Propagation of weak waves in a strong electromagnetic wave

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The interaction of plane electromagnetic waves and gravitational waves is investigated. It is proved on the basis of the Einstein-Maxwell equations that an empty space-time that admits of an isotropic absolutely parallel Killing field, is described by the Peres solution. The equations for small perturbations of the gravitational and electromagnetic fields against the background of a strong plane electromagnetic wave are examined in the Newman-Penrose formalism. These equations are reduced to a system of two second-order equations relative to the tetrad components of the Weyl tensor Ψ_0 and the Maxwell tensor F_0. Explicit solutions are obtained for the cases when weak plane gravitational and electromagnetic waves are incident on a strong wave.

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INTRODUCTION

We consider in this article the propagation of weak electromagnetic and gravitational waves in the field of a strong electromagnetic waves. For short-wave perturbations of the background of a strong electromagnetic wave, the question is quite clear. We encounter here all the effects that manifest themselves in the propagation of short waves in a vacuum in the presenee of external electromagnetic fields,[1] viz., the partial and total mutual conversion of gravitational and electromagnetic waves over a certain characteristic length, rotation of the plane of polarization relative to a parallel-transferred tetrad, conservation of the combined energy flux in the gravitational and electromagnetic waves, and the reaction of the short waves on the background over which they propagate.

In the case of plane waves, there is a particularly pronounced tendency of the beam of null-geodesics to become focused without rotation, a tendency inherent in the Einstein theory of gravitation: a beam of parallel null-geodesic, after passing through a nonlinear plane wave, intersects itself on a certain caustic surface, and the rays from an instantaneously flashing and dimming source are gathered, after passing through a plane wave, in a certain point or on a spacelike curve.[2,3] It can be assumed that inasmuch as the geometric-optics method is not valid on caustics, it is necessary, while remaining within the framework of the linear equations for the perturbations, merely to refine in the vicinity of the caustics the procedure used for the expansion in the reciprocal frequencies (as is done, for example, in phenomena involving diffraction of light[4] or in the case of an arbitrary hyperbolic system[5]).

The problem of the linear interaction of two plane θ-like electromagnetic waves was recently solved in[6]. An exact solution was obtained also for the problems involving collisions of a plane θ-like electromagnetic and a δ-like gravitational wave, as well as two θ-like neutrino waves.[7,8]

It is shown in Sec. 1 that the solutions of the electrovacuum equations, which admits of an isotropic absolutely parallel Killing vector, reduced with the aid of coordinate transformations to the Peres solution. In Sec. 2, the Einstein–Maxwell equations, linearized with respect to the background of a plane electromagnetic wave with circular polarization, are reduced to closed second-order equations for certain combinations of perturbations of the electromagnetic and gravitational fields. In Sec. 3 we obtain an exact solution of the linearized equations of Sec. 2 for the case when weak plane electromagnetic and gravitational waves pass through a strong electromagnetic wave.

Compared with the exact results already available on this subject, we investigate in the present article plane waves with trailing edges. In addition, the employed small-perturbation method enables us to obtain solutions also in the case when the weak wave is of arbitrary shape.

We note in connection with our results that the concept of the cosmic primordial radiation as an aggregate of plane electromagnetic waves with a Planck intensity distribution in frequency is no longer valid at high radiation intensity. The gravitational and electromagnetic radiations begin to interact with each other, since (as shown below) one electromagnetic wave passing through another is partially converted into a gravitational wave. The nonlinear wave interaction described is partially converted into a gravitational wave. The nonlinear wave interaction described below for the collision of plane electromagnetic waves is a classical effect, in contrast to the photon-photon interactions studied in quantum electrodynamics:

$$\gamma+\gamma\to\gamma'+\gamma'',\quad \gamma+\gamma\to e^++e^-,\quad \gamma+\gamma\to\mu^++\mu^-.$$

We note also that the observed processes take place in pure form only at low energy densities, when no importance attaches to the quantum effects of the gravitational interactions between photons, effects considered, e.g., by Vereshkov and Poltavtsev.[9]

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1. REDUCTION OF THE METRIC OF AN EMPTY SPACE THAT ADMITS OF AN ABSOLUTE PARALLEL ISOTROPIC VECTOR FIELD TO A PERES METRIC

