Singularities of thermodynamic quantities near tricritical points

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(Submitted December 28, 1976)
Zh. Eksp. Teor. Fiz. 72, 2299–2308 (June 1977)

A parametric equation of state, valid both near a second-order phase transition and at the tricritical point, is obtained on the basis of the renormalization-group method. The obtained equation of state describes, in particular, crossover phenomena that occur in the system as an isolated transition point is approached. The singularities of various thermodynamic quantities are calculated at the transcritical point. It is shown (with the heat capacity as an example) that these singularities can be different at $T > T_t$ and $T < T_t$ (T_t is the transition point.)

PACS numbers: 05.70.Ce, 05.70.Jk

1. The Landau theory of phase transitions[1] has a high degree of universality, allowing us, at various values of its parameters, to describe both first order and second order phase transitions. Modern representations[2] connect the second order phase transition (in contrast with first order) with the appearance in the system of fluctuations of certain thermodynamic quantities, which increase as the critical point is approached. However, it turns out that in many cases, near first order transition points, explicit signs of strongly developed fluctuations are observed,[3] and manifest themselves, in particular, in the anomalous increase in the specific heat, susceptibility and so on, in the vicinity of these points. It can therefore be thought that there exists a rather wide class of transitions of first order (with a significant heat of transition), similar in their essential features at the ordinary critical points and admitting of a common description with them, but on the base of fluctuation theory.

In this connection, there is great interest in the study of the Landau critical points (the tricritical points), at which the curve of transitions of second order undergoes a transition into the curve of transitions of first order. It is known that the fixed point of a renormalization group corresponding to the ordinary critical point is unstable if the constant of the quaternary interaction u in the Hamiltonian of the many-body system becomes less than zero. The vanishing of this constant leads to the appearance of the tricritical point. Just such a situation occurs in the He^3–He^4 solution.[4,5]

We have previously[6] proposed a model with two order parameters φ_1 and φ_2, which describes the phase transitions in this system. In the present work, we have constructed a phase diagram of the mixture He^3–He^4 within the framework of the proposed model. The possibility is shown of a unified (universal) description of this system in the entire region, including the tricritical point and the region of first order transitions. An equation is constructed of the state of the He^3–He^4 solution, which, in particular, describes the transition from critical to tricritical behavior as the tricritical point is approached. A comparison of the results with experiment is carried out. The difference in the critical points of higher order from the critical points is discussed in the case of multi-component mixtures.

2. As has been shown in Ref. 6, the effective Hamiltonian of the two-component system near the λ line takes the following form at a fixed value of the chemical potential μ of the impurity:

$$H_{\text{eff}} = \int d^dk \left\{ \frac{1}{2}[r(\mu)+k^2]|\varphi(k)|^2 + \frac{u}{4!}\prod_{i=1}^{4}\varphi(k_i)\delta\left(\sum_{i=1}^{4}k_i\right) + \frac{v}{6!}\prod_{i=1}^{6}\varphi(k_i)\delta\left(\sum_{i=1}^{6}k_i\right)\right\} - h\varphi(k=0). \tag{1}$$

Here $r(\mu) = [T - T_c(\mu)]/T_c(\mu)$ is the dimensionless departure of the temperature from the critical $T_c(\mu)$; $u(\mu)$ is the effective bare constant of the quaternary interaction, which changes its sign at some value $\mu = \mu_t$, which also corresponds to the tricritical point; h is the field thermodynamically conjugate to the order parameter φ (which orders the field). In the case of a λ transition in the He^3–He^4 system, the atoms of He^3 play the role of an equilibrium impurity; the field h in this system does not have a direct physical meaning.

We transform in (1) to the new variable:

$$\varphi(k) = \eta(k) + \sigma_0\delta(k).$$

The effective Hamiltonian in this case takes the form

$$\begin{aligned} H_{\text{eff}} = \int d^dk \Bigg[& \frac{1}{2}\left(r + \frac{u}{2}\sigma_0^2 + \frac{v}{4!}\sigma_0^4 + k^2\right)|\eta(k)|^2 \\ & + \frac{1}{3!}\left(u\sigma_0 + \frac{v}{6}\sigma_0^3\right)\prod_{i=1}^{3}\eta(k_i)\delta\left(\sum_{i=1}^{3}k_i\right) \\ & + \frac{1}{4!}\left(u + \frac{v}{2}\sigma_0^2\right)\prod_{i=1}^{4}\eta(k_i)\delta\left(\sum_{i=1}^{4}k_i\right) \\ & + \frac{v}{5!}\sigma_0\prod_{i=1}^{5}\eta(k_i)\delta\left(\sum_{i=1}^{5}k_i\right) + \frac{v}{6!}\prod_{i=1}^{6}\eta(k_i)\delta\left(\sum_{i=1}^{6}k_i\right)\Bigg] \\ & - \left(h - r\sigma_0 - \frac{u}{3!}\sigma_0^3 - \frac{v}{5!}\sigma_0^5\right)\eta(k=0) + H(\sigma_0). \end{aligned} \tag{2}$$

