Superradiance kinetics of a γ-ray laser

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An analysis is made of the kinetics of a single-pass mirror-free γ-ray laser. The conditions governing the operation of this laser are obtained. The possibility of Dicke superradiance in the γ-ray range is estimated. The analysis is carried out for single-mode emission from a system of two-level nuclei.

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The difficulty of constructing a resonator for electromagnetic waves in the γ-ray range makes it necessary to consider a needle-shaped active element forming a highly directional γ-ray beam as a result of amplification of spontaneous radiation emitted by nuclei. The present paper is concerned with the kinetics of such a mirror-free single-pass γ-ray laser, following the results reported in[1–4]. A common feature of these papers is an analysis of the kinetics on the basis of semiclassical equations to which, because of the homogeneity of the field equations, one has to add certain inhomogeneous boundary or initial conditions. Thus, strictly speaking, these papers are concerned with the kinetics of the amplification of a signal of some specific shape, whereas the greatest interest lies in the kinetics of amplification of the spontaneous radiation emitted by nuclei in an initially excited medium.

A very attractive idea is the Dicke superradiance in which the collective interaction of nuclei via a common electromagnetic field reduces strongly the deexcitation time of a system of excited nuclei. The promising nature of this superradiance in the specific case of a γ-ray laser was discussed some time ago in[5–7] and then in[3,4] and again in[8]. In our opinion, the fullest analysis is made in[3,4] but even there the problem is treated only approximately because the theory of the effect has not yet been developed sufficiently. The furthest progress in the theory of superradiance (superfluorescence) was made in[9–12]. The theory of single-mode superradiance is given in[9,10], the changes resulting from multimoding are considered in[11], and the theory is compared with the experimental results in[12]; this comparison shows that a suitable selection of the constants occurring in the relevant equations produces a satisfactory agreement with the experimental data if the semiclassical approximation for the quantum equations in the single-mode theory is used. A comparison of the semiclassical approximation to the quantum equations with the corresponding semiclassical (nonquantum) equations will be made below.

We shall study qualitative changes in the nature of the emitted pulses which occur when the parameters of the radiating system are varied. In particular, we shall give an analytic solution of the single-mode theory of superradiance for the characteristic parameters in our problem, which sheds new light on the process; for example, we shall determine the exact boundary separating oscillatory superradiance from the single-pulse emission.

1. FIELD DYNAMICS

We shall consider a needle-like crystal in which initially all the nuclei are in an excited state. The states of these nuclei are at first completely uncorrelated. The Hamiltonian of the system comprising atoms and the field is

$$\mathcal{H}=\mathcal{H}_f+\mathcal{H}_n+\mathcal{H}_e+\mathcal{H}_{int}^{(nf)}+\mathcal{H}_{int}^{(ne)}+\mathcal{H}_{int}^{(fe)}, \tag{1}$$

where $\mathcal{H}_f$ is the Hamiltonian of the field, $\mathcal{H}_n$ is the Hamiltonian of the nuclear subsystem of the crystal, $\mathcal{H}_e$ is the Hamiltonian of the electron subsystem of the crystal, and $\mathcal{H}_{int}^{(\alpha\beta)}$ are the Hamiltonians of the interactions of various subsystems with one another.

The equation of motion of the general density matrix is

$$i\hbar\frac{\partial\mathcal{R}}{\partial t}=[\mathcal{H},\mathcal{R}]+i\hbar\hat{\gamma}\mathcal{R},$$

where the second term describes the interaction of this system with a thermal reservoir. We shall be interested in the density matrix of the "nucleus + field" subsystem, which is related to the density matrix $\mathcal{R}(n,f,e,t)$ by the following equation:

$$\rho(n,f,t)=\mathrm{Tr}_e\{\mathcal{R}(n,f,e,t)\}$$

and has the equation of motion

$$i\hbar\frac{\partial\rho}{\partial t}=[\mathcal{H}_n+\mathcal{H}_f+\mathcal{H}_{int}^{(nf)},\rho]+\mathrm{Tr}_e\{[\mathcal{H}_{int}^{(ne)},\mathcal{R}]\}+\mathrm{Tr}_e\{[\mathcal{H}_{int}^{(fe)},\mathcal{R}]\}+i\hbar\hat{\gamma}\rho.$$

The equation of motion for the density matrix ρ in the interaction representation is

$$i\hbar\frac{\partial\tilde{\rho}}{\partial t}=[\tilde{\mathcal{H}}_{int},\tilde{\rho}]+i\hbar(\Gamma_1+\Gamma_2+\hat{\gamma}\tilde{\rho}), \tag{2}$$

where

$$\tilde{\rho}=U^{-1}\rho U,\quad \tilde{\mathcal{H}}_{int}=U^{-1}\mathcal{H}_{int}^{(nf)}U,\quad U=\exp\{-i(\mathcal{H}_n+\mathcal{H}_f)t/\hbar\},$$

Γ_1 is the term describing the interaction of the electron subsystem with the field, and Γ_2 is the term describing the interaction with the nuclei.