There are several published solutions of the Einstein equations corresponding to nonlinear plane gravitational waves. All these solutions—those of Takeno, Petrov, Landau and Lifshitz, Kaigorodov and Pestov, Peres (see the references in Zakharov's book[10])—have an absolutely parallel isotropic vector field. We shall show that all these solutions are reducible by a coordinate transformation to the Peres metric and are of type N (after Petrov). It can be shown that the metric tensor of an Einstein space that admits of an absolutely parallel isotropic vector field l^α can be represented in the general case in the form (see, e.g., [11])

$$(g_{\alpha\beta})=\begin{bmatrix} A & 1 & S & 0\\ 1 & 0 & 0 & 0\\ S & 0 & -E & 0\\ 0 & 0 & 0 & -1\end{bmatrix}, \tag{1.1}$$

where A, S, and E are functions of the coordinates x^0, x^2, x^3, and in this coordinate system we have $l^\alpha=\delta_1^\alpha$. To investigate this metric we shall use the Newman–Penrose formalism.[12] We choose a tetrad field in the form

$$\begin{gathered} l^\alpha=(0,1,0,0),\quad l_\alpha=(1,0,0,0),\\ n^\alpha=(1,-A/2,0,0),\quad n_\alpha=(A/2,1,S,0),\\ m^\alpha=(0,S/\sqrt{2E},-1/\sqrt{2E},-i/\sqrt{2E}),\quad m_\alpha=(0,0,\sqrt{E/2},i/\sqrt{2}),\\ \bar m^\alpha=(0,S/\sqrt{2E},-1/\sqrt{2E},i/\sqrt{2E}),\quad \bar m_\alpha=(0,0,\sqrt{E/2},-i/\sqrt{2}) \end{gathered}$$

and put $x^0=u$, $x^1=v$, $x^2=x$, $x^3=y$. The Ricci rotation coefficients are then expressed in terms of the functions A, S, and E as follows:

$$\begin{gathered} \lambda=\frac{1}{4}\frac{\partial}{\partial u}\ln E,\quad \mu=\frac{1}{4}\frac{\partial}{\partial u}\ln E-\frac{i}{2\sqrt{E}}\frac{\partial S}{\partial y},\\ \nu=\frac{1}{\sqrt{2E}}\frac{\partial S}{\partial u}-\frac{1}{2\sqrt{2E}}\frac{\partial A}{\partial x}-\frac{i}{2\sqrt{2}}\frac{\partial A}{\partial y},\\ \alpha=\beta=-\frac{i}{4\sqrt{2}}\frac{\partial\ln E}{\partial y},\quad \gamma=-\frac{i}{4\sqrt{E}}\frac{\partial S}{\partial y}. \end{gathered}\tag{1.2}$$

The remaining rotation coefficients vanish identically. In our case the nontrivial Einstein equations are the following (see[10]):

$$\begin{gathered} 2\Phi_{11}=(D-\rho+\bar\rho+\varepsilon)\gamma+(\delta+2\beta-\tau-\bar\alpha-\bar\pi)\alpha-(\bar\delta+\pi+\bar\tau+\bar\beta)\beta\\ -(\Delta-2\gamma-\bar\gamma+\mu-\bar\mu)\varepsilon-\mu\rho-\pi\tau+\nu k+\lambda\sigma,\\ \Phi_{12}=(\delta-\tau+\bar\alpha+2\beta)\gamma-(\Delta+\mu+\bar\gamma)\beta-\mu\tau+\nu\sigma+\varepsilon\bar\nu-\alpha\bar\lambda,\\ 2\Phi_{21}=(D+\bar\rho+\varepsilon-\rho+3\varepsilon)\nu+(\delta+3\beta-\tau-\bar\alpha-\bar\pi)\lambda\\ -(\bar\delta+\alpha+2\pi+\bar\tau+\bar\beta)\mu-(\Delta-\bar\gamma-\bar\mu)\pi-\pi\gamma,\\ \Phi_{22}=(\delta+3\beta-\tau+\bar\alpha)\nu-(\Delta+\mu+\gamma+\bar\gamma)\mu+\pi\bar\nu-\lambda\bar\lambda. \end{gathered}$$

Here

$$D=l^\alpha\partial_\alpha,\quad \Delta=n^\alpha\partial_\alpha,\quad \delta=m^\alpha\partial_\alpha,\quad \bar\delta=\bar m^\alpha\partial_\alpha.$$

These equations, with (1.2) taken into account, take the form

$$\begin{gathered} 0=(\delta-\bar\delta)\alpha+4\alpha^2,\quad 0=\delta\gamma-\Delta\alpha-2\lambda\alpha,\\ 0=(\delta-\bar\delta)\lambda+4\alpha\lambda-2\bar\delta\gamma,\quad \Phi_{22}=\delta\nu-\Delta\mu+2\alpha\nu-\mu^2-\lambda\bar\lambda. \end{gathered}$$

