The excitation spectrum of the electrons of a semiconductor in the resonance field of a standing electromagnetic wave

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The spectrum of the quasi-particle excitations of the electrons of a semiconductor in the resonance field of a standing electromagnetic wave is investigated. In contrast to the earlier investigated case of a running wave, when the corrections to the excitation spectrum were of the order of $kv/\lambda \ll 1$ (k is the wave vector, v is the electron velocity at the Fermi surface, and λ is the frequency of electron interband transitions under the action of the field), the spectrum in a standing wave is substantially different even when the parameter kv/λ is small. In particular, the gap in the density of states is replaced by a smooth decrease of the density of states. This is due to the fact that the mixing of the states in the valence and conduction bands occurs not at one value of the quasi-momentum, as obtained in a running-wave field. The obtained results are of interest not only for semiconductor lasers, where the standing-wave situation is usually realized, but also in connection with the investigation of superconductivity in the Fröhlich model. In this case the role of the interaction with the electromagnetic field is played by the electron-phonon interaction.

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It was shown in Ref. 1 that if the strong-field condition $\lambda\tau \gg 1$ ($\lambda = dE$, E is amplitude of the wave-field intensity, d is the dipole moment of the transition, and τ is the electron-phonon collision time) is fulfilled, then a gap arises in the electron spectrum of the semiconductor. Subsequently, it was theoretically and experimentally shown in a number of papers that the strong-field condition is realized at relatively low powers in semiconductors irradiated by external fields, as well as in semiconductor lasers (SL). The change in the spectrum leads to important effects in SL theory, in particular, to generation saturation.[2]

In the previous papers the analysis was performed for a running wave. Since for a strong field we can assume that $kv \ll \lambda$ (k is the wave vector and v is the velocity at the Fermi surface), the corrections to the spectrum that are due to the spatial inhomogeneity of the field are of the order of kv/λ, and they can be neglected. However, in the case of a standing electromagnetic wave, a case which is usually realized in SL, the situation turns out to be substantially different. As shown in the present paper, the spectrum undergoes substantial changes even when the parameter kv/λ is small. The spatial dispersion can be neglected only if $kL \ll 1$, where L is the sample dimension. If the condition $1/L < k \ll \lambda/v$ is fulfilled, then the gap in the density of states is washed out and replaced by a smooth decrease of the density of states.

It should be noted that the obtained results are of interest in connection with the investigation of the possibility of superconductivity in the Fröhlich model.[3] The role of the interaction with the electromagnetic field is played there by the electron-phonon interaction.

In the paper we investigate in detail the spectrum of the quasi-particle excitations of the electrons of a semiconductor in the resonance field of a standing electromagnetic wave.

§1. THE BASIC EQUATIONS

The Hamiltonian for the electrons and holes of a semiconductor located in a homogeneous monochromatic electromagnetic field admits in the resonance approximation, as was shown in Ref. 1, diagonalization with the aid of the transformation of the electron operators into quasi-particle operators. In the more general case of a resonance field with an arbitrary coordinate dependence, we can, after making the necessary generalizations, also diagonalize the Hamiltonian. For this purpose it is convenient to use two-component field operators similar to the spinor operators in the theory of superconductivity (see, for example, Ref. 4):

$$\psi(\mathbf{r}) = \begin{pmatrix} \psi_c(\mathbf{r}) \\ \psi_v(\mathbf{r}) \end{pmatrix}, \tag{1}$$

where ψ_c and ψ_v are the operators of electron annihilation in the conduction and valence bands. After the appropriate unitary transformation that transfers the time dependence from the Hamiltonian to the state vectors (which results in the energy reference point being changed in each band[1]), the Hamiltonian for electrons interacting with an inhomogeneous electromagnetic field can be written in the form

$$H = \int dV \psi^+ (H_0\sigma_3 + \lambda(\mathbf{r})\sigma_1)\psi. \tag{2}$$

Here

$$H_0 = -\Delta/2m - \mu_0, \quad \mu_0 \equiv p_0^2/2m = {}^1/_2(\omega - E_g), \quad \lambda(\mathbf{r}) = \mathbf{dE}(\mathbf{r}),$$

d is the dipole matrix element of the interband transition, E_g is the energy gap, the electric-field intensity is given in the form $\mathbf{E}(\mathbf{r})e^{-i\omega t} + \text{c.c.}$, $\sigma_{1,3}$ are Pauli matrices, and the electron and hole masses are assumed, without loss of generality, to be equal.

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The Hamiltonian (2) is a quadratic form in the field operators, and can be diagonalized with the aid of the linear transformation

$$\psi(\mathbf{r})=\sum_{\nu}\alpha_{\nu}\varphi^{\nu}(\mathbf{r}), \tag{3}$$

where α_ν is the annihilation operator for a quasi-particle in the single-particle quantum state ν with the wave function $\varphi^\nu=\{\varphi_1^\nu, \varphi_2^\nu\}$, the totality of which forms a complete orthonormal set. Requiring that $H=\sum\varepsilon_\nu\alpha_\nu^+\alpha_\nu$, we obtain for φ_1 and φ_2 the equations:

$$H_0\varphi_1+\lambda(\mathbf{r})\varphi_2=\varepsilon\varphi_1, \quad -H_0\varphi_2+\lambda^*(\mathbf{r})\varphi_1=\varepsilon\varphi_2. \tag{4}$$

The system (4) is similar to the Bogolyubov equations in the theory of superconductivity.