We choose σ_0 so that the cubic term in (2) vanishes. As a result, the Hamiltonian (2) takes a form that is iso-

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morphic to the Hamiltonian of the system near the second-order transition point. The lines of the second order transitions are determined from the equations

$$u\sigma_0+v\sigma_0^3/6=0, \quad r_{\text{eff}}=r+u\sigma_0^2/2+v\sigma_0^4/4!=0, \quad h_{\text{eff}}=h-r\sigma_0-u\sigma_0^3/3!-v\sigma_0^5/5!=0. \tag{3}$$

At $u>0$, we obtain the λ line:

$$\sigma_0=r=h=0. \tag{4}$$

At $u<0$, the solutions of the system (3) describe two second-order transition lines:

$$\sigma_0=\pm\left(-\frac{6u}{v}\right)^{1/2}, \quad r=\frac{v}{4!}\sigma_0^4, \quad h=\frac{v}{45}\sigma_0^5. \tag{5}$$

At the point $u=0$, thus, three lines of second-order phase transition converge. Near these lines, the free energy of the system is of the form

$$F(r,h,\mu)=r_{\text{eff}}^{2-\alpha_c}f[h_{\text{eff}}r_{\text{eff}}^{-\beta_c-\gamma_c}]+F_{\text{per}}(r,h,\mu). \tag{6}$$

The critical exponents α_c, β_c, γ_c in Eq. (6) characterize the behavior of the specific heat, the co-existence curve and the susceptibility near second-order transition points.

It is seen from the expression (6) that in the plane $h=0$, there also exists a first order transition line, determined by the condition $h_{\text{eff}}=0$.

We have considered the "symmetric" case, when there is the symmetry $\varphi_1\to-\varphi_1$ relative to one of the order parameters. It can be shown that there is no such symmetry. Such a situation is realized, for example, in multi-component mixtures. Here, together with the individual lines of critical points, there will be a single critical surface, bounded by the lines of final critical points. The point of intersection of the lines of final critical points is a critical point of higher order.[7] This is a single isolated point on the critical surface. The basic differences of this point from the symmetric tricritical point considered above are the following: there are no three separate lines of critical points, it is impossible to separate the critical and noncritical order parameters, i.e., the singularities of all the corresponding thermodynamic quantities relating to the different order parameters are the same.

3. We now find the singularities of behavior of the thermodynamic quantities near the tricritical point. For this purpose, as will be shown below, it is convenient to make use of an equation of state in parametric form, proposed by Migdal.[8] We separate in the Hamiltonian (1) those terms corresponding to the average value of the order parameter $\varphi_0=\varphi(k=0)$. Then the Hamiltonian takes the form

$$H_{\text{eff}}=\frac{r\varphi_0^2}{2}+\frac{u\varphi_0^4}{4!}+\frac{v\varphi_0^6}{6!}-h\varphi_0+\frac{1}{2}\int d^dk[\tilde{r}(\mu)+k^2]|\varphi(k)|^2+\Delta H, \tag{7}$$

where

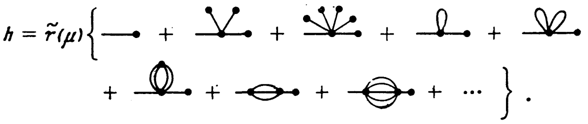
$$\tilde{r}(\mu)=r+{}^1/{}_2u\varphi_0^2+v\varphi_0^4/4!,$$

$$\Delta H=\int d^dk\left[\frac{1}{3!}\left(u\varphi_0+\frac{v}{6}\varphi_0^3\right)\prod_{i=1}^{3}\varphi(k_i)\delta\left(\sum_{i=1}^{3}k_i\right)+\frac{1}{4!}\left(u+\frac{v}{2}\varphi_0^2\right)\prod_{i=1}^{4}\varphi(k_i)\delta\left(\sum_{i=1}^{4}k_i\right)+\frac{v}{5!}\varphi_0\prod_{i=1}^{5}\varphi(k_i)\delta\left(\sum_{i=1}^{5}k_i\right)+\frac{1}{6!}v\prod_{i=1}^{6}\varphi(k_i)\delta\left(\sum_{i=1}^{6}k_i\right)\right]. \tag{8}$$

