Dynamics of anisotropic homogeneous generalizations of the Friedmann cosmological models

V. A. Ruban

Leningrad Institute for Nuclear Physics, USSR Academy of Sciences
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The paper deals with the dynamics, particularly the character of the singularity and the process of isotropization of anisotropic cosmological models of the "diagonal" Bianchi types I, V, and IX (axially symmetric) filled with thermalized matter at rest with pressure $P = n\epsilon$ ($0^S. n \leq 1$) and collisionless radiation (consisting of oppositely moving fluxes of ultrarelativistic free particles). The investigation is based on analytic solutions obtained in this paper. The latter are interpreted as perturbed flat, open or closed Friedmann metrios with infinitely (maximally in the type IX case) long gravitational waves against an isotropic background. The essential role of extremely "stiff" matter with an equation of state $P = \epsilon$ near the singularity is brought to light. This equation of state imitates the dynamical influence of a free ($\Box\psi = 0$) scalar field $\psi = \psi(\tau)$. General relativistic analogs are indicated for the homogeneous vacuum solutions and anisotropic models for $T = T_i^i = \epsilon - 3P$ for the types I, V, IX (axially symmetric) for the scalar-tensor Dicke cosmology.

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In connection with the hypothesis of an initial anisotropic state of the Universe, intensive investigations are being carried out of the dynamics of more general models than the Friedmann model, namely spatially homogeneous cosmological models of Bianchi types I–IX with gravitating matter, aimed at finding the conditions for isotropization in the expansion and at singling out the anisotropic versions which do not contradict the observed isotropy of the Metagalaxy for red shifts $z \lesssim 10 - 10^4$ (cf., e.g.,[1–11]).

In the present paper we consider simple homogeneous generalizations of the flat, open, and closed Friedmann models, namely the Bianchi types I, V, and IX (axially symmetric) with diagonal frame (tetrad) metrics $\gamma_{ab} = X_a^2(\tau)\,\delta_{ab}$ in a synchronous comoving frame, in the presence of an ideal fluid at rest with the equations of state most interesting for cosmology[1)]: $P = n\varepsilon$ ($n = 0$,[4] 1/3, 1) and collisionless radiation of the type of oppositely directed fluxes of free gravitons and neutrinos.[12] For these anisotropic models we list and briefly discuss only the new analytic solutions of the Einstein equations in the presence of matter, which are used for a study of the dynamical behavior near the singularity, and the process of isotropization. In particular, we discuss the type I with $P = n\varepsilon$ ($0 \leqslant n \leqslant 1$) as well as the case of a mixture of an ultrarelativistic component with $P = \varepsilon/3$ and a maximally "stiff" component with $P = \varepsilon$, without mutual interaction; the type V with $P = n\varepsilon$; the (axially symmetric) type IX with $P = \varepsilon$. In the types I and IX (axially symmetric) we indicate the asymptotic behavior near the singularity for gravitating fluids with $P = n\varepsilon$ ($0 \leqslant n \leqslant 1$) and collisionless radiation, and also consider the essential influence of a maximally "stiff" fluid with $P = \varepsilon$, which appears as the equivalent of the homogeneous scalar field in the Dicke cosmology (cf.[14]).

In addition to the cosmological applications, the exact solutions for the isotropizing models of type I, V, and IX (which may have claims to describing the early, essentially anisotropic, stages of expansion of the Universe) are also of independent interest. They may be interpreted as perturbations of the Friedmann metric with an anisotropic mode of a tensor field of the type of an infinitely (or maximally, for type IX) long gravitational wave superimposed on the isotropic background; thus, with the help of the exact solutions one can easily analyze the behavior of this wave at all stages, and hence study the nonlinear interaction of "long" ($\lambda \gtrsim c\tau$—the causal horizon)[15,16] waves with various material sources.

1. In the Bianchi types I and V the homogeneous spatial sections V_3 have isotropic vanishing or negative curvature, respectively, and on account of this the Einstein equations admit of first integrals, which after separation, from the metric $X_a = R(\tau)\exp[\beta_a(\tau)]$, of the volume factor $V = R^3 = XYZ$ (which characterizes the general expansion) and of the shift anisotropy of the Hubble velocities $\sigma_a = \dot\beta_a$, take the form[4]

$$\dot R^2 = \frac{\varkappa\varepsilon}{3} R^2 + \frac{\Sigma^2}{R^4} - k + \frac{\Lambda R^2}{3}, \quad \dot\beta_a = \frac{\Sigma_a}{R^3}, \quad k = 0, -1, \qquad (1)$$
$$\Sigma_1 + \Sigma_2 + \Sigma_3 = 0, \quad \Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 = 6\Sigma^2, \quad \Sigma_a = \text{const.}$$

Taking into account the law of conservation of energy ($T^k_{i;k} = 0$) in a synchronous comoving coordinate system for an adiabatically expanding fluid with $P = n\varepsilon$, written in the form

$$\frac{\dot\varepsilon}{P+\varepsilon} = -\frac{3\dot R}{R} = -\frac{\dot V}{V}, \quad \varkappa\varepsilon = \frac{3\mathscr{M}^{1+3n}}{R^{3(1+n)}}, \quad \mathscr{M} = \text{const} > 0 \qquad (2)$$

the problem reduces the integration of the generalized Friedmann equation

$$\dot R^2 = \left(\frac{\mathscr{M}}{R}\right)^{1+3n} + \frac{\Sigma^2}{R^4} - k + \frac{\Lambda R^2}{3}, \quad X_a = R\exp\left\{3q_a\Sigma\int\frac{d\tau}{R^3(\tau)}\right\}$$
$$q_a = \Sigma_a/3\Sigma = {}^2/_3\sin\psi_a, \quad -\pi/6 \leqslant \psi_1 = \psi \leqslant \pi/2, \qquad (3)$$
$$\psi_2 = \psi + 2\pi/3, \quad \psi_3 = \psi + 4\pi/3.$$

In the model of type V the restriction $T_1^0 = 0 - YZ = X^2 = R^2$ causes the set of constant exponents q_a to be fixed: $q_1 = 0$, $q_2 = 3^{-1/2}$, $q_3 = -3^{-1/2}$ ($\psi = 0$). The difference be-

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tween (3) and the anisotropic Friedmann models is due to the homogeneous anisotropic mode of the free tensor field ($\Sigma \neq 0$), which can be treated as an infinitely long gravitational wave in a nonstationary flat ($k=0$) or hyperbolic ($k=-1$) space V_3 of isotropic background.[16]

The anisotropy of Hubble rates of deformation $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab} = \frac{1}{2}(\dot{\beta}_1^2 + \dot{\beta}_2^2 + \dot{\beta}_3^2)$ will, according to the Raychaudhuri equation[6]

$$\ddot{R}/R = -{}^1/_6\varkappa(\varepsilon+3P) - {}^2/_3\sigma^2$$

yield a contribution to the density of active mass and will influence the rate of volume expansion (3) similar to an ideal fluid with extremely "stiff" equation of state $P_g = \varepsilon_g$. Therefore the effective density of the energy of the homogeneous mode (as well as of a long ($\lambda \gtrsim c$) gravitational wave) $\varkappa\varepsilon_g = \sigma^2 = 3\Sigma^2/R^6$ will increase most rapidly for a general compression ($R(\tau) \to 0$). This mode dominates over the contribution of matter with $P < \varepsilon$, which leads to an anisotropic vacuum stage and an initial singularity with a Kasner asymptotic collapse of V_3 into a line:

$$X_a \propto \tau^{p_a}, \quad V = R^3 \propto \tau \to 0, \quad \varkappa\varepsilon \propto V^{-(1+n)} \propto \tau^{-(1+n)} \to \infty,$$
$$p_a = {}^1/_3(1+2\sin\psi_a), \quad p_1+p_2+p_3 = p_1^2+p_2^2+p_3^2 = 1. \tag{4}$$

This asymptotic behavior does not depend on the presence of matter, and is determined only by the free gravitational tensor field ($\Sigma_a \neq 0$).