The interaction Hamiltonian $\tilde{\mathcal{H}}_{int}$ can be expressed in terms of time-independent Schrödinger operators:

$$\tilde{\mathcal{H}}_{int}=-\sum_k\{g_k(t)a_k^+R_k^-+\mathrm{H.c.}\}, \tag{3}$$

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where $\mathbf{k} \equiv (\mathbf{k}, \lambda)$,

$$g_{\mathbf{k}}(t) = (2\pi\hbar/V\omega_{\mathbf{k}})^{1/2}\langle - |\hat{\mathbf{j}}^{+}(\mathbf{k})\mathbf{e}^{(\lambda)}| + \rangle \exp\{i(\omega_{\mathbf{k}} - \omega_0)t\}, \quad (4)$$

$\hat{\mathbf{j}}^{+}(\mathbf{k})$ is the Fourier component of the positive-frequency part of the current density operator,

$$R_{\mathbf{k}}^{-} = \sum_{i=1}^{N} \sigma_{-}^{(i)} \exp\{i\mathbf{k}\mathbf{r}_{0i}\},$$

σ_{-}, σ_{+}, and σ_z are the Pauli spin operators.

The collective nuclear operators satisfy the commutation relations

$$[R_{\mathbf{k}}^{+}, R_{\mathbf{k}'}^{-}] = R_{z,\mathbf{k}'-\mathbf{k}}, \quad [R_{\mathbf{k}}^{\pm}, R_{z,\mathbf{k}'}] = \mp 2R_{\mathbf{k}\neq\mathbf{k}'}^{\pm}, \quad (5)$$

where

$$R_{z,\mathbf{k}} = \sum_{i=1}^{N} \sigma_z^{(i)} \exp\{i\mathbf{k}\mathbf{r}_{0i}\}.$$

The equation of motion for the density matrix (2) with the interaction Hamiltonian described by Eqs. (3) and (4) yields the following system of equations for the determination of the number of photons $n_{\mathbf{k}} = \langle a_{\mathbf{k}}^{+} a_{\mathbf{k}} \rangle$ in a mode $\mathbf{k}$:

$$\left.\begin{aligned}
&\frac{d}{dt} n_{\mathbf{k}} + \frac{1}{\tau_{\mathbf{k}}} n_{\mathbf{k}} = \frac{i}{\hbar}[g_{\mathbf{k}}(t)\langle a_{\mathbf{k}}^{+} R_{\mathbf{k}}^{-}\rangle - g_{\mathbf{k}}^{*}(t)\langle a_{\mathbf{k}} R_{\mathbf{k}}^{+}\rangle],\\
&\frac{d}{dt}\langle a_{\mathbf{k}}^{+} R_{\mathbf{k}'}^{-}\rangle + \frac{1}{2}\left(\frac{1}{\tau_{\mathbf{k}}} + \frac{1}{T_2}\right)\langle a_{\mathbf{k}}^{+} R_{\mathbf{k}'}^{-}\rangle\\
&= -\frac{i}{\hbar}\left[\sum_{\mathbf{k}''} g_{\mathbf{k}''}^{*}(t)\langle a_{\mathbf{k}}^{+} a_{\mathbf{k}''} R_{z,\mathbf{k}'-\mathbf{k}''}\rangle + g_{\mathbf{k}}^{*}(t)\langle R_{\mathbf{k}}^{+} R_{\mathbf{k}'}^{-}\rangle\right],\\
&\frac{d}{dt}\langle a_{\mathbf{k}} R_{\mathbf{k}'}^{+}\rangle + \frac{1}{2}\left(\frac{1}{\tau_{\mathbf{k}}} + \frac{1}{T_2}\right)\langle a_{\mathbf{k}} R_{\mathbf{k}'}^{+}\rangle\\
&= \frac{i}{\hbar}\left[\sum_{\mathbf{k}''} g_{\mathbf{k}''}(t)\langle a_{\mathbf{k}''}^{+} a_{\mathbf{k}} R_{z,\mathbf{k}''-\mathbf{k}'}\rangle + g_{\mathbf{k}}(t)\langle R_{\mathbf{k}'}^{+} R_{\mathbf{k}}^{-}\rangle\right],\\
&\frac{d}{dt}\langle R_{\mathbf{k}}^{+} R_{\mathbf{k}}^{-}\rangle + \frac{1}{T_2}\langle R_{\mathbf{k}}^{+} R_{\mathbf{k}}^{-}\rangle = -2\frac{i}{\hbar}\sum_{\mathbf{k}'} g_{\mathbf{k}'}(t)\langle a_{\mathbf{k}'}^{+} R_{\mathbf{k}'}^{-}\rangle\\
&+ \frac{i}{\hbar}\sum_{\mathbf{k}'}[g_{\mathbf{k}'}(t)\langle a_{\mathbf{k}'}^{+} R_{\mathbf{k}}^{-} R_{z,\mathbf{k}'-\mathbf{k}}\rangle - g_{\mathbf{k}'}^{*}(t)\langle a_{\mathbf{k}'} R_{\mathbf{k}}^{+} R_{z,\mathbf{k}-\mathbf{k}'}\rangle],\\
&\frac{d}{dt}\langle R_z\rangle + \frac{1}{T_1^{*}}(\langle R_z\rangle - \langle R_z^{0}\rangle) = -2\frac{i}{\hbar}\\
&\times\sum_{\mathbf{k}}[g_{\mathbf{k}}(t)\langle a_{\mathbf{k}}^{+} R_{\mathbf{k}}^{-}\rangle - g_{\mathbf{k}}^{*}(t)\langle a_{\mathbf{k}} R_{\mathbf{k}}^{+}\rangle],
\end{aligned}\right\} \quad (6)$$