The first three equations form a system for the unknown functions E and S. This system reduces to the form

$$\begin{gathered} \frac{\partial^2\ln E}{\partial y^2}+\frac{1}{2}\left(\frac{\partial\ln E}{\partial y}\right)^2=0,\quad \frac{\partial}{\partial y}\left(\frac{E^{-1/2}\partial S}{\partial y}\right)=0,\\ \frac{\partial}{\partial x}\left(\frac{E^{-1/2}\partial S}{\partial y}\right)+2\frac{\partial^2 E^{1/2}}{\partial u\partial y}=0. \end{gathered}\tag{1.3}$$

Integrating (1.3), we get

$$\begin{gathered} E^{1/2}=a(u,x)y+b(u,x),\quad S=c(u,x)[a(u,x)y^2+2b(u,x)y]+d(u,x),\\ \partial a/\partial u+\partial c/\partial x=0. \end{gathered}$$

We list now the sequence of coordinate transformations that reduces the metric (1.1) to the Peres metric. We write out first the transformation and then the form assumed by the square of the integral after the indicated transformation. In those cases when the independent variables are not explicitly indicated, the functions depend on u and x. The new coordinates will be marked by a tilde over the letter, but in the metric form the tilde will be left out.

We assume that $a\neq 0$.

1. The transformation is

$$\tilde x=\int_{x_0}^{x} a\,dx,$$

and the metric

$$ds^2=Adu^2+2dudv+[c(y+b)^2+d]dudx-(y+b)^2dx^2-dy^2.$$

2. The transformation

$$\tilde v=v+\psi,\quad \psi(u,x)=\int_{x_0}^{x}(d-cb^2)dx,$$

the metric

$$ds^2=Adu^2+2dudv+c(y+b)^2dudx-(y+b)^2dx^2-dy^2.$$

3. The transformation

$$\tilde x=y\cos x-\int_{x_0}^{x} b\sin x\,dx,\quad \tilde y=-y\sin x-\int_{x_0}^{x} b\cos x\,dx,$$

the metric

$$ds^2=Adu^2+2dudv+f(u,x,y)dudx+g(u,x,y)dudy-dx^2-dy^2.$$

4. The transformation

$$\tilde v=v+\psi(u,x,y),\quad \psi(u,x,y)=\int_{y_0}^{y} g(u,x,y)dy,$$

the metric

$$ds^2=Adu^2+2dudv+a(u,x,y)dudx-dx^2-dy^2. \tag{1.5}$$

The metric (1.5) is of the same form as (1.1) and (1.4) with $a=0$ and $b=1$. Therefore, taking into account the third equation of (1.4), we have

$$a(u,x,y)=2c(u)y+d(u,x).$$

5. The transformation

$$\tilde{v}=v+\psi(u,x,y),\quad \psi(u,x,y)=c(u)xy+\int_{x_0}^{x} d(u,x)\,dx,$$

the metric

$$ds^2=A\,du^2+2du\,dv+c(u)(y\,dx-x\,dy)\,du-dx^2-dy^2. \tag{1.6}$$

6. On changing to the polar coordinates

$$x=r\cos\varphi,\quad y=r\sin\varphi,$$

the metric (1.6) takes the form

$$ds^2=A\,du^2+2du\,dv+2c(u)r^2du\,d\varphi-dr^2-r^2d\varphi^2.$$

7. For the transformation

$$\tilde{\varphi}=\varphi-\int_{u_0}^{u} c(u)\,du,$$

the metric is

$$ds^2=A\,du^2+2du\,dv-dr^2-r^2d\varphi^2. \tag{1.7}$$

Let now $a\equiv 0$. Then the transformation

$$\tilde{x}=\int_{x_0}^{x} b\,dx$$

reduces the metric to the form (1.5), so that this metric is also the Peres metric (1.7).