In the course of the unlimited expansion ($R(\tau) \to \infty$) the gravitating matter and the isotropic curvature V_3 (which are inessential in the neighborhood of the singularity) become the determining factors of the dynamics and guarantee the asymptotic isotropization. This means that after the termination of the vacuum stage the models of type I (for $P < \varepsilon$) and of type V go over into the corresponding Friedmann solutions $R_F(\tau)$ with small damped corrections:

$$X_a(\tau) \approx R_F(\tau)[1 + h_a(\tau)], \quad |h_a| \ll 1. \tag{5}$$

Such homogeneous anisotropic additions $h_a(\tau)$ to the Friedmann metric are, according to the classification of E. Lifshitz,[15] tensor-type perturbations of the free gravitational field $h_\alpha^\beta = \nu(\tau)G_\alpha^\beta$, and are analogous in the linear approximation with respect to the structure to the Weyl curvature tensor (the Petrov type I) and to the dynamical behavior

$$\nu(\tau) \propto \int d\tau/R_F^3(\tau)$$

to long gravitational waves ($\lambda \gtrsim c\tau$) superimposed on an isotropic quasi-euclidean or open model. A similar description for the models of type I and V agrees with the interpretation of the metrics of the Bianchi type VII,[10] which represent monochromatic circularly polarized waves of finite wavelength ($q \neq 0$) against an isotropic Friedmann background, namely a standing wave in flat space for the type VII_0 and two oppositely traveling waves in hyperbolic space of constant curvature for the type VII_h, respectively. In the limit of infinitely long waves ($q=0$) the type VII_0 goes over into type I, and the type VII_h into type V.

The metric of the Bianchi type IX can also be represented in the form of a superposition of an isotropic closed part and a set of standing gravitational waves of maximally possible length $\lambda = 2/3\pi R_F$ (tensorial harmonics with wave number $l=3$) compatible with V_3 being closed.[16] For the types VII and IX the spatial curvature is anisotropic (in distinction from types I and V), so that one must attribute to gravitational waves of finite length an energy-momentum pseudotensor with anisotropic stresses. In the purely wave situation ($\lambda \ll c\tau$) such short-wave perturbations of the Weyl curvature tensor are equivalent to a set of massless particles—gravitons—against a large-scale background of an averaged metric, and if the distribution is isotropic they influence the dynamics similarly to an ultrarelativistic gas with $\tilde{P}_g = \tilde{\varepsilon}_g/3$.[17]

Thus, with the aid of the exact solutions for the anisotropic models of types I, V, and IX, which are interpreted as perturbed isotropic metrics, it is easy to study the dynamical behavior and the influence of homogeneous tensor modes of the free field,[2)] modes which appear as the limit of long gravitational waves with $\lambda \gtrsim c\tau$ on a Friedmann background in the nonlinear regime.

2. The Bianchi type I always admits a diagonalization of the frame (tetrad) metric (if $T_a^b = T_a\delta_a^b$):

$$-ds^2 = -d\tau^2 + X^2(\tau)dx^2 + Y^2(\tau)dy^2 + Z^2(\tau)dz^2 \tag{6}$$

and contains as special cases homogeneous plane-symmetric T-models with an additional axial symmetry ($X=Y$) and a quasi-Euclidean Friedmann model ($X=Y=Z=R_F$). Sufficiently many exact solutions are known for the metric (6) for various material sources: the ideal fluid with $P=0$,[4] $P=\varepsilon/3$,[5] $P=\varepsilon$[5,7,13]; free particles in a kinetic description,[12] a scalar field[14] and a "primary" magnetic field in combination with matter ($P=0$)[2]; these solutions have served as a basis for analyses of astrophysical processes and possible observational consequences for such a parabolic version ($\rho_0 = \rho_c = 3H_0^2/8\pi G \simeq 5\times10^{-30}$ g/cm^3) of anisotropic cosmology (cf.[12]).

An analytic solution for the type I metric (6) can also be obtained for a more general equation of state $P = n\varepsilon$ ($0 \leq n < 1$) of matter if one introduces in (3) the dimensionless parameter η by means of the substitution

$$d\tau/\tau_0 = (R/R_0)^{3n}d\eta.$$

We then have for $\Lambda = 0$:

$$\left(\frac{R}{R_0}\right)^3 = [\eta(\eta+\eta_0)]^{1/(1-n)}, \quad \frac{\tau}{\tau_0} = \Phi(\eta) = \int_0^\eta [\eta(\eta+\eta_0)]^{n/(1-n)}d\eta,$$
$$\frac{X_a}{X_a^{(0)}} = \eta^{(1+2s_a)/3(1-n)}(\eta+\eta_0)^{(1-2s_a)/3(1-n)}, \quad s_a = \sin\psi_a, \tag{7}$$
$$\tau_0 = \frac{2R_0^{3(1+n)/2}}{3\mathscr{M}^{(1+3n)/2}(1-n)}, \quad \eta_0 = \frac{2\Sigma}{\mathscr{M}^{(1+3n)/2}R_0^{3(1-n)/2}}, \quad R_0^3 = X_1^{(0)}X_2^{(0)}X_3^{(0)};$$

here τ_0, η_0, and X_a^0 are integration constants.[3)] In the limit $n=0$ this solution goes over into the Heckman-Schücking metric for "dustlike" matter with $P=0$ ($\tau_0 = 4\Sigma/3\mathscr{M}$, $\eta_0=1$)[4] and for $n=1/3$ it yields a different parametrization for the standard "hot" (big bang) model of type I, which is radiation-dominated with $P=\varepsilon/3$.[5]

We analyze the dynamical behavior of an infinitely long "gravitational wave" on a flat Friedmann background and its interaction with an ideal fluid with the help of Eq. (7), assuming an equation of state $P=n\varepsilon$ with $0\leqslant n<1$. The latter encompasses all physically admissible asymptotic forms of the equation of state of matter ($0<P<\varepsilon$) and contains the variants which are most important for cosmology:

1) $P=0$—nonrelativistic matter, dominating in the source term at relatively late stages of expansion of the Metagalaxy;

2) $P=\varepsilon/3$—equilibrium radiation and a mixture of ultrarelativistic particles, dominating in the radiation and lepton eras in the "hot" variant. The ultrarelativistic equation of state $P=\varepsilon/3$ is possibly applicable also during the hadronic stage, particularly within the framework of the quark-parton picture of strong interactions at high energy.[18]

The case $P=0$ may turn out to be a good approximation also for the early "hot" hadron era if Hagdoern's hypothesis[19] on the existence of a limiting temperature for the strongly excited vacuum, $T_{\max}=m_\pi c^2/k\simeq 2\times 10^{12}$ K, is valid.