where

$$\frac{1}{\tau_{\mathbf{k}}} = \frac{c}{l_{abs}(\mathbf{k})} + \frac{c}{L(\mathbf{k})},$$

$l_{abs}(\mathbf{k}) = l_B$ for the Bragg modes and $l_{abs}(\mathbf{k}) = l_{0B}$ in other cases, $L(\mathbf{k})$ is the length of the investigated crystal in the direction of the mode $\mathbf{k}$, $1/T_2$ is the width of a Mössbauer line, $1/T_1^{*} = 1/\tau_c + 1/T_1$, $1/\tau_c$ is the rate of internal conversion, $1/T_1$ is the radiative width of a level, and $\langle R_z^{0}\rangle$ is the equilibrium value of the difference between the populations.

The relaxation times are introduced above on the assumption that the interaction between the nuclear and electron subsystems reduces to internal conversion processes and can be allowed for phenomenologically by the relaxation time τ_c. The interaction of the electron subsystem with the field can be reduced to various processes of the absorption and scattering of γ rays by electrons, $1/l_{abs} = n\sum_i \sigma_i$, where σ_i are the photoabsorption, Compton scattering, and other cross sections. The interaction of the nuclear subsystem with the thermal reservoir is allowed for, as usual, by introducing relaxation times T_1 and T_2. Since we are quantizing the field in a volume equal to the volume of the sample, we must allow for the interaction of the "nucleus + field inside the crystal" with the thermal reservoir consisting of free noninteracting quanta. This requires the substitution $c/l_{abs} \to c/l_{abs} + c/L(\mathbf{k})$.

We can easily see that the system (6) is not closed and, therefore, the exact solution of the quantum problem in which allowance is made for the interaction of all the modes is hardly possible.

2. EQUATIONS FOR SINGLE-MODE AND BRAGG SUPERRADIANCE

We shall now return to the commutation relationships (5) and compare the diagonal elements of the operator $R_{z,\mathbf{k}-\mathbf{k}'}$ with the nondiagonal elements ($\mathbf{k} \neq \mathbf{k}'$). The average value of the operator $R_{z,\mathbf{k}-\mathbf{k}'}$ in the state in which the system is at a moment t is

$$\langle t|R_{z,\mathbf{k}-\mathbf{k}'}|t\rangle = \sum_{i'} \exp[i(\mathbf{k}-\mathbf{k}')\mathbf{r}_{0i'}] - \sum_{i''} \exp[i(\mathbf{k}-\mathbf{k}')\mathbf{r}_{0i''}],$$

where i' labels the nuclei excited at a moment t, and i'' labels the unexcited nuclei. Thus, if we allow for the complete spatial homogeneity of our initial conditions and equations, we can say that the above average disappears for $\mathbf{k} - \mathbf{k}' \neq \mathbf{K}$ ($\mathbf{K}$ is the reciprocal lattice vector).

