

The exponential decrease of the Green function means that $G(p)$ does not have a singularity at $p^2 = 0$ in momentum space. The singularity at the point $p^2 = m^2$ has the power form (17) with exponent less than zero but greater than -1. The imaginary part of the Green function is nonzero only for $p^2 > m^2$. This implies that fermion states with energy less than m^2 do not exist.

The branch point of the Green function points to the existence of a continuous spectrum that begins at an energy equal to m^2. The states of this continuous spectrum are a superposition of a massive fermion and a certain number of massless bosons. These bosons are bound states of an even number of fermions. They are the analog of Goldstone particles.

CONCLUSION

Thus, in the model considered a fermion mass has arisen as a result of the mutual interaction of the fermions. Spontaneous symmetry breaking does not occur in this case. The Green function (17) can be represented in the form

$$G(p) \sim {}^1/_2 (p^2 - m^2)^{e^2/\pi}[G_0(p) + i\gamma_5 G_0(p) i\gamma_5].$$

This implies that the spectrum of fermion states is degenerate with respect to parity.

We note that massive fermions do not arise in any order of perturbation theory in the four-fermion interaction constant for the original model.

The pre-exponential factors and the power exponent in the Green function (31) were represented in the form of series in $N^{-1} \sim e^2$. Terms exponentially small in e^2 were omitted. These terms arise, firstly, because we treat the field θ simply as a boson field whereas it is an angular variable and varies from 0 to 2π, and, secondly, because, in performing the integration in (5) by the method of steepest descents, we take into account only the one stationary point corresponding to the uniform solution of the equation. It is possible to believe that these terms do not affect the qualitative solutions in the model under consideration. The exact solutions of nonrelativistic one-dimensional models with four-fermion interaction[11] can serve as a justification for this. In these models it is found that the spectrum of the fermion excitations has a gap.

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