The intermediate barotropic forms $P=n\varepsilon$ ($0\leqslant n\leqslant 1/3$) allow one to imitate a mixture of nonrelativistic matter ($P=0$) and equilibrium radiation ($P=\varepsilon/3$), and the case of "stiff" equations of state with $1/3<n<1$ can in principle be realized during the "hot" hadronic era and may be preferable for superdense baryonic matter in the "cold" model.[20]

In the vacuum era ($\eta\ll\eta_0$), which is determined by the strong homogeneous "gravitational wave" ($\lambda\to\infty$):

$$\begin{aligned} V=R^3&\approx 3\Sigma\tau\left\{1+\frac{2}{2-n}\left(\frac{\tau}{\tau_F}\right)^{1-n}\right\},\\ \varkappa\varepsilon&\approx\frac{4}{3\tau_F^{1-n}\tau^{1+n}}\left[1-\frac{2(1+n)}{2-n}\left(\frac{\tau}{\tau_F}\right)^{1-n}\right],\\ X_a&\propto\tau^{p_a}\left[1+\frac{2(1-n)-4s_a}{3(1-n)(2-n)}\left(\frac{\tau}{\tau_F}\right)^{1-n}\right],\\ \tau_F&=\frac{2}{3}\frac{(2\Sigma)^{(1+n)/(1-n)}}{\mathscr{M}^{(1+3n)/(1-n)}}, \end{aligned}\tag{8}$$

the common expansion from the "linear" Kasner singularity $\tau=0$, (4), is accompanied by a contraction in one of the directions ($p_1<0$, $p_{2,3}>0$). The gravitating matter at rest, which for $P<\varepsilon$ behaves as a test object on the strongly anisotropic background of the dominating homogeneous mode of the free tensor field ($\Sigma\neq 0$) slows down and stops this contraction, converting it into an expansion with subsequent asymptotic isotropization. In (8) τ_F is the characteristic time of the end of the vacuum stage or the beginning of the isotropic Friedmann stage ($\eta\simeq\eta_0$) when the dynamical influence of the gravitating matter ($\mathscr{M}\neq 0$) and of the anisotropy of velocities of deformations ($\Sigma\neq 0$) is of the same order.

The special axially symmetric variant of (7) with the kinematic set of Kasner exponents $p_1=1$, $p_2=p_2=p_3=0$ ($\psi=\pi/2$) (which represents the analog of the Newtonian collapse of an oblate ellipsoid into a "pancake" at $P=0$[21]) exhibits a degenerate behavior: it does not in fact have a vacuum stage, since there is no free gravitational field (the invariants of the Weyl curvature tensor are $\alpha_a=-p_bp_c/\tau^2=0$), and the corresponding Taub-Kasner metric in vacuum (4) transforms to the Minkowski form. Such a kinematical singularity of the degeneracy of V_3 into a "pancake" becomes physical on account of the presence of matter (ε, $p\propto\tau^{-(1+n)}\to\infty$):

$$\begin{aligned} X&\propto\tau\left[1-\frac{2(1+n)}{3(1-n)(2-n)}\left(\frac{\tau}{\tau_F}\right)^{1-n}\right]\to 0,\\ Y=Z&\propto\left[1+\frac{2}{3(1-n)}\left(\frac{\tau}{\tau_F}\right)^{1-n}\right]\to\text{const}. \end{aligned}\tag{8a}$$

The gravitation of matter is always dynamically important, and the time derivatives

$$\left|\frac{\dot X}{X}\right|,\ \left|\frac{\ddot Y}{Y}\right|,\ \left|\frac{\dot X\dot Y}{XY}\right|\propto\tau^{-(1+n)}\to\infty$$

have the same order of magnitude as the contribution of matter (and are not of order τ^{-2}, as in the vacuum stage (4)); for $P\neq 0$ the Hubble velocities and accelerations increase without bound as V_3 collapses into a "pancake."[4)]

The process of isotropization and the dynamics of the quasi-isotropic stage in (7) ($\eta\gg\eta_0$) depends essentially on the equation of state of the gravitating matter. On account of the slower decrease of the energy density of matter during its adiabatic expansion ($\varepsilon\propto R^{-3(1+n)}$), compared to the contribution of the vacuum anisotropy of the rates of deformation $\sigma^2\propto R^{-6}$ at $P<\varepsilon$, this matter is in fact responsible for the transition of the type I models to a quasi-Euclidean Friedmann metric with $P=n\varepsilon$, of the form

$$X_a\approx R_F(\tau)=\left\{\frac{3(1+n)}{2}\mathscr{M}^{(1+3n)/2}\tau\right\}^{2/3(1+n)},\quad \varkappa\varepsilon_F\approx\frac{4}{3(1+n)^2\tau^2}\tag{9}$$

In the quasi-isotropic stage ($\tau\gg\tau_F$) the small corrections to the Friedmann solution (9), necessitated by the influence of the homogeneous anisotropic mode ($\Sigma\neq 0$), yield in the linear approximation only perturbations of the metric if $P\neq 0$:

$$\begin{aligned} h_a=\frac{\delta X_a}{R_F}&\approx-\frac{2s_a}{3(1-n)}\frac{\eta_0}{\eta}\approx-\frac{2s_a\eta_0}{3(1-n)}\left(\frac{R_0}{R_F}\right)^{3(1-n)/2}\\ &=-\frac{2s_a}{3(1-n)}\left(\frac{\tau_F}{\tau}\right)^{(1-n)/(1+n)} \end{aligned}\tag{10}$$

and the volume of the elements of V_3 does not change under such an anisotropic deformation ($s_1+s_2+s_3=0$), so that the energy density does not deviate from the Friedmann law.

It is known[1,2] that short gravitational waves ($\lambda\ll c\tau$) do not interact with a homogeneously distributed ideal fluid, and their amplitude decreases according to the adiabatic law $|h_\alpha^\beta|\propto R_F^{-1}\propto\tau^{-2/3(1+n)}$ (corresponding to the behavior of the energy density of the ultrarelativistic graviton gas $\tilde P_g=\tilde\varepsilon_g/3\propto R_F^{-4}$). In distinction from the short waves, long-wave ($\lambda \gtrsim c\tau$) anisotropic perturbations of the metric of the type (10) are damped out somewhat faster if $P<\varepsilon/3$, owing to interaction with matter (in particular, for $P=0$[15]).

For $P=\varepsilon/3$ the amplitude of long gravitational waves (10) varies according to the same law as for short waves ($|h_\alpha^\beta| \propto \nu \propto R_F^{-1} \propto \tau^{-1/2}$), and this agreement is closely related to the conformal invariance of the linearized Einstein equations for weak gravitational waves against the isotropic Friedmann background in the case when the trace of the energy-momentum tensor vanishes: $T=\varepsilon-3P=0$.[17]

In the second approximation, the nonlinear interaction of a homogeneous anisotropic mode with a fluid at rest produces a deviation of the time-dependence of the volume factor $V=R^3=XYZ$ and of the energy density from the Friedmann law (9) at $n\neq 1/3$, given by

$$\frac{\delta V}{V_F}=-\frac{(1+n)\delta\varepsilon}{\varepsilon_F}\approx\frac{1}{4(3n-1)}\left(\frac{\eta_0}{\eta}\right)^2$$
$$\approx\frac{\eta_0^2}{4(3n-1)}\left(\frac{R_0}{R_F}\right)^{3(1-n)}\approx\frac{1}{4(3n-1)}\left(\frac{\tau_F}{\tau}\right)^{2(1-n)/(1+n)}, \tag{11}$$

and appearing as a result of the reaction of the long wave via its effective energy density $\varepsilon_g=P_g$ on the behavior of $R(\tau)$ in (3).