We shall consider the situation in which an avalanche develops along the axis of a needle-like crystal. The condition for needle-like shape is $L\lambda/d^2 \sim 1$, where L is the length of the sample and d is its transverse size. In this case the radiation is emitted in the first diffraction mode so that only two waves exist in the crystal and these travel in opposite directions along the crystal axis. The commutation relationships (5) considered subject to the above discussion become

$$[R_{\mathbf{k}}^{+}, R_{\mathbf{k}'}^{-}] = R_z\delta_{\mathbf{k},\mathbf{k}'}, \quad (7)$$

where $R_z \equiv R_{z,0}$.

The physical meaning of the commutation relations (7) lies in the assumption of a negligibly weak interaction between the non-Bragg modes. The application of the relationships (7) modifies in the following way the system of equations (6):

$$\left.\begin{aligned}
&\frac{dn}{dt} + \frac{n}{\tau} = \langle F\rangle,\\
&\frac{d\langle F\rangle}{dt} + \frac{1}{2}\left(\frac{1}{\tau} + \frac{1}{T_2}\right)\langle F\rangle = \frac{1}{T_0^{2}}(\langle \hat{n}R_z\rangle + \langle I\rangle),\\
&\frac{d\langle I\rangle}{dt} + \frac{1}{T_2}\langle I\rangle = \langle FR_z\rangle - \langle F\rangle,\\
&\frac{d\langle R_z\rangle}{dt} + \frac{1}{T_1^{*}}(\langle R_z\rangle - \langle R_z^{0}\rangle) = -2\langle F\rangle.
\end{aligned}\right\} \quad (8)$$

where

$$F=\frac{i}{\hbar}\sum_{\mathbf{k}}[g_{\mathbf{k}}(t)a_{\mathbf{k}}^{+}R_{\mathbf{k}}^{-}-g_{\mathbf{k}}^{*}(t)a_{\mathbf{k}}R_{\mathbf{k}}^{+}],$$

$$I=\sum_{\mathbf{k}}R_{\mathbf{k}}^{+}R_{\mathbf{k}}^{-},\quad \frac{1}{T_0^2}=\frac{2|g_{\mathbf{k}}|^2}{\hbar^2}=\frac{2|g_{-\mathbf{k}}|^2}{\hbar^2}.$$

The system (8) is derived using the fact that the symmetry of the problem leads to the equality

$$n_{\mathbf{k}}=n_{-\mathbf{k}}=n/2,\quad n=\sum_{\mathbf{k}}n_{\mathbf{k}},$$

and also that $\omega_{\mathbf{k}}-\omega_0=0$.

We shall now consider superradiance under the Bragg diffraction conditions. For γ rays of energies of a few tens or hundreds of kiloelectron-volts we can usually expect the two-wave diffraction. The Bragg pairs whose reflection planes are parallel to the axis of the crystal are in the most favorable position. We shall now consider this case. The equivalence of the two directions parallel and antiparallel to the axis is obtained for two pairs of waves traveling opposite to one another and interacting only via the inversion depletion.

We shall now consider what simplifications result from allowance for the symmetry of the problem. First of all, it is clear that $R_{\mathbf{k}_1}^{\pm}=R_{\mathbf{k}_2}^{\pm}$, where $\mathbf{k}_2=\mathbf{k}_1+\mathbf{K}$. Thus, bearing in mind the foregoing discussion, we can rewrite the commutation relationships (5) in the form

$$[R_{\mathbf{k}\alpha}^{+}, R_{\mathbf{k}\beta}^{-}]=R_z,\quad [R_{\mathbf{k}\alpha}^{+}, R_{-\mathbf{k}\beta}^{-}]=0, \tag{9}$$

where α and β can have the values 1 or 2. Secondly, it follows from the symmetry of the problem that $a_1^{+}a_1=a_2^{+}a_2$ so that $a_1=a_2e^{i\varphi}$. For the Borrmann modes, we have $\varphi\approx\pi$. Allowance for all these factors gives a system of equations for the Bragg modes, which is identical with the system (8), with the exception of the second equation, provided we make the substitutions

$$l_{abs}=l_{0n}\to l_{abs}=l_B,\quad I\to I_1,\quad I_1={}^1/_2 I.$$

The second equation then has the form

$$\frac{d\langle F\rangle}{dt}+\frac{1}{2}\left(\frac{1}{\tau_1}+\frac{1}{T_2}\right)\langle F\rangle=\frac{1}{T_0^2}(1-p)[\langle\hat{n}R_z\rangle+2\langle I_1\rangle], \tag{10}$$

where

$$p=\frac{\langle+|\hat{\mathbf{j}}^{-}(\mathbf{k}_1)\mathbf{e}^{(1)}|-\rangle\langle-|\hat{\mathbf{j}}^{+}(\mathbf{k}_2)\mathbf{e}^{(2)}|+\rangle}{|\langle-|\hat{\mathbf{j}}^{+}(\mathbf{k})\mathbf{e}^{(\lambda)}|+\rangle|^2}.$$