2. EQUATION FOR SMALL PERTURBATIONS AGAINST THE BACKGROUND OF A PLANE ELECTROMAGNETIC WAVE

A plane wave in general relativity theory means space-time with a metric

$$ds^2=A(u,x,y)\,du^2+2du\,dv-dx^2-dy^2, \tag{2.1}$$

$$\partial^2A/\partial x^2+\partial^2A/\partial y^2=4\Phi_{22}. \tag{2.1}$$

For a pure gravitational wave (PGW) we have $\Phi_{22}=0$ and a particular solution of (2.2) is

$$A(u,x,y)=a(u)(x^2-y^2)+b(u)xy,$$

while for a pure electromagnetic wave (PEW)

$$A(u,x,y)=a(u)(x^2+y^2),\quad a(u)\geqslant 0.$$

We shall use the tetrad field

$$\begin{gathered}
l^\alpha=(0,1,0,0),\quad l_\alpha=(1,0,0,0),\\
n^\alpha=(1,-A/2,0,0),\quad n_\alpha=(A/2,1,0,0),\\
m^\alpha=(0,0,1/\sqrt{2},i/\sqrt{2}),\quad m_\alpha=(0,0,-1/\sqrt{2},-i/\sqrt{2}),\\
\bar{m}^\alpha=(0,0,1/\sqrt{2},-i/\sqrt{2}),\quad \bar{m}_\alpha=(0,0,-1/\sqrt{2},i/\sqrt{2}).
\end{gathered}$$

In this tetrad field, the space-time (2.1) has the following nonzero characteristics:

a) the Ricci rotation coefficient

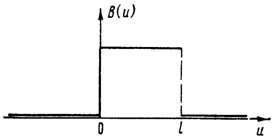

FIG. 1. Profile of strong plane electromagnetic wave.

$$\nu=2^{-1/2}(A_{,x}-iA_{,y}),$$

b) the tetrad component of the Weyl tensor

$$\Psi_4=\bar{\delta}\nu=\tfrac{1}{4}(A_{,xx}-A_{,yy}-2iA_{,xy}), \tag{2.3}$$

c) the tetrad component of the Ricci tensor

$$\Phi_{22}=-\tfrac{1}{2}R_{22}=\delta\nu=\tfrac{1}{4}(A_{,xx}+A_{,yy}).$$

An electromagnetic wave in a PEW can be defined by a 4-potential

$$A_\alpha=(\varphi,0,0,0),\quad \varphi=B(u)(x\cos\omega u+y\sin\omega u).$$

In this case

$$\begin{gathered}
A(u,x,y)=\frac{\varkappa B^2(u)}{16\pi}(x^2+y^2),\quad \nu=\frac{\varkappa B^2(u)}{16\pi\sqrt{2}}(x-iy),\\
\Psi_4=0,\quad \Phi_{22}=\frac{\varkappa}{8\pi}F_2\bar{F}_2=\frac{\varkappa}{16\pi}B^2(u),\\
F_2=-F_{\alpha\beta}n^\alpha\bar{m}^\beta=-2^{-1/2}B(u)e^{-iu\omega}.
\end{gathered} \tag{2.4}$$

Consider the case when $B(u)$ is a step function (Fig. 1). We obtain an equation for small perturbations against the background of a PEW. The small perturbations of the various quantities will be designated by the same letter, but primed. We introduce the symbol $b=B(\varkappa/16\pi)^{1/2}$.

We linearize the following:

a) Maxwell's equations in terms of the tetrad components:

$$DF_1-\bar{\delta}F_0=(\pi-2\alpha)F_0+2\rho F_1-kF_2, \tag{2.5}$$

$$\delta F_1-\Delta F_0=(\mu-2\gamma)F_0+2\varepsilon F_1-\sigma F_2; \tag{2.6}$$

b) the Bianchi identities:

$$\begin{aligned}
-(D-4\rho-2\varepsilon)\Psi_1+(\bar{\delta}-4\alpha+\pi)\Psi_0-3k\Psi_2={}&-(D-2\bar{\rho}-2\varepsilon)\Phi_{01}\\
&+(\delta-2\bar{\alpha}-2\beta+\pi)\Phi_{00}+2\sigma\Phi_{01}-2k\Phi_{11}-\bar{k}\Phi_{02},
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
(\Delta-4\gamma+\mu)\Psi_0-(\delta-4\tau-2\beta)\Psi_1-3\sigma\Psi_2={}&-(D-\bar{\rho}+2\bar{\varepsilon}-2\varepsilon)\Phi_{02}\\
&+(\delta+2\bar{\pi}-2\beta)\Phi_{01}-\bar{\lambda}\Phi_{00}+2\sigma\Phi_{11}-2k\Phi_{12};
\end{aligned} \tag{2.8}$$

c) the formula for Ψ_0:

$$\Psi_0=(D-3\varepsilon+\bar{\varepsilon}-\rho-\bar{\rho})\sigma-(\delta-3\beta-\tau-\bar{\alpha}+\pi)k. \tag{2.9}$$