In the field of the running wave $\lambda(\mathbf{r})=\lambda e^{i\mathbf{k}\cdot\mathbf{r}}$, which mixes the state with momentum $\mathbf{p}$ in the valence band and the state with momentum $\mathbf{p}+\mathbf{k}$ in the conduction band, the Eqs. (4) are exactly soluble.[5] The two energy branches of the spectrum, distinguished by the indices α and β, have the form

$$\varepsilon_{\alpha,\beta}=\frac{1}{2}\{\xi(\mathbf{p})-\xi(\mathbf{p}-\mathbf{k})\pm((\xi(\mathbf{p})+\xi(\mathbf{p}-\mathbf{k}))^2+4\lambda^2)^{1/2}\}, \tag{5}$$
$$\xi(\mathbf{p})=p^2/2m-\mu_0.$$

If we neglect collisions ($\lambda\tau\gg1$), then the reduced density of single-particle states for the spectrum in question has the form (see Fig. 1)

$$\rho_\alpha(\varepsilon)=\rho_\beta(-\varepsilon)=\frac{1}{kv}\begin{cases}F(\varepsilon)-F(-\varepsilon), & \varepsilon-\lambda>{}^1/_2kv,\\ F(\varepsilon), & |\varepsilon-\lambda|<{}^1/_2kv,\end{cases} \tag{6}$$

where

$$F(\varepsilon)=[(\varepsilon+{}^1/_2kv)^2-\lambda^2]^{1/2}, \quad p_0=mv.$$

It can be seen from this that, when $kv<2\lambda$, the spectrum contains a gap of width $2\lambda-kv$. When $kv>2\lambda$, the gap is absent. In the limit $kv\ll\lambda$, the results go over into the corresponding expressions for a uniform field. The effect of scattering on $\rho(\varepsilon)$ was thoroughly studied in an earlier paper.[6]

As is well known, the presence of a gap leads to the absence of absorption of a weak signal at frequencies close to the frequency of the strong field, to the absence of recombination emission at these same frequencies,[5] as well as to the existence of a limiting value for the field generated by the laser.[2] Knowledge of the energy spectrum is very important for the understanding of the processes occurring in SL. In practice, in quite a number of cases the situation is realized when the electromagnetic field in the SL is, to a good degree of accuracy, the field of a standing wave. Therefore, in the following sections we shall consider precisely this situation.

§2. GENERAL PROPERTIES OF THE ENERGY SPECTRUM IN THE FIELD OF A STANDING WAVE

The solution of the system (4) for the field of the standing wave $\lambda(\mathbf{r})=\lambda\cos\mathbf{k}\cdot\mathbf{r}$ meets with serious mathematical difficulties. In this section we shall consider the general properties of the spectrum, and at the same time construct a perturbation theory with the parameter λ/kv. We shall use the well-known—in the theory of linear differential equations with periodic coefficients—Hill method[7] (see also Melikyan's paper[8]), and we shall seek the solution in the form

$$\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=e^{ipr}\sum_n\begin{pmatrix}a_n\\ b_n\end{pmatrix}e^{inkr}. \tag{7}$$

The set of equations for the coefficients breaks up into two independent subsystems. Into one of them enter $a_{2s}\equiv x_{2s}$ and $b_{2s+1}\equiv x_{2s+1}$:

$$\begin{aligned}&x_{2s}+\frac{\Lambda}{(p+2s)^2-p_1^2}(x_{2s-1}+x_{2s+1})=0,\\ &x_{2s+1}-\frac{\Lambda}{(p+1+2s)^2-p_2^2}(x_{2s}+x_{2s+2})=0,\end{aligned} \tag{8}$$

where

$$\Lambda=\frac{\lambda m}{k^2}, \quad p_{1,2}^2=\frac{2m}{k^2}(\mu\pm\varepsilon), \quad \mu=\mu_0-\frac{p_\perp^2}{2m}, \quad p=p_z/k.$$

For definiteness, the wave vector $\mathbf{k}$ is directed along the z axis and the quasi-momentum $\mathbf{p}=\{\mathbf{p}_\perp, p_z\}$. The second subsystem is obtained from (8) by means of the substitution $p\to p+1$ and redesignation of the unknown quantities, or by means of the substitution $\varepsilon\to-\varepsilon$.

The infinite determinant $D(p)$ of the homogeneous system (8), considered as a function of p, possesses the following properties:

1) $D(p)$ is a meromorphic function, having simple poles at the points $\pm p_1+2s$ and $\pm p_2+2s+1$;

2) $D(p)$ is an even periodic function with period 2;

3) $D(p)\to1$ as $|p|\to\infty$.