3. SEMICLASSICAL APPROXIMATION

A semiclassical approximation can give a satisfactory agreement with the experimental data in the optical range. Let the field variables be the c numbers, so that the system (8) becomes (we shall drop the angular brackets and write averages as if they were the quantities themselves):

$$\left.\begin{aligned}&\frac{dn}{dt}+\frac{n}{\tau}=F,\quad \frac{dF}{dt}+\frac{1}{2}\left(\frac{1}{\tau}+\frac{1}{T_2}\right)F=\frac{1}{T_0^2}(nR_z+I),\\&\frac{dI}{dt}+\frac{1}{T_2}I=FR_z,\quad \frac{dR_z}{dt}+\frac{1}{T_1}(R_z+N)=-2F,\\&n(0)=0,\quad F(0)=0,\quad I(0)=N,\quad R_z(0)=N.\end{aligned}\right\} \tag{11}$$

We shall consider the initial stage of the deexcitation process when the number of nuclei which have decayed is still small. Retaining terms which are linear in respect of n, we find from the system (11) that

$$\frac{d^2n}{dt^2}+\left(\frac{3}{2\tau}+\frac{1}{2T_2}\right)\frac{dn}{dt}+\frac{1}{2\tau}\left(\frac{1}{\tau}+\frac{1}{T_2}\right)n-\frac{n}{T^2}=\frac{1}{2T^2} \tag{12}$$

which gives the following condition for the existence of an unstable solution:

$$\frac{1}{T^2}>\frac{1}{2\tau}\left(\frac{1}{\tau}+\frac{1}{T_2}\right),\quad \frac{1}{T^2}=\frac{2N}{T_0^2}. \tag{13}$$

In those cases when $T_2\ll\tau$, the condition (13) becomes

$$2\alpha_0>1/l_{abs}, \tag{14}$$

where $\alpha_0=\lambda^2N/2\pi\Gamma T_1V$ is the amplitude gain.

However, in the case of generation of coherent γ rays of interest to us, we have a different situation in which $\tau\ll T_2$, so that the condition (13) becomes

$$\lambda^2Nc/2\pi T_1V>1/2\tau^2. \tag{15}$$

The meaning of this last condition is easily understood. In Eq. (14) the coefficient α_0 is the resonance gain but under our conditions this resonance gain is attained in a time comparable with the excited-state lifetime but the deexcitation process occurs, subject to Eq. (15), in a time much shorter than the isomer lifetime.

The gain calculated allowing for the time dependence of the process is

$$\alpha=\frac{\lambda^2}{2\pi}\frac{N}{T_1V}\frac{\Gamma}{(\Delta\omega)^2+\Gamma^2};$$

because $\Delta\omega\lesssim 1/\tau$, the inequality (15) becomes

$$T_2\gg 1/2\alpha c. \tag{16}$$

This condition means that the time taken by a photon to travel the amplification length should be much shorter than the dephasing time. Of the three relaxation times which we are introducing, the shortest one is τ and its values are very different from the other relaxation times so that ignoring them, we find from the system (11) a new system of equations

$$\left.\begin{aligned}&\frac{d^2n}{dt^2}+\frac{3}{2\tau}\frac{dn}{dt}+\frac{n}{2\tau^2}=\frac{1}{T_0^2}\left(nR_z+N-\frac{R_z^2-N^2}{4}\right),\\&\frac{dR_z}{dt}=-2\left(\frac{dn}{dt}+\frac{n}{\tau}\right),\\&n(0)=0,\quad \frac{dn}{dt}(0)=0,\quad R_z(0)=N.\end{aligned}\right\} \tag{17}$$

The nature of the solution of the system (17) depends on the ratio of the characteristic times of the problem τ and T. We shall introduce a parameter β which governs the square of the ratio of the characteristic times $\beta=\tau^2/T^2$. Here, τ is the lifetime of a photon in the medium under discussion. The meaning of the time T is easily understood when we go to the limit $\tau\to\infty$. In this

case the system (17) describes the exchange of energy between the medium and the field in an infinitely long rod. The process is periodic and the period is of the order of T. The appearance of the first peak is given by

$$t_0 = T \ln N = \frac{\tau}{\sqrt{\beta}} \ln N \tag{18}$$

and the maximum radiation intensity is

$$\left.\frac{dR_z}{dt}\right|_{max} = -\frac{N}{T}\frac{4}{3\sqrt{3}} = -\frac{N\sqrt{\beta}}{\tau}\frac{4}{3\sqrt{3}}. \tag{19}$$

Thus, T is the characteristic time of the exchange of energy between the medium and the field.