Linearizing (2.5) – (2.9) we obtain

$$DF_1'-\bar{\delta}F_0'=-k'F_2, \tag{2.5'}$$

$$\delta F_1'-\Delta F_0'=-\sigma'F_2, \tag{2.6'}$$

$$-D\Psi_1'+\bar{\delta}\Psi_0'=-D\Phi_{01}'+\delta\Phi_{00}', \tag{2.7'}$$

$$\Delta\Psi_0'-\delta\Psi_1'=-D\Phi_{02}'+\delta\Phi_{01}', \tag{2.8'}$$

$$\Psi_0'=D\sigma'-\delta k'. \tag{2.9'}$$

Here

$$\Phi_{00}' = -\frac{\varkappa}{2}T_{11},\quad \Phi_{01}' = -\frac{\varkappa}{2}T_{13},\quad \Phi_{02}' = \frac{\varkappa}{8\pi}F_0'\bar{F}_2 - \frac{\varkappa}{2}T_{33},$$

$T_{\alpha\beta}$ is an energy-momentum tensor of non-electromagnetic origin.

We eliminate F_i' and Ψ_i' from (2.5′)—(2.9′) and use the fact that for a scalar function f we have

$$\Box f = 2(D\Delta - \delta\delta)f.$$

Then

$$\Box F_0' = -\sqrt{2}Be^{-iu\omega}\Psi_0', \tag{2.10}$$

$$\Box\Psi_0' = -\frac{B\varkappa}{4\pi\sqrt{2}}e^{iu\omega}D^2F_0' + \varkappa L; \tag{2.11}$$

where $L = D^2T_{33} + \delta^2T_{11} - 2D\delta T_{13}$.

We take the Fourier transforms:

$$\tilde{A} = \int e^{ikv}F_0'dv,\quad \tilde{B} = \int e^{ikv}\Psi_0'dv.$$

Then Eqs. (2.10) and (2.11) go over into the system

$$\left[2ik\frac{\partial}{\partial u} - b^2r^2k^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\tilde{A} = 4\left(\frac{2\pi}{\varkappa}\right)^{1/2} be^{-iu\omega}\tilde{B}, \tag{2.12}$$

$$\left[2ik\frac{\partial}{\partial u} - b^2r^2k^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\tilde{B} = \left(\frac{\varkappa}{2\pi}\right)^{1/2} bk^2\tilde{A}e^{iu\omega} + \varkappa\tilde{L}. \tag{2.13}$$

If we change over in the region $\{u\in(0,l)\}$ to the new variables

$$D_j = \left(\frac{\varkappa}{2\pi}\right)^{1/2} e^{iu\omega}\tilde{A} + \eta_j\tilde{B}\quad (j=1,2),\quad \eta_{1,2} = \frac{1}{bk}[\omega \pm (\omega^2+4b^2)^{1/2}], \tag{2.14}$$

then the system (2.12) and (2.13) breaks up into two independent equations

$$\left[2ik\frac{\partial}{\partial u} - b^2r^2k^2 - \frac{4b}{\eta_j} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]D_j = \varkappa\eta_j\tilde{L}\quad (j=1,2). \tag{2.15}$$

An analogous situation obtains in the Nordstrom-Reissner field, where, as shown in[13], the Einstein-Maxwell system for small perturbations also breaks up into independent second-order equations for certain combinations of the perturbations of the metric and of the electromagnetic-field components. We take this opportunity to point out some errors in the formulas of the preceding article.[13] The correct expressions for $\varphi_0(\omega)$ and A on p. 1246 [p. 619 of the translation] are

$$\varphi_0(\omega) \approx 8i\omega(l+1)l\ln(-i\omega),$$

$$A = m\sum_{k,n=0}^{l}\frac{(l+k)!(l+n)!(-1)^{k+n+1}}{k!n!(l-k)!(l-n)!(k+n+2)!}\sum_{p=1}^{k+n+2}\frac{1}{p}.$$

Accordingly, it is necessary to eliminate the logarithmic term from formulas (5.2)—(5.5) of[13].[1)]

In the regions $\{u>l\}$, $\{u=l\}$, A and B are the solutions of the system of equations

$$\left[2ik\frac{\partial}{\partial u} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\tilde{A} = 0, \tag{2.16}$$

$$\left[2ik\frac{\partial}{\partial u} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]\tilde{B} = \varkappa\tilde{L}, \tag{2.17}$$

while in the region $\{0<u<l\}$ they are determined from

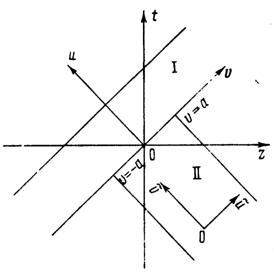

FIG. 2. Collision of a strong and weak wave; I—strong wave, II—weak wave.