It follows from these properties that $D(p)$ can be represented in the form

$$D(p)=1+A\frac{\sin\pi p_1}{\cos\pi p_1-\cos\pi p}+B\frac{\sin\pi p_2}{\cos\pi p_2+\cos\pi p}, \tag{9}$$

where the quantities A and B do not depend on p, and can be expressed in terms of $D(0)$ and $D(1)$. It follows from the structure of the determinants $D(0)$ and $D(1)$ that they are obtainable from each other by means of the substitution $p_1\rightleftarrows p_2$, so that it is sufficient to compute only one of them. (This same property is possessed by the quantities A and B.)

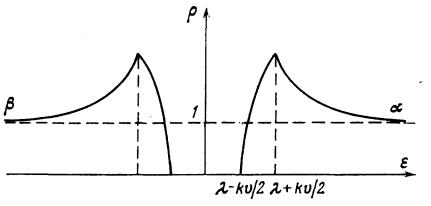

FIG. 1. Relative density of states of the quasi-particle excitations in the field of a running wave for $kv<2\lambda$.

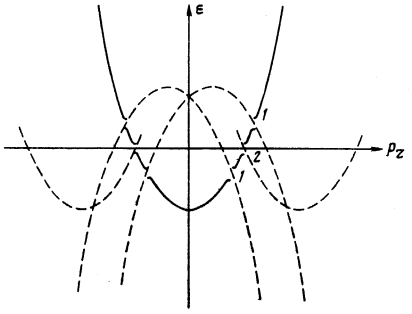

FIG. 2. Schematic shape of the excitation spectrum for a standing-wave field. For simplicity, only the low-lying branches of the spectrum are shown: 1) single-quantum resonance splittings, 2) Bragg splitting.

The equation $D(p) \equiv D(p, \varepsilon) = 0$ determines the branches, $\varepsilon(p)$, of the energy spectrum that are connected with the system of equations (8). The function $\varepsilon(p)$, as can be seen from the properties of $D(p)$, is an even periodic function with period 2. The determinant $\tilde{D}(p) \equiv \tilde{D}(p, \varepsilon) = D(p+1, \varepsilon)$ for the second subsystem will give the branches $\tilde{\varepsilon}(p) = \varepsilon(p+1)$. Furthermore, the relation $D(p+1, \varepsilon) = D(p, -\varepsilon)$ is valid. Therefore, the resulting excitation spectrum is an even periodic function of p with period 1, and is symmetric under the substitution $\varepsilon \to -\varepsilon$.

The energy spectrum $\varepsilon(p)$ has singularities connected with the presence of points of intersection of the unperturbed terms

$$\varepsilon_c(p_z) = -\mu + p_z^2/2m, \quad \varepsilon_v(p_z) = \mu - p_z^2/2m.$$

Intersections of the first type correspond to the resonance mixing by the field of the states ε_c and ε_v with momenta, p_z, differing from each other by an odd number of k, i.e.,

$$\varepsilon_c(p_z) = \varepsilon_v(p_z - (2s+1)k). \tag{10}$$

In this case the parameters p_1 and p_2 entering into $D(p)$ differ from each other by an odd number. This leads to pairwise fusion of the poles corresponding to them and to the appearance of gaps in the spectrum. Intersections of the second kind correspond to the mixing by the field of the states ε_c (or ε_v) for opposite momentum values, differing from each other by an even number of k. This condition is analogous to the Bragg condition for reflection during motion in a periodic field. At these points the parameter p_1 (or p_2) is equal to a whole number. The poles of $D(p)$ that are connected with this parameter merge in pairs, and a gap appears in the corresponding branch of the spectrum.

For a sufficiently weak field, we can restrict ourselves to the consideration of a few terms of the expansion of $D(0)$ in a series in powers of λ^2. This leads to the following equation for the spectrum:

$$1 + \frac{\pi\Lambda^2}{[1-(p_1+p_2)^2][1-(p_1-p_2)^2]} \left\{ \frac{\sin \pi p_1}{p_1} \frac{1+p_1^2-p_2^2}{\cos \pi p_1 - \cos \pi p} + \frac{\sin \pi p_2}{p_2} \frac{1+p_2^2-p_1^2}{\cos \pi p_2 + \cos \pi p} \right\} = 0. \tag{11}$$

In the most interesting parameter region, where $\mu \sim \mu_0 \gg k^2/m$, this equation is applicable if

$$\lambda \ll k(2\mu/m)^{1/2} = kv \sim kv_0. \tag{12}$$

In the vicinity of one of the points of intersection of the terms ε_c and ε_v for $s = 0$ in (10), which corresponds to a single-quantum resonance, $p_2 \approx p_1 - 1$. Therefore, carrying out the necessary expansions in (11) near $\varepsilon \approx \frac{1}{2}kv$ and $p \approx -\frac{1}{2} + mv/k$, we obtain for $\varepsilon(p)$ the result (5). Thus, there exists resonance splitting, $\Delta\varepsilon = \lambda$, near points of this kind. These splittings arise as a result of the independent resonant action of the two running waves forming the standing wave. Equation (11) is not suitable for the computation of the spectrum near the points corresponding to resonances of higher multiplicity, since it is necessary to take account of further terms of the expansion of the determinant in powers of λ^2 right up to $(\lambda^2)^{2s+1}$. The splitting at these points is proportional to λ^{2s+1}. The many-quantum resonances will be considered in the following section with the aid of the quasi-classical approximation.