For the case $P=\varepsilon/3$, Eq. (7) (where the integral at $n=1/3$ can be expressed in terms of elementary functions) implies that in the expansion (11) there appears a logarithmic contribution ($\eta \gg \eta_0$, $\tau \gg \tau_F = 8\Sigma^2/3\mathscr{M}^3$):

$$-\frac{\delta\varepsilon}{\varepsilon_F}\approx\frac{4}{3}\frac{\delta V}{V_F}\approx\frac{\eta_0^2}{\eta^2}\left(\ln\frac{\eta}{\eta_0}+C\right)\approx\left(\frac{R_0\eta_0}{R_F}\right)^2\left[\ln\frac{R_F}{R_0\eta_0}+C\right]$$
$$\approx\frac{\tau_F}{2\tau}\left[\ln\frac{\tau}{\tau_F}+C'\right]. \tag{12}$$

For $P=0$, the models (7) have a qualitatively different asymptotic behavior of the isotropization ($\tau \gg \tau_F = 4\Sigma/3\mathscr{M}$):

$$\frac{\delta X_a}{R_F}\approx\frac{(1-2s_a)\tau_F}{3\tau},$$
$$\frac{\delta V}{V}\approx-\frac{\delta\rho}{\rho_F}\approx\frac{\tau_F}{\tau}\approx\eta_0\left(\frac{R_0}{R_F}\right)^{3/2}, \tag{13}$$

which yields even in the linear approximation a change of the volume of the elements of V_3 and of the density of "dustlike" matter. However such fictitious homogeneous perturbations can be removed by shifting the origin of time $\tau+\tau_0\to\tilde{\tau}$, so that the anisotropic additions to the flat Firedmann metric are again described by (10) with $n=0$, in agreement with law of decay of long waves for $P=0$[15]:

$$\nu\propto \text{const}+C/R_F^{3/2}\propto\text{const}+C'/\tau.$$

3. a) The special case $P=\varepsilon$ is not covered by the solution (7) and requires separate consideration, since such a maximally "stiff" matter becomes dynamically important near the singularity, and the anisotropic models have no vacuum stage. Although the problem of reality of the equation of state $P=\varepsilon$ in the "hot" variant during the hadronic era is not clear (cf.[21]), the limiting asymptotic form $P=\varepsilon$ is admissible in principle, e.g., for superdense strongly degenerate baryon matter at low temperatures and is usually used for the description of the early stages of expansion in the "cold" charge-asymmetric variant of the initial state of the Universe.[20]

In addition, gravitating extremely "stiff" matter imitates the dynamical influence of a scalar massless field, insofar as it appears only as an additional material source, as, for instance, in the conformal presentation of the scalar-tensor theory of Dicke for a vanishing trace of the energy-momentum tensor, $T=\varepsilon-3P=0$.[14] The canonical energy-momentum tensor of a scalar massless field

$$\Lambda_i^k = {}^1/_4\zeta(\Psi_{,i}\Psi^{,k}-{}^1/_2\delta_i^k\Psi_{,n}\Psi^{,n}),$$
$$\zeta=(2\omega+3)/\varkappa_0=\text{const}>0, \tag{14}$$

has a hydrodynamic structure similar to the case of an ideal fluid with

$$P_s=\varepsilon_s={}^1/_4\zeta\Psi_{,i}\Psi^{,i},\quad u_i=\Psi_{,i}/(\Psi_{,n}\Psi^{,n})^{1/2},\quad u_iu^i=1,$$

so that in the homogeneous case $\Psi=\Psi(\tau)$ the scalar free field ($\Box\Psi=0$) is equivalent to extremely "stiff" matter at rest with $P_s=\varepsilon_s=\frac{1}{4}\zeta\dot{\Psi}^2$. Consequently, the homogeneous cosmological models in general relativity, filled with motionless maximally "stiff" matter with $P_s=\varepsilon_s$, can be considered as an analog of the corresponding vacuum solutions for the conformal presentation of the scalar-tensor theory of Jordan-Brans-Dicke, and they are easily transformed to the original version with a variable gravitational coupling "constant" $G\sim\varphi^{-1}(\tau)$, $\Psi=\ln\varphi$ (cf.[14] for details).

Starting with this analogy between the general-relativistic theory and the conformal presentation of the Dicke scalar-tensor cosmology, it is easy to clarify the essential influence of a homogeneous mode of the free scalar field on the character of the initial singularity and the dynamics of the anisotropic expansion, if one considers the Bianchi type I models which contain in addition to ordinary matter with $P=n\varepsilon$ ($0\leqslant n<1$) also a maximally "stiff" fluid of the form

$$\varkappa\varepsilon_s=\varkappa P_s=\frac{3\mathscr{M}_s^4}{R^6}=\frac{\zeta}{4}\dot{\Psi}^2,\quad \dot{\Psi}=\frac{\dot{\varphi}}{\varphi}=\frac{S}{R^3},$$
$$\mathscr{M}_s^4=\frac{2\omega+3}{12}S^2=\text{const}.$$

For such a mixture of two noninteracting components, the Einstein equations (3) admit of an analytic solution of the type (7) with the substitutions

$$\Sigma\to\tilde{\Sigma}=(\Sigma^2+\mathscr{M}_s^4)^{1/2},\quad s_a\to\tilde{s}_a=(\Sigma/\tilde{\Sigma})\sin\psi_a. \tag{7a}$$

Owing to the maximal "stiffness," the "scalar fluid" ($\mathscr{M}_s\neq 0$) predominates over matter with $P<\varepsilon$ for unlimited contraction, and acting on a par with the anisotropic tensor mode ($\Sigma\neq 0$), it alters the Kasner asymptotic behavior (4) in agreement with the exact solution for the model of type I for $P_s=\varepsilon_s=\frac{1}{4}(\dot{\varphi}/\varphi)^2$ [5,6,13,14]:

$$X_a=X_a^{(0)}(\tau/\tau_0)^{\tilde{p}_a},\quad V=V_0(\tau/\tau_0),$$
$$\varkappa\varepsilon_s=3\mathscr{M}_s^4/V^2=\mu/\tau^2,\quad \varphi/\varphi_0=(\tau/\tau_0)^{2B},$$
$$\tilde{p}_1+\tilde{p}_2+\tilde{p}_3=1,\quad \tilde{p}_1^2+\tilde{p}_2^2+\tilde{p}_3^2=1-2\mu, \tag{15}$$
$$\mu=\mathscr{M}_s^4/3(\Sigma^2+\mathscr{M}_s^4)=(2\omega+3)B^2=\text{const}.$$

This generalized asymptotic behavior with $\tilde{p}_a = (1/3)(1 + 2\tilde{s}_a)$ at $0 < \mathscr{M}_s^4 < 3\Sigma^2$ $(0 < \mu < \frac{1}{4})$, in addition to the collapse of V_3 into a line, contains also an anisotropic point collapse, which becomes at $\mathscr{M}_s^4 > 3\Sigma^2$ $(\mu > \frac{1}{4})$ the only possible type and in the limit $\mu \to \frac{1}{3}$ $(\mathscr{M}_s^4 \gg 3\Sigma^2)$ approaches the isotropic one. We note that in the axially symmetric variant there is for $\mathscr{M}_s^4 = 3\Sigma^2$ $(\mu = \frac{1}{4})$ an anomalous type of collapse into a "barrel" ($\tilde{p}_1 = 0$, $\tilde{p}_2 = \tilde{p}_3 = \frac{1}{2}$), and the kinematic regime of the collapse of V_3 into a "pancake" becomes utterly impossible at $P_s = \varepsilon_s$.

In the course of the expansion the dynamical influence of the matter with $P \gg \varepsilon$ (which is negligible near the singularity) increases compared to the more rapidly decreasing contributions of the "scalar fluid" and the anisotropy of the Hubble velocities, which has as a result the transition to the solution (7) and its subsequent isotropization. It should be stressed that when the "scalar fluid" with $P_s = \varepsilon_s$ dominates over the tensor mode ($\mu > \frac{1}{4}$) the strong anisotropic stage is absent and the dynamics of the expansion is always close to the quasi-isotropic regime, even near the singularity.