(2.14) and (2.15). $\tilde{A}$ and $\tilde{B}$ must be continuous at the points of the surfaces $\{u=0\}$, $\{u=l\}$.

3. COLLISION OF A STRONG PEW WITH WEAK PGW AND PEW

A. Let a weak PGW (II) be incident on a strong PEW (I) (see Fig. 2). Assume that in terms of the coordinates u and v connected with the weak wave, its metric is of the form

$$ds^2 = 2d\tilde{u}d\tilde{t} + \varepsilon[\theta(v+a) - \theta(v-a)]xyd\tilde{u}^2 - dx^2 - dy^2.$$

Using (2.4), we can easily calculate

$$\tilde{\Psi}_4 = -{}^1/_2\varepsilon i[\theta(v+a) - \theta(v-a)].$$

It follows, however, from the transformation formulas for Ψ_i that in the region $\{u<0\}$, in terms of the coordinates u and v, we have

$$\Psi_0 = \bar{\tilde{\Psi}}_4 = {}^1/_2\varepsilon i[\theta(v+a) - \theta(v-a)].$$

Therefore

$$\tilde{B}|_{u=0} = \int e^{ikv}\Psi_0 dv = \frac{\varepsilon i}{k}\sin ka, \tag{3.1}$$

$$\tilde{A}|_{u=0} = 0. \tag{3.2}$$

We change over in (2.15) to polar coordinates and seek a solution that does not depend on the polar angle φ:

$$\left[2ik\frac{\partial}{\partial u} - b^2r^2k^2 - \frac{4b}{\eta_j} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]D_j = 0.$$

We separate the variables u and r:

$$D_j = \int E_j(\lambda, u)f(\lambda, r)d\lambda, \tag{3.3}$$

f satisfies the equation

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - b^2r^2k^2 - \lambda\right]f = 0.$$

This equation is well known in the theory of the confluent hypergeometric function (see[14], Vol. 1). It has two linearly independent solutions, but we need here only one of them:

$$f = e^{-b|k|r^2}\Phi\left(\frac{1}{2} + \frac{\lambda}{2b|k|}, 1, 2b|k|r^2\right).$$

Using the expansion of the confluent hypergeometric function in zero-order Bessel functions, we obtain the solution of (3.3) in the form

$$D_j=\exp(b|k|r^2-iba_ju)\int_0^{+\infty} e^{-t^2}J_0(2(2b|k|)^{1/2}rt)\,U_j(t\exp(-ibu\,\mathrm{sign}\,k/2))\,dt, \tag{3.4}$$

where $U_j(j=1,2)$ should be obtained from the initial conditions at $u=0$, while $\alpha_j=2/k\eta_j$. The initial conditions (3.1) and (3.2) as well as formula (2.14) yield

$$\eta_1U_2(t)-\eta_2U_1(t)=0,\quad D_1|_{u=0}-D_2|_{u=0}=(\eta_1-\eta_2)\frac{\varepsilon\iota}{k}\sin ka=p. \tag{3.5}$$

Substituting (3.4) in (3.5), using the Hankel transformation (see[14], Vol. 2) and formula 6.631.4 of[15]

$$\int_0^{+\infty} x^{\nu+1}e^{-\alpha x^2}J_\nu(\beta x)\,dx=\frac{\beta^\nu}{(2\alpha)^{\nu+1}}\exp\left(-\frac{\beta^2}{4\alpha}\right),\quad \mathrm{Re}\,\alpha>0,\quad \beta>0,\quad \mathrm{Re}\,\nu>-1, \tag{3.6}$$

we obtain U_j:

$$U_j(t)=4p\gamma_jte^{-t^2},\quad \gamma_j=\eta_j/(\eta_1-\eta_2). \tag{3.7}$$

We introduce the notation

$$\rho=\mathrm{tg}\frac{bu}{2},\quad \mu=\exp(-ibu\,\mathrm{sign}\,k/2),\quad \nu=\frac{2\mu}{\mu^2+1}=\frac{1}{\cos(bu/2)}.$$