Near the singularity connected with the Bragg reflection for the ε_c term, the parameter $p_1 \approx s$ ($s \sim mv/k \gg 1$). If in this case p_2 is not close to a whole number, then the second term in the curly brackets in Eq. (11) can be neglected. Then there arises a splitting in the ε_c term near $\varepsilon \approx -\mu + k^2 s^2/2m$ and $p \approx s$:

$$\Delta\varepsilon \sim \lambda^2/kv \ll \lambda. \tag{13}$$

At these energies the ε_v term does not, generally speaking, split up. An analogous splitting for ε_v occurs when p_2 is close to a whole number.

A schematic shape of the spectrum in the extended zone scheme is shown in Fig. 2. For simplicity, we show only the branches of the spectrum connected with the determinant $D(p)$. The branches of the spectrum connected with $\tilde{D}(p)$ are obtained by means of the substitution $\varepsilon \to -\varepsilon$. In the reduced zone scheme the spectrum is obtainable by means of a periodic continuation with respect to p. Figure 3 shows the shape of the

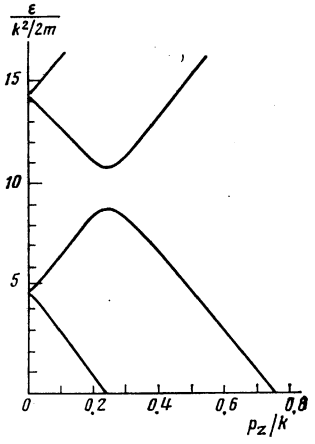

FIG. 3. Shape of the spectrum determined by Eq. 11 for $\Lambda = 1$ and $2m\mu/k^2 = 95.15$.

spectrum determined by Eq. (11) for specific values of the parameters.

Notice that, for $|\mu| \ll \mu_0$ and $\lambda \ll kv_0$, Eq. (11) is valid for $\varepsilon \gg \lambda(\lambda m/k^2)^{1/3}$. For it to be applicable in the region of smaller ε, we need a more rigid condition on the field: $\lambda \ll k^2/m$.

§3. THE QUASI-CLASSICAL APPROXIMATION

If the distance, v/λ, which an electron traverses during the time of its transition between the valence and conduction bands is small compared to the field-inhomogeneity period $1/k$, i.e., if $\lambda \gg kv$, then the electron motion in the field of the standing wave $\lambda(z) = \lambda \cos kz$ can be treated quasi-classically. Separating out the dependence on the transverse components of the momentum, and going over to the variable $\tau = kz$, we can represent (4) in the form

$$\begin{aligned} \varphi_1'' + \frac{2m}{k^2}(\mu-\varepsilon)\varphi_1 - \frac{2m}{k^2}\lambda(\tau)\varphi_2 = 0, \\ \varphi_2'' + \frac{2m}{k^2}(\mu+\varepsilon)\varphi_2 + \frac{2m}{k^2}\lambda(\tau)\varphi_1 = 0. \end{aligned} \tag{14}$$

Let us apply a previously proposed method.[9] Let us effect a rotation through an angle $\theta(\tau)$:

$$\begin{aligned} \varphi_1 = \chi_1 \operatorname{ch}\frac{\theta}{2} + \chi_2 \operatorname{sh}\frac{\theta}{2}, \quad \varphi_2 = \chi_1 \operatorname{sh}\frac{\theta}{2} + \chi_2 \operatorname{ch}\frac{\theta}{2}, \\ \operatorname{th}\theta = \lambda(\tau)/\varepsilon, \quad \lambda(\tau) = \lambda\cos\tau. \end{aligned} \tag{15}$$

Neglecting the small terms containing the derivatives of θ, we arrive at two independent equations for $\chi_{1,2}$:

$$\chi_{1,2}'' + \Phi_{1,2}^2(\tau)\chi_{1,2} = 0, \quad \Phi_{1,2} = \left\{\frac{2m}{k^2}[\mu \pm (\varepsilon^2 - \lambda^2(\tau))^{1/2}]\right\}^{1/2}. \tag{16}$$