In the γ-ray range we have $\beta < 1$. In this case we can easily obtain an analytic solution of the system (17). We shall divide it into three stages. In the first stage the signal grows exponentially. The solution of the system (17) is of the form

$$\left.\begin{aligned} n(t) &= \frac{2\beta}{1+4\beta} e^{-t/2\tau}\left[\operatorname{ch}\left(\sqrt{1+4\beta}\,\frac{t}{2\tau}\right) - 1\right], \\ U_1(t) &= \frac{2(1+2\beta)}{1+4\beta} e^{-t/2\tau} \operatorname{ch}\left(\sqrt{1+4\beta}\,\frac{t}{2\tau}\right) \\ &+ \frac{2}{\sqrt{1+4\beta}} e^{-t/2\tau} \operatorname{sh}\left(\sqrt{1+4\beta}\,\frac{t}{2\tau}\right) + \frac{4\beta}{1+4\beta} e^{-t/2\tau} - 2, \end{aligned}\right\} \tag{20}$$

where $U_1(t) = N - R_z(t)$. The system (20) readily yields an approximate expression for the time of appearance of the first radiation peak:

$$t_0 = \frac{2\tau}{\sqrt{1+4\beta} - 1} \ln N = \tau_0 \ln N. \tag{21}$$

When the quantity $U_1(t)$ reaches values of the order of $N/10$, we have the stage of very fast emptying of the upper levels. Retaining in the system (17) with small values of β the terms linear in β, we find that

$$\left.\begin{aligned} n(t) &= \frac{\beta N}{4}\frac{N^2 - R_0^2}{NC(t) - R_0 S(t)}, \quad R_z(t) = N\frac{R_0 C(t) - NS(t)}{NC(t) - R_0 S(t)}, \\ C(t) &= \operatorname{ch}\frac{\beta(t-t_1)}{2\tau}, \quad S(t) = \operatorname{sh}\frac{\beta(t-t_1)}{2\tau}, \end{aligned}\right\} \tag{22}$$

where R_0 is the difference between the populations at the time t_1 corresponding to the matching of the solutions.

Since the strong absorption ($\beta < 1$) prevents even the first radiation peak from reaching the value of N, the other peaks are much smaller than N or they may not appear at all. Thus, for the range of values of the parameter β considered here, the third stage is characterized by the fact that the function $R_z(t)$ then remains negative. Depending on the value of β, there are two types of solution for this stage.

1. In the first case we can have oscillatory superradiance ($\beta > 1/4$):

$$n(t),\ U_2(t) \sim e^{-t/2\tau}\left(A_{1,2} + B_{1,2}\cos\frac{\sqrt{4\beta-1}}{2\tau}t + C_{1,2}\sin\frac{\sqrt{4\beta-1}}{2\tau}t\right), \tag{23}$$

where the values of the coefficients A, B, and C are found from the matching conditions.

2. In the second case we have single-pulse superradiance ($\beta \leq 1/4$):

$$\left.\begin{aligned} n(t) &= \frac{\beta U_0}{2}\exp\left[\frac{\sqrt{1-4\beta}-1}{2\tau}(t-t_2)\right], \\ U_2(t) &= U_0 \exp\left[\frac{\sqrt{1-4\beta}-1}{2\tau}(t-t_2)\right], \end{aligned}\right\} \tag{24}$$

where $U_2(t) = N + R_z(t)$, and U_0 is the value of $U_2(t)$ at the matching moment t_2.

Thus, in the oscillatory superradiance case the solution is a linear combination of exponentially damped sinusoids and cosinusoids varying with a period of

$$T_h = 4\pi\tau/\sqrt{4\beta-1}. \tag{25}$$

It follows from Eqs. (20) and (22)–(24) that $n(t)$ has its maximum value during the second stage. This value is attained at a moment

$$t_0 = \frac{\tau}{\beta}\ln N \tag{26}$$

and it is equal to

$$n_{max}(t_0) = \beta N/4. \tag{27}$$

Since we are considering the problem in terms of the semiclassical approximation, we can consider rigorously photons inside the active volume and photons which have been emitted. In the case of nonabsorbing crystals ($L \ll l_{abs}$) the number of the latter photons is given by

$$\bar{n}(t) = [N - R_z(t)]/2 - n(t). \tag{28}$$

Thus, the intensity of the emitted radiation is

$$d\bar{n}/dt = n/\tau. \tag{29}$$

Consequently, the oscillations of the radiation intensity follow exactly the oscillations of the photon density inside the "resonator." Using Eq. (27), we find that the maximum radiation intensity is