Since at $T = \varepsilon - 3P = 0$ the material sources do not generate a scalar field and it can only be free, the general-relativistic models filled with a mixture of noninteracting maximally stiff ($P_s = \varepsilon_s$) and ultrarelativistic ($P = \varepsilon/3$) components are equivalent to the scalar-tensor variants in the conformal presentation of Dicke,[14] as, e.g., the model of type I with $P = \varepsilon/3$, for which the general solution can be written in parametric form different from (7a) ($d\tau = R^2 d\xi / 2\mathscr{M}$):

$$R = \frac{\tilde{\Sigma}}{\mathscr{M}} \operatorname{sh}\frac{\xi}{2}, \quad \tau = \frac{\tilde{\Sigma}^2}{4\mathscr{M}^3}(\operatorname{sh}\xi - \xi), \quad \varkappa\varepsilon = \frac{3\mathscr{M}^2}{R^4}, \quad \varkappa\varepsilon_s = \frac{3\mathscr{M}_s^4}{R^6},$$
$$X_a = \frac{\tilde{\Sigma}}{\mathscr{M}} \operatorname{sh}\frac{\xi}{2}\left(\operatorname{th}\frac{\xi}{4}\right)^{2\tilde{s}_a}, \quad \frac{\varphi}{\varphi_0} = \left(\operatorname{th}\frac{\xi}{4}\right)^{\delta}, \quad \delta = \frac{2\mathscr{M}_s^2}{\tilde{\Sigma}}\left(\frac{3}{2\omega+3}\right)^{1/2}. \tag{7a'}$$

b) In the investigation of the dynamics of anisotropic models, in addition to the matter at thermodynamical equilibrium described as an ideal fluid with $P = n\varepsilon$ $(0 \leqslant n \leqslant 1)$, one must also take into account the presence of weakly interacting particles such as gravitons or neutrinos, with mean free paths $L \gtrsim c$, the contribution of which may predominate at early stages of the expansion of the Universe, and even at the present epoch.[12] On account of the kinetic acceleration ($P_a X_a = \text{const}$) during the sharply anisotropic background, such collisionless radiation has a distribution function in momenta P_a which is strongly prolate along the direction of contraction. It can be regarded as two fluxes of relativistic particles moving in opposite directions, e.g., along the x_1 axis, with an energy-momentum tensor of the form

$$T_0^0 = -T_1^1 = \Pi, \quad T_2^2 = T_3^3 = 0, \quad \varkappa\Pi = \mathscr{M}'/X^2YZ, \quad \mathscr{M}' = \text{const}. \tag{16}$$

From the analysis of solutions of the type I[12,7] it follows that the free particles (16) do not influence the "linear" Kasner singularity (4), but remove the Friedmann point collapse (even in the presence of a fluid with $P \leqslant \varepsilon/3$) and the regime of collapse of V_3 into a "pancake." On account of their gravitational influence, the process of isotropization is slowed down, and at the stage when $P = \varepsilon/3$ the anisotropy is practically conserved, decreasing only logarithmically. However, at $P > \varepsilon/3$ quasiisotropic collapse becomes possible, matter dominates ($\Pi/\varepsilon \sim \tau^{2(3n-1)/3(1+n)} \to 0$), and there appears also a special type of anisotropic point collapse, where the dynamical influences of the "stiff" matter with $P = n\varepsilon$ $(n > 1/3)$ and of free particles (16) are of the same order[7]:

$$X \propto \tau^{2n/(1+n)} \to 0, \quad YZ \propto \tau^{2(1-n)/(1+n)} \to 0,$$
$$\varkappa\varepsilon = \frac{2(1-n)}{(1+n)^2\tau^2}, \quad \varkappa\Pi = \frac{(1-n)(3n-1)}{(1+n)^2\tau^2}. \tag{17}$$

This regime yields also an asymptotic behavior of unlimited expansion in a model of type I, which on account of the influence of the free particles is not isotropized for $P > \varepsilon/3$.

In the case of a mixture of a maximally "stiff" ("scalar") fluid with $P_s = \varepsilon_s$ and free particles (16) with $T = 0$, one can obtain an analytic solution if one takes[12,7]

$$\tau = \int_0^{\xi} X(\xi)\, d\xi.$$

Then

$$\varkappa\varepsilon_s = 3\mathscr{M}_s^4/R^6, \quad \varkappa\Pi = \mathscr{M}'/X^2YZ,$$
$$X/X_0 = \xi^{\tilde{p}_1/(1-\tilde{p}_1)} e^{\xi}, \quad Y/Y_0 = \xi^{\tilde{p}_2/(1-\tilde{p}_1)}, \quad Z/Z_0 = \xi^{\tilde{p}_3/(1-\tilde{p}_1)}.$$

The free particles do not change the dynamics of the initial expansion, and the generalized asymptotic behavior (15) remains in force near the singularity $\xi \to 0$. For a general expansion, however, the high pressure along the axis of acceleration of the particles ($\tilde{p}_1 < 0$) is damped with a transition to the quasi-inertial regime (cf. (8a)):

$$X \propto \tau(\ln\tau)^{\tilde{p}_1/(1-\tilde{p}_1)} \to \infty, \quad Y \propto (\ln\tau)^{\tilde{p}_2/(1-\tilde{p}_1)} \to \infty, \quad Z \propto (\ln\tau)^{\tilde{p}_3/(1-\tilde{p}_1)} \to \infty$$

4. The Bianchi type V is a generalization of the open variant of the Friedmann model with negative spatial curvature ($k = -1$) and low mean matter density ($\varepsilon_0 < \varepsilon_c = 3H_0^2/8\pi G$) and is characterized by the diagonal metric in a comoving coordinate system of the form

$$-ds^2 = -d\tau^2 + R^2(\tau)\{dx_1^2 + e^{2x_1}[\mathscr{L}^2(\tau)\,dx_2^2 + \mathscr{L}^{-2}(\tau)\,dx_3^2]\}. \tag{18}$$

For the model of type V with immobile matter the equations (1)–(3) can be integrated in terms of elliptic functions for some $n = 0$, $1/3$, $1/3$, 1 and for the equation of state $P = \varepsilon$ and $P = \varepsilon/3$ one may propose a simpler parametric form for the exact solutions in terms of elementary functions if one carries out the substitution

$$\tau = \int_0^{\eta} R(\eta)\, d\eta.$$

In the case $P_s = \varepsilon_s$, $\varkappa\varepsilon_s = 3\mathscr{M}_s^4/R^6$ we have

$$R^2(\eta) = (\Sigma^2 + \mathscr{M}_s^4)^{1/2} \operatorname{sh} 2\eta, \quad \mathscr{L}^2(\eta) = (\operatorname{th}\eta)^d,$$
$$d = [3\Sigma^2/(\Sigma^2 + \mathscr{M}_s^4)]^{1/2}. \tag{19}$$

Maximally stiff matter is dynamically essential near the singularity and determines its character on a par with the homogeneous anisotropic mode of the free gravitational field, leading to an asymptotic behavior of the type (15) with a set of exponents $\tilde{p}_1 = 1/3$, $\tilde{p}_{2,3} = 1/3 \pm \Sigma/\sqrt{3}\,\tilde{\Sigma}$. Owing to the isotropic negative curvature of V_3 the models of type V always isotropize in the case of infinite expansion; even in vacuum ($\mathscr{M}_s = 0$) and at $P = \varepsilon$ they (in distinction from the model of type I) go over in the limit into the Friedmann-Milne model with

$$R_{FM} \propto \tau \to \infty \quad (\eta \gg 1, \ \tau \gg \tau_{FM} = [(\Sigma^2 + \mathscr{M}_s^4)/4]^{1/4})$$

and with small corrections of the form ($\mathscr{M}_s = 0$)

$$-\frac{\delta Y}{R_{FM}} = \frac{\delta Z}{R_{FM}} = -\delta\mathscr{L} = \sqrt{3}\left(\frac{\tau_{FM}}{\tau}\right)^2 \propto \frac{1}{R_{FM}^2}, \qquad \frac{\delta V}{V_{FM}} = -\left(\frac{\tau_{FM}}{\tau}\right)^4 \propto \frac{1}{R_{FM}^4} \tag{20}$$

which are due to the homogeneous anisotropic mode against the nonstationary hyperbolic background V_3.