Using the quasi-classical solutions to Eqs. (16), let us represent the linearly independent solutions to the basic system (14) in the form

$$\begin{aligned} u, \tilde{u} = \begin{pmatrix} \operatorname{ch}(\theta/2) \\ \operatorname{sh}(\theta/2) \end{pmatrix} \Phi_1^{-1/2} \exp\left\{\pm i\int \Phi_1 d\tau\right\}, \\ w, \tilde{w} = \begin{pmatrix} \operatorname{sh}(\theta/2) \\ \operatorname{ch}(\theta/2) \end{pmatrix} \Phi_2^{-1/2} \exp\left\{\pm i\int^{\tau} \Phi_2 d\tau\right\}. \end{aligned} \tag{17}$$

The functions $\Phi_{1,2}$ have in the complex τ plane two types of branch points, which are repeated periodically. The branch points of the first kind are connected with the zeros of the interior root. On bypassing these points the solutions u and w go over into each other. This corresponds to resonant mixing of the ε_c and ε_v terms. The branch points of the second kind are determined by the zeroes of the exterior root. The bypassing of these points leads to the conversion of the solutions u and $\tilde{u}$ (w and $\tilde{w}$) into each other, which corresponds to the Bragg conditions for reflection. Consequently, the general solution is a superposition of four types of waves that get converted into each other at the branch points. The problem consists in the search for those superpositions of these waves that would, with allowance for the indicated conversions, have the Bloch form.

As will be shown below, for $\mu \gg \lambda$, the influence of the branch points of the second kind is negligible, i.e., we can restrict ourselves to the consideration of only the resonant conversions. In this case we should consider the cases $\varepsilon > \lambda$ and $\varepsilon < \lambda$ separately.

For $\varepsilon > \lambda$ the branch points of the interior root are complex:

$$\tau_n = \pi n \pm i\tau_0, \quad \operatorname{ch}\tau_0 = \varepsilon/\lambda.$$

Let us introduce in the interval $(\pi n,\ \pi(n+1))$ the functions u_n and w_n for which the lower limit of the integration in (17) coincides with the left end of the interval, and let us represent the sought solution in the interval in the form

$$\varphi = A_n u_n + B_n w_n. \tag{18}$$

Continuing the functions u_n and w_n into the next interval first along the real axis and then along a path bypassing the branch points located respectively in the upper and lower half-planes, where these functions decrease exponentially (in the same way as is done in the problem of quasi-classical above-the-barrier reflection[10]), we obtain the following unimodular transformation matrix for the coefficients A_n and B_n:

$$U_n = e^{iS_0}\begin{pmatrix} e^{iS}(1+R^2)^{1/2} & i(-1)^{n+1}Re^{iS} \\ i(-1)^n Re^{-iS} & e^{-iS}(1+R^2)^{1/2} \end{pmatrix}. \tag{19}$$

Here

$$S_0 \pm S = \int_0^{\pi} \Phi_{1,2} d\tau, \quad S_0 \approx \pi\frac{mv}{k}, \quad S \simeq \frac{2\varepsilon}{kv} E\left(\frac{\lambda}{\varepsilon}\right), \tag{20}$$

where E is the complete elliptic integral.

The quantity R is the coefficient of above-the-barrier reflection:

$$R = \exp\left\{-\int_0^{\tau_0} [\Phi_1(i\tau) - \Phi_2(i\tau)]d\tau\right\} \approx \exp\left\{-\frac{2\varepsilon}{kv}[K(x) - E(x)]\right\}, \tag{21}$$

where $x = (1 - \lambda^2/\varepsilon^2)^{1/2}$. An analogous reflection coefficient R is encountered in, in particular, the problem of the many-photon resonance in a two-level system.[11,12]

If $\varepsilon \gg \lambda$ (which is actually equivalent to a weak field),

$$R \approx (e\lambda/4\varepsilon)^{2\varepsilon/k_0},$$

and the exponent at the point of intersection of the terms (10) is equal to the multiplicity of the corresponding resonance, i.e., $2\varepsilon/kv = 2s+1$. For ε not too close to λ, the quantity $R \ll 1$. This condition of applicability of the quasi-classical approximation is violated only in the immediate proximity to λ, when $\varepsilon - \lambda \lesssim kv$.

The continuation of the solution over the complete period of 2π is given by the matrix $U = U_{n+1}U_n$, which has the eigenvalues

$$\sigma = \exp\left\{i\frac{2\pi}{k}p_z\right\} = \exp\{2i(S_0 \pm S_\varepsilon)\}, \quad \sin S_\varepsilon = (1+R^2)^{1/2}\sin S. \tag{22}$$

Let us take as the coefficients A_n and B_n the eigenvectors of the matrix U. Then the wave function (18)

acquires, after the continuation over the period, the phase factor (22), i.e., has the Bloch form, so that p_z is a quasi-momentum. The energy spectrum is determined by the relation

$$\pm p_z = mv \pm kS_\varepsilon/\pi. \tag{23}$$

The minus sign in front of p_z in (23) corresponds to the second pair of Bloch functions, which are constructed in similar fashion from $\tilde{u}$ and $\tilde{w}$.