$$\left.\frac{d\bar{n}}{dt}\right|_D = \frac{N\beta}{4\tau} = \frac{N}{T_{SF}}, \tag{30}$$

where

$$T_{SF} = 8\pi V T_1/N\lambda^2 L. \tag{31}$$

The use of strongly absorbing crystals ($L \gg l_{abs}$) is not recommended because a crystal is damaged before the radiation reaches its maximum intensity. In a crystal of critical length ($L = l_{abs}$) the maximum radiation intensity is obviously 4 times as high as in a nonabsorbing crystal of the same length. Since in the latter case half the emitted photons are absorbed in the crystal, we have even in this case the problem of removal of high-energy photo-electrons and filling of the resultant vacancies with low-energy electrons. Consequently, the most effective extraction of the stored energy is achieved by the use of crystals with $L < l_{abs}$. All this is

valid only as long as both these lengths are less than the coherence length $l_c = cT$, i.e., in systems with $\beta < 1$. If $l_c < L < l_{abs}$, the maximum radiation intensity is less than the Dicke intensity (30) and it is given by Eq. (19).

It follows from Eqs. (30) and (26) that a reduction in the parameter β reduces the maximum radiation intensity and increases the delay time of the first pulse. It is intuitively clear that the faster the rate of extraction of the stored energy, the better the directionality of the radiation output from a real system. Consequently, the problem of optimization of the parameters of the resultant pulses reduces to the problem of increasing the parameter

$$\beta = \frac{N\lambda^2 l_{abs}}{2\pi V}\frac{\tau}{T_1}. \tag{32}$$

If we assume that $\lambda \sim 3\times 10^{-9}$ cm, $N/V \sim 10^{23}$ cm^{-3}, and $l_{abs} \sim 0.1$ cm, we obtain

$$\beta = (10^{-7} - 10^{-8})/T_1.$$

It should be noted[13] that far from the frequencies close to the absorption edge in the K shell we have $l_{abs} \propto (\hbar\omega)^{3.5}$; thus, β rises strongly with increasing γ-ray energy; $\beta \propto (\hbar\omega)^5$.

4. CONCLUSIONS

Interaction of separate radiators with one another in the deexcitation process is due to two effects: firstly, it is due to the stimulated radiation described by the term nR_z and, secondly, it is due to the induced polarization of the moment of the nuclear transition. The second effect is described by the cross terms of the operator I:

$$I_{sp} = \sum_{\mathbf{k}}\sum_{i\neq j}\sigma_+^{(i)}\sigma_-^{(j)}\exp\{i\mathbf{k}(\mathbf{r}_{0i}-\mathbf{r}_{0j})\}.$$

When these interactions are weak, the system (8) describes the spontaneous decay of N noninteracting nuclei, which occurs at the rate

$$\left.\frac{d\bar{n}}{dt}\right|_{sp} = \frac{\beta}{\tau} = 2\frac{N}{T_1}\frac{\Delta\Omega}{4\pi}\frac{l_{abs}}{L}, \tag{33}$$

where $\Delta\Omega = \lambda^2/A$ (A is the transverse cross sectional area of the crystal). The coefficient 2 appears because the radiation is emitted from both ends. Thus, Eq. (33) describes spontaneous decay into a diffraction mode.

In considering the deexcitation process we are assuming that all the characteristic times of this process are considerably shorter than T_2. This means throughout the deexcitation process that the nuclear system "remembers" the phase of the emitted photons, so that the system reaches a state of collective spontaneous emission at a moment t_0. The time t_0 is the longest characteristic time. Consequently, the critical density of the excited nuclei in superradiance effects is found from the condition $t_0 \lesssim T_2$. Using Eqs. (26) and (32), we obtain

$$n_{cr}/\Gamma T_1 > 2\pi \ln N/\lambda^2 l_{abs}. \tag{34}$$

Substituting the same values of the parameters as in Eq. (32) we find that $n_{cr}/\Gamma T_1 < 10^{21}$. Such densities of excited nuclei have not yet been created experimentally.