The model of type V, at $P = \varepsilon/3$, $\varkappa\varepsilon = 3\mathscr{M}^2/R^4$, and a metric of the form (18), with $d\tau = R(\eta)d\eta$, admits of the analytical solution

$$R^2(\eta) = \mathscr{M}^2 \operatorname{sh}^2\eta + \Sigma \operatorname{sh} 2\eta, \qquad \mathscr{L}^2(\eta) = \left\{\frac{\mathscr{M}^2}{\mathscr{M}^2 + 2\Sigma}\left[1 + \frac{2\Sigma}{\mathscr{M}^2}\operatorname{cth}\eta\right]\right\}^{-\sqrt{3}}. \tag{21}$$

After the vacuum (4) stage (with $p_1 = 1/3$, $p_{2,3} = 1/3 \pm 1/\sqrt{3}$) is over, the model approaches the open Friedmann metric on account of the joint action of gravitating matter with $P = \varepsilon/3$ in the parabolic stage (as in the type I model, (7), (10), (12) if $\eta_0 = 2\Sigma/\mathscr{M}^2 \ll 1$) and of the curvature of the homogeneous Milne stage of the type (20) for $\eta_0 \gtrsim 1$, when the matter is dynamically unimportant and may be considered a test object against the vacuum background (19) with $\mathscr{M} = 0$.

In the case of a model of type V filled with a mixture of noninteracting ultrarelativistic ($P = \varepsilon/3$) and maximally "stiff" ($P_s = \varepsilon_s$) components, one can also write down an analytic solution of the form (19) and (21), where $\Sigma \to \tilde{\Sigma} = (\Sigma^2 + \mathscr{M}_s^4)^{1/2}$. Its behavior is intermediate between (19) during the early stage near the singularity $\eta = 0$ of type (15) (controlled by the dominating "scalar" fluid with $P_s = \varepsilon_s$), and (21) at a later epoch, when ultrarelativistic matter with $P = \varepsilon/3$ and the curvature, which are inessential near the singularity, become the determining factors in the dynamics of the expansion.

For $P = 0$ the solution can be written in terms of elliptic functions, but qualitatively the dynamics of such models differs little from (17) for $P = \varepsilon/3$, although nonrelativistic matter ($P = 0$) leads to a faster isotropization than radiation with $P = \varepsilon/3$.

Thus, the behavior of the homogeneous anisotropic mode in a model of type V during the parabolic stage coincides with that of a model of type I and is undistinguishable from the case of long waves in a flat Friedmann model,[15] but during the Milne stage it decays faster, according to the law

$$|h| = \delta\mathscr{L} \propto (\tau_{FM}/\tau)^2 \propto R_{FM}^{-2} \tag{22}$$

on account of the influence of the curvature, acting in analogy with a fluid with $P_k = -(1/3)\varepsilon_k$ (cf. (10) for $n = -1/3$).[5)]

5. The Bianchi type IX is the only homogeneous modification of the closed Friedmann model ($k = +1$, $\varepsilon_0 > \varepsilon_c$) with closed spatial sections V_3 having the topology of a 3-sphere S^3 and has been considered principally near the singularity during the vacuum oscillatory stage, neglecting the gravity of matter.[1-3] For a diagonal frame (tetrad) metric when the proper rotation of matter is excluded, one has also investigated the full dynamics of triaxial models[9,11] and found that a gravitating fluid with $P \leqslant \varepsilon/3$ can guarantee only approximate isotropization owing to the finiteness of the total expansion in the model of type IX. The spatial curvature remains as a rule anisotropic, although in typical regimes there is a tendency of equalization of the two principal values of the curvature tensor of V_3 and the model approached a perturbed axially symmetric model (up to the maximal expansion, which is replaced then by a phase of overall contraction to the second singularity).

A special version of the type IX metric with additional axial symmetry

$$-ds^2 = -d\tau^2 + X^2(\tau)[d\psi + \cos\vartheta\, d\varphi]^2 + Y^2(\tau)[d\vartheta^2 + \sin^2\vartheta\, d\varphi^2] \tag{23}$$

has anisotropic principal curvatures V_3

$$K_1^1 = \frac{X^2}{2Y^4}, \quad K_2^2 = K_3^3 = \frac{1}{Y^2} - \frac{X^2}{2Y^4}, \quad K = \frac{2}{Y^2} - \frac{X^2}{2Y^4} \tag{24}$$

and includes as a special case $X = Y = R_F/\sqrt{2}$ the closed Friedmann metric. As a cosmological model the axially symmetric type IX has been studied by a number of authors[4,22,11]; Taub has found an exact vacuum metric which can be written in the following canonical form:

$$\begin{gathered} -ds^2 = -U^{-1}(T)\,dT^2 + U(T)[d\tilde{\psi} + 2N\cos\vartheta\, d\varphi]^2 + (T^2 + N^2)[d\vartheta^2 + \sin^2\vartheta\, d\varphi^2], \\ 0 < U(T) = 2(MT + N^2)/(T^2 + N^2) - 1. \\ T_- \leqslant T \leqslant T_+. \quad T_\pm = M \pm (M^2 + N^2)^{1/2}, \end{gathered} \tag{25}$$

which encompasses only the homogeneous T-region V_4.[13] It represents a vacuum anisotropic modification of the closed Friedmann cosmology with one excited standing gravitational wave in the lowest homogeneous mode. The self-interaction of this wave imitates material sources and guarantees the closed character of spatial sections V_3. The boundaries $T = T_\pm$ in the synchronous system (23) manifest themselves as a "kinematic" Kasner collapse of anisotropically hyperspheres $\tilde{S}_3$ (T = const) getting deformed into a "disk"

$$X \propto (\tau - \tau_\pm) \to 0. \quad Y(\tau_\pm) = \text{const} \neq 0.$$

At these pseudosingularities $U(T_\pm) = 0$, which, like the Schwarzschild sphere, represent in reality null-hypersurfaces and only express the incompleteness of the T-system (23), the invariants of the curvature tensor of V_4 are finite. Therefore the metric (25) can be analytically continued into stationary inhomogeneous R-regions V_4, where $U(T) < 0$ and the time coordinate T and the spatial "angular" coordinate ($0 \leqslant \tilde{\psi} \leqslant 4$) interchange their roles. The stationary Newman-Unti-Tamburino (NUT) form obtained from (25) by a simple relabeling $T \to R$, $\tilde{\psi} \to t$ appears as a generalization of the Schwarzschild metric ($N\equiv 0$) and contains an additional invariant parameter $N\neq 0$, to be identified with the "quasimagnetic" type of gravitational mass.[23]

As it is filled with gravitating fluid, the T-region (23) becomes a normal cosmological model with two singularities (ε, $P\to\infty$) and the R-regions which have physical anomalies, such as closed timelike geodesics, are removed from V_4. In view of the instabilities of the properties of the Taub-NUT "universe" when matter is introduced into the T-region (and with respect to metric perturbations of the free gravitational field, also leading to a singularity in the vacuum) the axially symmetric type IX model is degenerate. Its dynamics must differ qualitatively from the more general triaxial case, since there is no vacuum stage and only the gravitation of matter determines the appearance and the behavior of the singularities. For such models with matter with $P=n\varepsilon$, no analytic solution could be found (except the isotropic Friedmann metric with $X=Y=R_F/\sqrt{2}$) and one usually restricts one's attention to a qualitative analysis of the dynamics and numerical integration of the most characteristic examples.[11,22]