Since $R \ll 1$, we have $S_\varepsilon \approx S$ in those regions where $|\sin S|$ is not close to unity, and the spectrum in the extended zone scheme is given by the following implicit expression:

$$\pm p_z = mv \pm \frac{2\varepsilon}{\pi v} E\left(\frac{\lambda}{\varepsilon}\right). \tag{24}$$

The influence of R becomes important near the resonances where $S \approx \pi(s+\frac{1}{2})$. There arise at these places in the spectrum the splittings

$$\Delta\varepsilon = \frac{kv}{2} R \Big/ K\left(\frac{\lambda}{\varepsilon}\right). \tag{25}$$

Until $\varepsilon \ll \mu$, the branch points of the second kind lie far off in the τ complex plane, and are therefore unimportant.

Let us consider the energy region $\varepsilon < \lambda \ll \mu$. In this case the branch points of the interior root in (16) lie on the real axis:

$$\tau_n = \pi n \pm \tau_0, \quad \cos\tau_0 = \varepsilon/\lambda.$$

The branch points of the exterior root can be neglected. In the interval $(\pi(n-1)+\tau_0,\ \pi n - \tau_0)$ the functions (17) oscillate. Let us continue them into the next interval, bypassing the branch point $\pi n - \tau_0$ on top and underneath. Each of them will get converted into a sum of an increasing, and a decreasing, function. Here we should rectify the coefficient attached to the decreasing term[13] to ensure the constancy of the particle flux. There are four reversal points in a complete period. We continue the solution, successively bypassing all these points, and obtain the following unimodular transformation matrix:

$$U = e^{2iS_0}\begin{pmatrix} \cos^2 S - \frac{1}{4}R^2\sin^2 S & (-1)^n\left(\frac{1}{4}+\frac{1}{R^2}\right)\sin 2S \\ (-1)^{n+1}\left(1+\frac{1}{4}R^2\right)\sin 2S & \cos^2 S - \frac{4}{R^2}\sin^2 S \end{pmatrix}. \tag{26}$$

The quantity R is analogous to the quasi-classical coefficient of penetration through a barrier

$$R = \exp\left\{-\operatorname{Im}\int_{-\tau_0}^{\tau_0}\Phi_1\,d\tau\right\} \approx \exp\left\{-\frac{2\lambda}{kv}\left[E(x)-(1-x^2)K(x)\right]\right\},$$
$$x = \left(1-\frac{\varepsilon^2}{\lambda^2}\right)^{1/2}. \tag{27}$$

Further,

$$S_0 \approx \pi mv/k,$$
$$S \approx \frac{2\lambda}{kv}\left[E(y)-(1-y^2)K(y)\right], \quad y = \varepsilon/\lambda. \tag{28}$$

The quantities figuring in (27) and (28) were computed for $\varepsilon < \lambda \ll \mu$.

Determining the eigenvalues of the matrix (26), and introducing the quasi-momentum p_z, we obtain the following expression for the spectrum:

$$\pm p_z = mv \pm \frac{k}{\pi}S_\varepsilon, \quad \sin S_\varepsilon = \left(\frac{R}{4}+\frac{1}{R}\right)\sin S. \tag{29}$$

At energies ε not too close to λ, the quantity R is exponentially small. Therefore, the spectrum is a sequence of narrow allowed bands in the vicinity of energies where $S \approx \pi n$. In particular, in the region $\varepsilon \ll \lambda$ the allowed bands lie in the vicinity of the energies

$$\varepsilon_n = (\lambda k v n)^{1/2}. \tag{30}$$

As the energy increases, the allowed-band spacing decreases, while the band width increases.

Let us estimate the relative role of the branch points of the first and second kinds. Let us, for simplicity, restrict ourselves to the case of small ε. It follows from (27) that

$$R = \begin{cases} \exp(-2\lambda/kv), & \varepsilon \ll \lambda \ll \mu, \\ \exp[-1{,}65\,(2m\lambda/k^2)^{1/2}], & \varepsilon < \mu \ll \lambda. \end{cases} \tag{31}$$

The bypassing of the complex (for $\varepsilon < \mu$) branch points of the exterior root $[\mu - (\varepsilon^2 - \lambda^2(\tau))^{1/2}]^{1/2}$ leads to the appearance of the exponential factor

$$R = \begin{cases} \exp\left(-\frac{2mv}{k}\ln\gamma\frac{\mu}{\lambda}\right), & \gamma \sim 1, \quad \varepsilon \ll \lambda \ll \mu, \\ \exp\left[-\frac{4}{3}\left(\frac{2m\lambda}{k^2}\right)^{1/2}\left(\frac{\mu}{\lambda}\right)^{3/2}\right], & \varepsilon \ll \mu \ll \lambda. \end{cases} \tag{32}$$

It can be seen from a comparison of (31) and (32) that the mixing of the ε_c (or ε_v) states under conditions of Bragg reflection becomes important ($\tilde{R} \gg R$) at small μ, i.e., when the transverse quasi-momentum components $p_\perp$ are close to mv_0.