In those cases when $T_2 \ll t_0$, the rise of the number of photons during the initial stage is again given by Eq. (20) but $n(t)$ does not rise to $N\beta/4$ but only to

$$n_{max} \sim \beta \exp(T_2/\tau_0) = \beta N^{T_2/t_0}.$$

Thus, the maximum attainable intensity of the radiation is

$$\left.\frac{d\bar{n}}{dt}\right|_{max} = \left.\frac{dn}{dt}\right|_{sp} N^{T_2/t_0}.$$

If we make allowance for the above processes of deexcitation of samples which are initially completely excited, we find that it is incorrect to assume that the difference between the populations varies weakly throughout the deexcitation process or to assume that it decays at rate characterized by the time constant T_1, as is sometimes assumed. In fact, it follows from the system (20) that the decay of the population inversion is much faster even during the initial moments.

One of the main assumptions in our discussion is a two-level model of initially completely excited nuclei. This model describes best the processes in a γ-ray laser based on long-lived isomers because the variants utilizing short-lived isomers are predicted to operate in three- or four-level schemes. However, our discussion is applicable also to the models in which the lifetimes of the active levels are approximately equal, which seems to be the most promising situation. If initially there are N_2^0 excited nuclei, the coefficient β decreases by a factor N/N_2^0. Allowance for the nuclear vibrations reduces to the multiplication of the parameter β by the probability of Mössbauer emission. We have considered also a single-mode γ-ray laser and have thus lost information on the spatial changes in the photon flux density. Consequently, $n(t)$ should be regarded as the photon density averaged over the whole volume of the sample.

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Semiclassical theory of cooperative radiation of a polyatomic system

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The kinetics of a pulse of cooperative radiation is considered with account taken of the homogeneous and inhomogeneous broadening of the spectrum. It is shown that in the case of sufficiently large homogeneous broadening, the Dicke superradiance is transformed into superluminescence, which can be described by the balance equations. The existence of an optimal gain length corresponding to the maximum intensity of the cooperative radiation is proved.

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I. INTRODUCTION

The cooperative character of the spontaneous emission of a polyatomic system is due to exchange of real and virtual transverse photons between atoms. Dicke,[1] followed by others,[2,3] have shown that for a system of N atoms in a small volume, with linear dimensions shorter than the wavelength, the radiation intensity is proportional to N. The proportionality of the intensity to the square of the number of particles shows that phase alignment of the atomic dipole takes place in the emission process, although the macroscopic moment may be equal to zero in the initial state.

The present paper is devoted to an investigation of the cooperative spontaneous emission of extended polyatomic system with account taken of the homogeneous and inhomogeneous broadening of the luminescence spectrum. Examples of such systems are pulsed gas lasers without mirrors.[4-6] Introduction of the broadenings is essential not only for the determination of the details of the structure of a cooperative-emission pulse, but also to find the conditions for the manifestation of the cooperative effect. The quantum theory of superradiance of extensive systems without allowance for the broadening of the spectrum was developed in[7-9]. The use of a more elaborate model makes a semiclassical approximation (a quantum description of the atomic system and a classical description of the electromagnetic field) more advantageous. In particular, it becomes possible to trace the connection between the theory that describes the Dicke superradiance and the balance equations that are used in the theory of pulsed lasers.[10] The transition to the latter is realized in the case of sufficiently large homogeneous broadening of the spectrum, which leads to loss of the phase memory of the atomic system. Pulsed cooperative emission, during the course of which the phase memory of the atomic system is preserved, will be called superradiance. On the other hand, if the relaxation of the off-diagonal elements of the density matrix of the atoms is effective enough, so that the conditions for the validity of the balance equations are satisfied, then the corresponding radiation regime will be called superluminescence.

2. THE SEMICLASSICAL APPROXIMATION

In the semiclassical approximation, a system of two-level atoms interacting with an electromagnetic field can be described by the system of equations for the single-atom density matrix $\|\rho_{ab}\|$ and the equation for the intensity of the electric field E due to the polarization $\mathbf{P}$ of the medium[11]:

$$\begin{aligned} i\hbar\dot{\rho}_{aa}&=V_{ab}\rho_{ba}-\rho_{ab}V_{ba},\\ i\hbar\dot{\rho}_{ba}&=\hbar\omega\rho_{ba}+V_{ba}\rho_{aa}-\rho_{bb}V_{ba}-i\hbar\frac{\rho_{ba}}{T_2},\\ \left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\varkappa'\frac{\partial}{\partial t}\right)\mathbf{E}&=\frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{P}. \end{aligned} \tag{1}$$

Here $V_{ba}=V_{ab}^*$ is the matrix element of the interaction; in the dipole element we have $V_{ba}=-\boldsymbol{\mu}\cdot\mathbf{E}$, where $\boldsymbol{\mu}$ is the dipole-moment matrix element corresponding to the transition from the excited state b to the ground state a. The density matrix is locally averaged over the positions of the atoms and is a function of the time, of the spatial coordinates, and of the natural frequency ω of the

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