However, in the special case of maximally "stiff" matter with $P=\varepsilon$ the Einstein equations admit of an exact general solution if one sets in (23) $d\tau=XY^2d\eta$[13]:

$$X^2=\frac{\alpha}{\operatorname{ch}\alpha\eta},\quad Y^2=\frac{\gamma^2}{4\alpha}\operatorname{ch}\alpha\eta\Big/\operatorname{ch}^2\frac{\gamma}{2}(\eta+\eta_0),\quad \varkappa\varepsilon=\frac{\beta^2}{4X^2Y^4},\qquad \alpha^2+\beta^2=\gamma^2,\quad \eta_0=\text{const}. \tag{26}$$

At $\beta^2=0$ this solution goes over into the Taub metric (25), so that the constants α and η_0 can be related to the parameters of the "dual" mass, M and N:

$$Y^2(\pm\infty)={}^1/_2\alpha\exp(\mp\alpha\eta_0)=T_\pm{}^2+N^2,\quad T_\pm=M\pm(M^2+N^2)^{1/2}.$$

As a special case, if $\beta^2=3\alpha^2$ and $\eta_0=0$, this solution contains the closed Friedmann model with $P=\varepsilon$ can be sufficiently close to it in dynamical behavior in weakly perturbed ($|\alpha-\beta/\sqrt{3}|<\alpha$, $\gamma\eta_0\ll 1$) anisotropic variants, particularly for time-symmetric solutions with $\eta_0=0$ ($M=0$). These models (26) have an initial and final singularity ($\eta\to\pm\infty$) which in synchronous time $\tau=\int XY^2d\eta$ is characterized by the same power-law asymptotic behavior as the anisotropic collapse of the form (15):

$$X\propto\tau^{\tilde p_1}\to 0,\quad Y\propto\tau^{\tilde p_2}\to 0,\quad \varkappa\varepsilon\approx\mu/\tau^2\to\infty,\qquad \tilde p_1=\frac{\alpha}{2\gamma-\alpha}=1-2\tilde p,\quad \tilde p_2=\frac{\gamma-\alpha}{2\gamma-\alpha}=\tilde p,\qquad 0<\tilde p<{}^1/_2,\quad \mu=\beta^2/(2\gamma-\alpha)^2<{}^1/_3, \tag{27}$$

in which gravitating matter with $P=\varepsilon$ dominates and the influence of the anisotropic curvature (23) is negligible; the curvature scalar becomes negative and unbounded if[6)] $p>1/3$, in spite of the S_3-topology of the closed V_3 sections.

A qualitative analysis of the Einstein equations for the axially-symmetric type IX models filled with gravitating fluid with $P=n\varepsilon$ $(0\leqslant n<1)$[8,11] shows that there are the following three kinds of singularities with power-law asymptotic behavior, depending on the equation of state $P=n\varepsilon$ of matter[13] (there can be no other asymptotic behaviors and limit cycles[11]):

a) A quasi-isotropic point collapse of the Friedmann type

$$\frac{X}{C_1}=\frac{Y}{C_2}\approx\tau^{2/3(1+n)}\to 0,\quad \varkappa\varepsilon=\frac{4}{3(1+n)^2\tau^2}\to\infty \tag{28}$$

in the special case when there is no vacuum anisotropy of the deformation velocities of V_3 ($\Sigma=0$) and the gravitating matter with $P=n\varepsilon$ by itself governs the dynamics. The influence of the spatial curvature may be neglected ($K\propto\tau^{-4/3(1+n)}\to\infty$) but its anisotropy ($C_1\neq C_2$) has a substantial influence during the expansion and leads to increasing deviations from the Friedmann regime, particularly during the phase of contraction with approach to the second singularity, which is now of a type different from the "disk."

b) A degenerate anisotropic collapse of V_3 into a point, according to the law

$$X=\frac{(1+2n-3n^2)^{1/2}}{2(1+n)}C^{(3+n)/2(1+n)}\tau^{(1-n)/2(1+n)}\to 0,\qquad Y=(C\tau)^{(3+n)/4(1+n)}\to 0,\quad \varkappa\varepsilon\approx\frac{5-n}{4(1+n)^2\tau^2}\to\infty, \tag{29}$$

when the dynamical influence of matter and the influence of the anisotropic spatial curvature $K_1^1=-K_2^2=-K_3^3\propto\tau^{-2}\to\infty$ (i.e., the influence of the "stress" tensor of the homogeneous standing wave) are of the same order of magnitude, with the effective "energy density" (the curvature scalar $K=-K_2^2\to-\infty$ for closed $\tilde S_3$-sections.

c) The general case of the anisotropic collapse into a "disk," when the spatial curvature is "switched off" near the singularity and the behavior of models of type IX, (23) coincides with the kinematic asymptotic behavior (8a) for the symmetric type I for $P=n\varepsilon$ ($n<1$):

$$X=X_0\tau\to 0,\quad Y=Y_0\tau_F\left[1+\frac{2}{3(1-n)}\left(\frac{\tau}{\tau_F}\right)^{1-n}\right]\to\text{const},\qquad \varkappa\varepsilon=\frac{4}{3\tau^{1+n}\tau_F^{1-n}}\to\infty. \tag{30}$$

In (30), in the same manner as for the Taub-NUT vacuum metric (25) the horizon of causal connection along the symmetry axis ($0\leqslant\psi\leqslant 4\pi$) is absent on the null-boundaries of the T-region $U(T_\pm)=0$, and photons can go around a closed $\tilde S_3$-section an infinite number of times as $\tau\to 0$. However, the gravitating matter transforms these boundary horizons of the type of the Schwarzschild sphere into genuine singularities: ε, $P\propto\tau^{-(1+n)}\to\infty$ (without changing their lightlike orientation if $P<\varepsilon$), so that an analytical continuation from the T-region into the R-regions becomes impossible.

In the axially symmetric models of type IX the gravitation of matter is always dynamically important, in particular for the most typical regime of initial expansion from the "disk" singularity (30). If matter predominates significantly over the curvature in the dynamics at $\tau\gtrsim\tau_F$ ($X_0^2\ll Y_0^4$, $Y_0^2\gg 1$), then at this matter-dominated stage it guarantees the approximate isotropization of the expansion and the approach of a quasi-isotropic parabolic ($k=0$, $\varepsilon_0\simeq\varepsilon_c$) regime (28), as in the case of the model of type I, (9). However, the curvature does not become automatically isotropic ($C_1 \neq C_2$) and leads again to an increase of the anisotropy of the Hubble expansion velocities. Only for special initial conditions ($X_0 \simeq Y_0$) does there appear an equalization of the principal curvatures of V_3 ($X \simeq Y$) together with the isotropization of the Hubble velocities, and such models of the type IX will be sufficiently close to the closed Friedmann model with $\varepsilon_0 > \varepsilon_c$ at a sufficiently long duration of the stage up to the maximum expansion. In the case when the curvature is switched on before the instant of isotropization τ_F ($X_0^2 \gg Y_0^4$) the intermediate regime (29) is realized when the gravity of matter and the anisotropy of curvature together determine the dynamics of expansion. By means of the asymptotic behaviors (28)–(30) determined near the initial singularity, a numerical calculation was carried out on an electronic computer for a number of characteristic parameters with $P=0$ and $P=\varepsilon/3$, confirming the results of a qualitative analysis of the dynamics. The models of type IX (23) usually have two "disk" singularities (30), as well as a vacuum T-region, although combinations of the "disk" collapse with a pointlike isotropic (28) or anisotropic (29) singularity are possible.