Allowance for all the branch points leads to quite a complicated expression for the spectrum. However, in the region of parameter values

$$k^2/m, \quad \varepsilon, \quad |\mu| \ll \lambda \tag{33}$$

we can develop another method of investigating the system of equations (14).

The quasi-classical solutions oscillate near the points $\pi(s+\frac{1}{2})$ in the intervals $|\Delta\tau| \lesssim \varepsilon/\lambda \ll 1$, while outside these intervals (in the subbarrier regions) these solutions contain exponential (increasing or decreasing) factors.

The penetrability of the barriers is given by the exponentially small quantity R, which also determines the allowed-band width. If we neglect this width, i.e., if we are interested only in the position of the allowed bands, then we can restrict ourselves in Eqs. (14) to the consideration of only the vicinities of $|\tau - \pi(s+\frac{1}{2})| \lesssim \varepsilon/\lambda$ and, to make up for this, successively take into account the influence of the branch points of the second

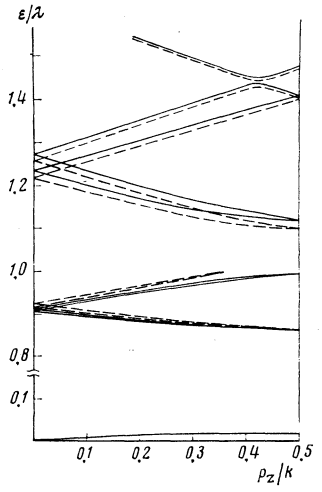

FIG. 4. The excitation spectrum for a standing-wave field in the reduced zone scheme for $\mu/\lambda=10$ and $kv/\lambda=0.5$.

kind, which can be either complex (when $\varepsilon<\mu$), or real (when $\varepsilon>\mu$). The thus obtained discrete energy levels are degenerate, since they correspond to states localized in the indicated intervals. Allowance for the overlap of the wave functions of these states, i.e., for the quantity R, leads to their being smeared into bands (a situation which is similar to the tight-binding approximation in solid-state theory).

If in (14) we expand $\cos\tau$ near $\pi/2$ and go over to the Fourier components

$$J_{1,2}(p)=\int e^{-ip\tau}(\varphi_1\pm i\varphi_2)\,d\tau,$$

then we obtain for them a system of first-order equations that leads to the following second-order equation for $J\equiv J_2$:

$$J''+[E^2-V(x)]J=0,\quad V(x)=(E_0-x^2)^2-2x,$$
$$E=\eta\varepsilon/\lambda,\quad E_0=\eta\mu/\lambda,\quad p=\eta x,\quad \eta=(2m\lambda/k^2)^{1/3}. \tag{34}$$

The form of the effective potential energy $V(x)$ depends essentially on the quantity E_0 (i.e., on μ). If $E_0\gg 1$, i.e., if $\lambda/\eta\ll\mu\ll\lambda$, then $V(x)$ represents two potential wells separated by a fairly large barrier. The energy spectrum for $\varepsilon\ll\mu$ can be obtained by considering the two wells independently. Expanding $V(x)$ near the minima, we obtain in the quasi-classical approximation for the energy levels the expression (30). These levels (except the $\varepsilon=0$ level) are two-fold degenerate. Allowance for the finite barrier penetrance removes this degeneracy. For the quasi-classical barrier penetration factor we obtain, as was to be expected, a quantity $\tilde{R}$ coinciding with the second expression in (32) (for $\varepsilon<\mu$, $\tilde{R}$ is connected with the bypassing of the complex branch points in the τ plane, or, correspondingly, with the bypassing of the real branch points in the plane $p=\eta x$). The degenerate levels undergo splittings of magnitude $\Delta\varepsilon\sim\tilde{R}$, which is significantly greater than the width of the band into which each of the levels is smeared when the factor R is taken into consideration.

As μ decreases, the potential barrier between the wells diminishes and generally disappears finally at $\mu=1.19\lambda/\eta$. Starting from some value of $\mu\sim\lambda/\eta$, and upon its subsequent decrease, the "ground" state energy ε^2 becomes strictly greater than zero and increases. Let us give the quasi-classical expression for the spectrum for $|E_0|\ll 1$:

$$\varepsilon\approx 2.35\lambda\eta^{-1}(n+1/2)^{2/3}. \tag{35}$$

Let us draw attention to the nontrivial field dependence of the energy levels: $\varepsilon\sim\lambda^{2/3}$.

At large negative $\mu(|E_0|\gg 1)$ we obtain for not too high levels the expression

$$\varepsilon\approx|\mu|+\frac{3}{5}\frac{k\lambda}{(2m|\mu|)^{1/2}}(n+1/2), \tag{35'}$$

i.e., the level spacing decreases with increasing $|\mu|$.

A numerical computation of the quasi-particle excitation spectrum was carried out on an electronic computer for some specific values of the parameters. The results are represented in Fig. 4 by the continuous curve. The dashed line represents the spectrum computed with the aid of the quasi-classical approximation. The quantitative agreement (with the exception of the regions where the conditions for quasi-classicality are strongly violated) is very good.