We obtain D_j from (3.4), and (3.6), and (3.7):

$$D_j=p\nu\gamma_j\exp(-iba_ju-ib\rho kr^2).$$

As $bu\to\pi$ we have $|D_j|\to\infty$ and therefore the necessary condition for staying within the framework of the linear approximation is

$$bl<\pi.$$

We solve Eqs. (2.16) and (2.17) at $u>l$. We seek solutions in the form

$$\bar{A},\bar{B}=\int C(\lambda,u)J_0(\lambda r)\,d\lambda.$$

The equation for C yields readily

$$\bar{A}=\int P(\lambda)J_0(\lambda r)\exp(-i\lambda^2u/2k)\lambda\,d\lambda,$$
$$\bar{B}=\int Q(\lambda)J_0(\lambda r)\exp(-i\lambda^2u/2k)\lambda\,d\lambda. \tag{3.8}$$

Here P and Q are functions that can be obtained by joining the solutions together at $u=l$. We denote

$$\nu|_l=\nu_0,\quad \rho|_l=\rho_0,$$

$$a_1=(2\pi/\varkappa)^{1/2}e^{-i\omega l}\gamma_1\gamma_2(e^{-iba_2l}-e^{-iba_1l}).$$

To determine $P(\lambda)$ we must solve the equation

$$\int P(\lambda)J_0(\lambda r)\exp(-i\lambda^2l/2k)\lambda\,d\lambda=a_1\nu_0p\exp(-i\rho_0bkr^2). \tag{3.9}$$

We use for this purpose the Hankel transformation and formula No. 6.631.6 of[15]:

$$\int_0^{+\infty} x^{\nu+1}e^{\pm i\alpha x^2}J_\nu(\beta x)\,dx=\frac{\beta^\nu}{(2\alpha)^{\nu+1}}\exp\left[\pm i\left(\frac{\nu+1}{2}\pi-\frac{\beta^2}{4\alpha}\right)\right],\quad \alpha>0,\quad -1<\mathrm{Re}\,\nu<1/2. \tag{3.10}$$

Substituting the solution (3.9) in (3.8) and turning again to (3.10), we get

$$\bar{A}=\frac{-A_0\varepsilon}{2(1-2\rho_0b(u-l))}\frac{1}{k^2}\{\exp[ik(r^2c+a)]-\exp[ik(r^2c-a)]\};$$
$$A_0=a_1\nu_0(\eta_1-\eta_2)k,\quad c=c(u)=(-1/\rho_0b+2(u-l))^{-1}.$$

The generalized function $1/k^2$ has been determined accurate to a function with a carrier in $\{0\}$, i.e., accurate to an arbitrary linear combination of a δ-function and its derivatives. We choose $1/k^2$ such that $F_0'\to 0$ and $v\to-\infty$. We put

$$1/k^2=(1/k^2)_0-i\pi\delta'(k),$$

where $(1/k^2)_0$ is the generalized function $1/k^2$.[10] Then, using the table of Fourier transforms,[16] we get

$$F_0'=\frac{1}{2\pi}\int e^{-ikv}\bar{A}\,dk=\frac{A_0\varepsilon}{4(1-2\rho_0b(u-l))}\begin{cases}0, & v<r^2c-a\\ v-r^2c+a, & |v-r^2c|\le a.\\ 2a, & v>r^2c+a\end{cases} \tag{3.11}$$

We obtain similarly

$$\Psi_0'=\frac{i\varepsilon B_0}{2(1-2\rho_0b(u-l))}\begin{cases}1, & |v-r^2c|\le a\\ 0, & |v-r^2c|>a'\end{cases} \tag{3.12}$$
$$B_0=\nu_0(\gamma_2e^{-iba_2l}-\gamma_1e^{-iba_1l}).$$

The behavior of F_0' and Ψ_0' is illustrated in Fig. 3.