Opposed fluxes of free particles (16) are incompatible with a quasi-isotropic asymptotic behavior of a point collapse (even in the presence of a fluid with $P \lesssim \varepsilon/3$) and exclude the possibility of kinematic collapse of V_3 into a "disk" (30). Owing to the considerable reaction on the metric near the boundary of the T-region ($\Pi \propto \tau^{-2} \to \infty$) they must destroy null-horizons (similar to a "scalar fluid" with $P_s = \varepsilon_s$) and in the axially symmetric type IX model (23) they lead to the uniquely possible regime of anisotropic point collapse of the form

$$X = X_0 \tau^{1/3} \to 0, \quad Y = Y_0 \tau^{2/3} \to 0, \quad \varkappa \Pi \approx 2/3\tau^2 \to \infty,$$
$$K_1^1 \approx -K_2^2 = -K_3^3 = 4/9\tau^2 \to \infty, \tag{31}$$

which is determined by the joint action of the anisotropic curvature and the free particles.

In the case of a mixture of a fluid with $nP = n\varepsilon$ ($0 \leq n \leq 1/3$) and free particles (16), the latter dominate near the singularities (31), if $n < 1/5$, and it dominates over matter and the asymptotic behavior (29) is realized at $n > 1/5$. At $n > 1/3$ a quasi-isotropic Friedmann collapse (28) becomes possible with $\Pi/\varepsilon \to 0$, and also a singular anisotropic regime of collapse into a point (17), in which the influence of the curvature (24) is negligible.

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1) The results of this paper were partially published before.[13]

2) We note that in the isotropic models the following anisotropic perturbation modes of the free gravitational tensor field are admissible: types I and VII_0 in the quasi-Euclidean ($k=0$) model, types V and VII_h in the open model ($k=-1$), and type IX in the closed model ($k=+1$).

3) The integral $\Phi(\eta)$ can be calculated explicitly for the series $n = m(1+m) = 0, 1/2, 2/3, \ldots$, and $n = (2m+1)/(2m+3) = 1/3, 3/5, \ldots$, ($m = 0, 1, 2, \ldots$).

4) When "dust" particles ($P=0$) are focused on a two-dimensional caustic there appears a flat "pancake" with infinite volume density but finite surface density. Since the relative acceleration of two flat Lagrangian slices is proportional to the specific gravitational mass between them, which is constant for $P=0$, the relativistic and Newtonian motions are similar. But if $P \neq 0$, the active mass between the slices changes on account of the work of the pressure forces, in agreement with the adiabatic compression of the medium $m(\tau) \propto \varepsilon V \propto V^{-n} \propto \tau^{-n} \to \infty$ and the mutual attraction of adjacent slices increases without bound during relativistic collapse into a "pancake".

5) In the formulation of Lifshitz[15] the gravitational waves in the open Friedmann model in the Milne stage (both short and long waves ($\lambda \gtrless R_{FM}$)) are damped according to the same law $\nu(\tau) \propto R_{FM}^{-1} \propto \tau^{-1}$. A faster decrease of the amplitude of the homogeneous mode for the type V model is related to the dilution effect of the wavefront for a wave of finite wavelength ($\lambda \gtrsim R_{FM} \sim c\tau$) on the curved hyperbolic background V_3 in the type VII_h,[10] which seems to be preserved in the limit $\lambda \to \infty$.

6) The cosmological model of type IX with maximally stiff matter (26) is equivalent to a solution of the simultaneous Einstein and D'Alembert equations in vacuum ($\Box\Psi = 0$) for the coupled gravitational tensor and massless scalar fields, if in agreement with (14) one substitutes $\beta^2 \to 12\mu_s^4 = (2\omega+3)S^2$, $\Psi(\eta) = \Psi_0 + S\eta$. It can be treated as an isolated T-region (23) with a homogeneous mode of the free scalar field $\Psi = \Psi(\tau)$ as material source of the geometry of V_4 acting similarly to a gravitating fluid with $P_s = \varepsilon_s = \frac{1}{4}\zeta(d\Psi/d\tau)^2$, $\varkappa\varepsilon_s = 3\mu_s^4/X^2Y^4$ and leads to two point-singularities of the type (27), completely removing the boundary null-horizons. In this respect the scalar massless field is qualitatively different from the vector electromagnetic field, which in the axially symmetric type IX model[24] is inscribed in the T-region in a nonsingular way, without destroying the boundary horizons $U(T_\pm) = 0$, and conserving the anomalous R-regions. We note that a spinor massless field of the neutrino type is incompatible with the "diagonal" metric (23) of the type IX, since its inclusion requires $T_{0\alpha} \neq 0$.[24]

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Gravitational antenna with SQUID as sensor

A. V. Gusev and V. N. Rudenko

Moscow State University
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A gravitational antenna with quantum magnetometer (SQUID) as sensor is considered. The factors that restrict the sensitivity of the SQUID in two regimes, without and with hysteresis, are analyzed. Expressions are obtained for the minimal detectable force connected with the intensity of gravitational radiation. Requirements are also formulated for the case of a sensor on an antenna with mechanical transformation of displacements.

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§1. INTRODUCTION

Negative results in experimental searches for gravitational waves[1,2] are stimulating the creation of a second generation of antennas with parameters that are close to the theoretical requirements.[3-9] For a resonance gravitational detector at the frequency $\omega_\mu \approx 10^4$ and linear dimension $l_g \approx 10^2$ cm the required sensitivity level is characterized by fluctuation variations of the vibration amplitude equal to $\Delta x_0 \approx 10^{-17}$ cm.[6,10] Concrete programs are underway to achieve this level by a high mechanical quality factor $Q_\mu \approx 10^{10}$ [3,11] or by lowering the temperature of the gravitational detector to $T_\mu \approx 3 \cdot 10^{-3}$ °K.[12] The main experimental difficulties are connected with measuring the small amplitude of the mechanical vibrations. It was suggested in[13,14] that a quantum magnetometer, a so-called SQUID,[1)] should be used.

The aim of the present paper is to analyze the SQUID as a sensing element of a gravitational antenna. For comparison, we shall use the characteristics of an antenna with capacitative parametric transducer,[10,11] which in the optimal regime has the sensitivity

$$(F_0)_{min} \approx \frac{3\sqrt{\pi}}{\hat{\tau}}\left(m\varkappa T\frac{\omega_\mu}{\omega}\right)^{1/2}; \tag{1}$$

here, m is the equivalent mass of the gravitational detector[9]; F_0 is the amplitude of the "external force", which is related to the flux density I_g of the gravitational radiation by $F_0 = m\omega_\mu l_g(8\pi Gc^{-3}I_g)^{1/2}$; T is the temperature of the transducer; ω is the pumping frequency; and τ is the duration of the radiation pulse.

§2. SUPERCONDUCTING DISPLACEMENT TRANSFORMER

On the antenna proposed in[13,14] the SQUID is coupled to the gravitational detector by means of a mechanical displacement transformer. The idea is due to Lavrent'ev[15] and reduces to a model of coupled oscillators: the gravitational detector with mass M and pendulum of small mass m fixed to it. At the natural frequency, the amplitude of the pendulum is $(M/m)^{1/2}$ times greater than the amplitude of the forced vibrations of the gravitational detector. If $Q_{tr} \geqslant Q_\mu$ (Q_μ and Q_{tr} are the Q factors of the detector and the pendulum), then for $T_\mu = T_{tr}$ the inherent fluctuations of the pendu-

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