Let us compute the density of states for the strong-field case ($\lambda\gg kv_0$). The dominant contribution to it is made by the range of momenta $p_\perp\lesssim mv_0$. Therefore, we can use for the energy spectrum the expressions (24) (in the case when $\varepsilon>\lambda$) and (28) and (29) (in the case when $\varepsilon<\lambda$). After simple computations we obtain up to terms of the order of $(kv_0/\lambda)^{1/2}$ the expression

$$\rho(\varepsilon)=\frac{2}{\pi}\begin{cases}\frac{\varepsilon}{\lambda}K\left(\frac{\varepsilon}{\lambda}\right), & \varepsilon<\lambda,\\ K\left(\frac{\lambda}{\varepsilon}\right), & \varepsilon>\lambda.\end{cases} \tag{36}$$

The logarithmic singularity at $\varepsilon=\lambda$ is connected with the inapplicability of the quasi-classical approximation in the region $|\varepsilon-\lambda|\lesssim kv_0$. The function $\rho(\varepsilon)$ is depicted in Fig. 5.

§4. THE GREEN FUNCTION

Such integral characteristics of the spectrum as the density of states can be very clearly obtained with the aid of the diagrammatic technique. Let us introduce the Green function

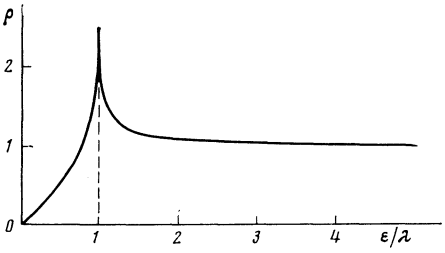

FIG. 5. Relative density of states of the quasi-particle excitations in the field of a standing-wave ($\lambda\gg kv$).

$$iG_v(x, x') = \langle T\{\psi_v(x)\psi_v^+(x')\}\rangle,$$
$$iG_c(x, x') = \langle T\{\psi_c(x)\psi_c^+(x')\}\rangle.$$

In the momentum representation the diagrammatic expansion for $G_v(\mathbf{p}, \mathbf{p}; \omega)$ has the form

$$G_v(\mathbf{p}, \mathbf{p}; \omega) = G_v^0(\mathbf{p}; \omega) + \left(\frac{\lambda}{2}\right)^2 \{G_v^0(\mathbf{p}; \omega) G_c^0(\mathbf{p}+\mathbf{k}; \omega) G_v^0(\mathbf{p}; \omega) + G_v^0(\mathbf{p}; \omega) G_c^0(\mathbf{p}-\mathbf{k}; \omega) G_v^0(\mathbf{p}; \omega)\} + \ldots, \quad (37)$$

where

$$iG_v^0(\mathbf{p}; \omega) = [\omega + \xi(\mathbf{p})]^{-1}, \quad iG_c^0(\mathbf{p}; \omega) = [\omega - \xi(\mathbf{p})]^{-1}. \quad (38)$$

The energy range in which the interaction of an electron with the field has a resonance character is of the order of λ. Because of the spatial inhomogeneity of the field, the change in the electron momentum during one interaction event is $\mathbf{k}$, while the energy change $\sim kv$. Therefore, the number of collisions that do not take the electron out of the region of resonance energies is $N \sim \lambda/kv$. In a strong field, $N \gg 1$. If the collisions have a stochastic character (i.e., are equally probable processes in each of which the electron momentum changes by $\pm\mathbf{k}$), then the change in the electron energy after the effective number of collisions $(N)^{1/2}$ is of the order of

$$kvN^{1/2} \sim (kv\lambda)^{1/2} \ll \lambda.$$

Therefore, we can set $\mathbf{k} = 0$ in (37). Summing the series, we obtain

$$iG_v(\mathbf{p}, \mathbf{p}; \omega) = \frac{1}{\omega + \xi(\mathbf{p})} \left\{\frac{\omega^2 + \xi^2(\mathbf{p})}{\omega^2 - \lambda^2 - \xi^2(\mathbf{p})}\right\}^{1/2}. \quad (39)$$

The Green function G_c is obtainable from (39) by means of the substitution $\xi(\mathbf{p}) \to -\xi(\mathbf{p})$.

The density of states computed with the aid of these Green functions, [14]

$$\rho(\omega) = -\frac{1}{\pi} \operatorname{Im} \sum_{\mathbf{p}} \{G_c(\mathbf{p}, \mathbf{p}; \omega) + G_v(\mathbf{p}, \mathbf{p}; \omega)\}, \quad (40)$$

coincides with the result (36).

The substantial difference between the electronic-excitation spectra of a semiconductor in a running-wave field and in a standing-wave field will lead to some difference in their optical properties. Because of the absence of a gap in the density of states (see Fig. 5), the transparency region in the absorption of a weak signal and in the recombination-emission spectrum will be effaced.

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