B. Let now a strong PEW of frequency ω_1 and amplitude B_1 be impinged on by a weak PEW of frequency ω_2 and amplitude B_2. According to (2.4), the weak PEW, in terms of the coordinates $\tilde{u}$ and $\tilde{v}$ connected with the wave (see Fig. 2) has a nonzero tetrad component of the Maxwell tensor

$$\tilde{F}_2=\frac{-B_2}{\sqrt{2}}\exp(-iv\omega_2-i\varphi_2).$$

From the transformation formulas for F_i it follows that in the region $\{u<0\}$, in terms of the coordinates u and v, we have

$$F_0=-\tilde{F}_2=2^{-1/2}B_2\exp(iv\omega_2+i\varphi_2).$$

Therefore

$$A|_{u=0}=\int e^{ikv}F_0dv=2^{1/2}B_2e^{i\varphi_2}\frac{\sin(k+\omega_2)a}{k+\omega_2},\quad B|_{u=0}=0.$$

By the same operations as in item A, but with different initial conditions at $u=0$, we arrive at the following expressions for F_0' and Ψ_0' in the region $\{u>l\}$:

$$F_0'=\frac{A_0\exp(i\omega_2v-i\omega_2r^2c)}{4(1-2\rho_0b_1(u-l))}[\theta(v-(a+r^2c))-\theta(v-(-a+r^2c))], \tag{3.13}$$

where

$$A_0=\nu_02^{1/2}B_2e^{i\varphi_2-i\omega_1l}(\gamma_1e^{-ib_1a_2l}-\gamma_2e^{-ib_1a_1l}), \tag{3.14}$$

$$\Psi_0'=\frac{iB_0e^{i\omega_2v}}{2(1-2\rho_0b_1(u-l))}[e^{i\omega_2a}\delta(v-(a+r^2c))-e^{-i\omega_2a}\delta(v-(-a+r^2c))]-$$

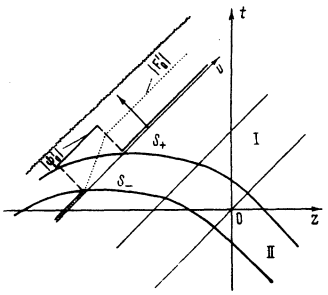

FIG. 3. Collision of a strong PEW with a weak PGW: I–strong wave, II–weak wave. $S_\pm : v = r^2c(u) \pm a$, wave line—the singularity $u = l + (2b)^{-1} \times \cot(bl/2)$.

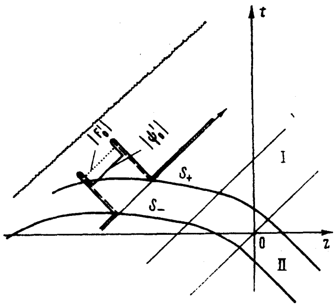

FIG. 4. Collisions of strong and weak PEW. The symbols are the same as in Fig. 3.

$$-\frac{\omega_2 B_0 \exp(i\omega_2 v - i\omega_2 r^2 c)}{2(1-2\rho_0 b_1(u-l))}[\theta(v-(a+r^2c))-\theta(v-(-a+r^2c))];$$

$$B_0 = \nu_0 \cdot 2^{1/2} B_2 e^{i\varphi_2} \frac{1}{k(\eta_1-\eta_2)}(e^{-ib_1\alpha_1 l} - e^{-ib_1\alpha_2 l})$$

The behavior of $|\Psi_0'|$ and $|F_0'|$ is illustrated in Fig. 4.

In the solutions (3.11)—(3.14) over the surfaces we have $S_\pm = 0$, where $S_\pm \equiv v - (r^2c \pm a)$, are isotropic, i.e., $(\nabla S_\pm)^2 = 0$. On these surfaces the gravitational and electromagnetic fields in the wave have discontinuous solutions. In the case of the passage of a PGW (item *A*), the intensity of the produced electromagnetic wave is continuous on these surfaces, and the gravitational field undergoes a discontinuity of second order. In accord with the general theory, [17] the intensity of the discontinuity $|\Psi_0'|^2$ satisfies the continuity equation

$$\nabla_\alpha(|\Psi_0|^2 l^\alpha) = 0, \quad l_\alpha \equiv \nabla_\alpha S_\pm. \tag{3.15}$$

In the case of passage of a PEW (item *B*), the electromagnetic field has a jump on the surfaces $v = r^2c \pm a$, while the gravitational field has a δ-function singularity. The expansions (3.14) agree with the results of our earlier study, [17] in which, in particular, we investigated the algebraic structure of a first-order discontinuity of the Weyl tensor on isotropic surfaces. Again, the coefficient of the δ-function singularity satisfies the intensity conservation law (3.15). The obtained formulas at values of u that cause the denominator $1 - 2\rho_0 b_1 (u - l)$ to vanish. The calculation of the invariants, however, shows that this singularity has in fact an unphysical character.

1) The authors of [13] thank A. A. Starobinskiĭ for calling their a attention to this circumstance.

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