The equation of state is obtained from the condition[9]

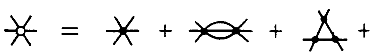
$$\langle\partial H_{\text{eff}}/\partial\varphi_0\rangle=0. \tag{9}$$

Averaging in (9) is carried out over the distribution function with the Hamiltonian (7). Expanding (9) in a series in ΔH and carrying out partial summation, we obtain



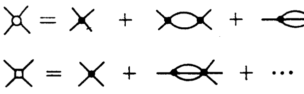
(10)

Here the lines with a point at the end denote φ_0, lines without the point denote the bare Green's function; $G(k)=[\tilde{r}(\mu)+k^2]^{-1}$. The sixfold vertices entering into (10) are equal to $\tilde{v}\equiv$



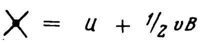
(11)

The four vertices in Eq. (10) turn out to be of two types u and u':



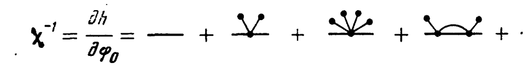
(12)

while the bare fourfold vertex is determined by the combination



(13)

where B is the part of the integral $\int d^dkG(k)$ that is independent of the closeness to the critical point. The first equation (12) determines the total fourfold vertex. Far from the tricritical point, the presence of vertices of the two types is unimportant, since the vertex determined by the equation similar to the second equation of (12), is eliminated by the elementary substitution $M^2=u\varphi_0^2$. Near the tricritical point, such a substitution is inconvenient, since u vanishes. We therefore transform to the equation of state in the variables χ, φ_0. We add to the relations (10)–(12) the equation for the susceptibility χ, which is found by direct differentiation of Eq. (10) with respect to φ_0:



(14)

We have written down only the most characteristic graphs in first and second order in the constants of four and six interactions. The first three graphs are obtained upon differentiation of the corresponding terms of Eq. (10) with respect to φ_0, connected with external ends, while the last graph is obtained by differentiation of the internal G functions with respect to φ_0. Eliminating the quantity $\tilde{r}(\mu)$ from Eqs. (10) and (14), we obtain the equation of state $h=h(\chi, \varphi_0)$, in which only the complete fourfold and sixfold vertices appear:

$$h=\varphi_0\chi^{-1}-\frac{1}{3}\tilde{u}\varphi_0^3-\frac{4}{5!}\tilde{v}\varphi_0^5. \tag{15}$$

Equation (15) is valid both near the critical point and the tricritical point. The quantities $\tilde{u}$ and $\tilde{v}$ change upon approach to the transition point, each tending to its limiting value. At small but non-vanishing values of the bare four vertex, Eqs. (11), (12) and (15) describe the transition from the tricritical to the critical point (crossover). As is seen from (11), the terms which are proportional to the quantity v, reduce in this case to account of higher orders in the constant of four interaction.

To find the equation of state in explicit form, it is necessary to find the functions $\tilde{u}$ and $\tilde{v}$ that enter into (15), i.e., to sum the series (11) and (12). For this purpose, the expansion is somewhat inconvenient, since it is connected with the expansion in a series in the small departure of the dimensionality of the space d from a dimensionality for which the fluctuations are Gaussian. In our case, there are two fixed points, one of which is Gaussian at $d=4$ (the critical point), and the other at $d=3$ (tricritical point). We make use therefore of the method suggested by S. L. Ginzburg.[10] This method allows us to calculate the critical exponents directly in three-dimensional space.

We emphasize that all the interior G functions in Eqs. (11) and (12) can be assumed to be renormalized and equal to $G(k)=(k^2+r_c^{-2})^{-1}$, where r_c is the correlation radius. Differentiating Eq. (12) with respect to r_c^{-2} for example, we obtain

$$\frac{d\tilde{u}}{dr_c^{-2}}=$$

$$+\cdots \tag{12'}$$

The wavy line in (12′) denotes differentiation of the Green's function with respect to r_c^{-2}. It can be shown by direct summation that the complete four and six vertices enter into the right side of (12′). The resultant equation represents the Ward identity for the fourfold vertex.

In obtaining the first equation (12), we have neglected the index η (the critical exponent which characterizes the departure of the correlation function from the Orstein-Zernike function). We have, within the framework of this approximation:

$$r_c^{-2}=A\chi^{-1},\quad 4\beta=\gamma \tag{16}$$

(where A is a constant). The relation (16) means that the values of the ratio β/γ does not change (with accuracy to within the index η) in the transition from the tricritical to the critical region. This allows us below to simplify the equation of state (15) by eliminating the strong dependence on the susceptibility in it. It is essential that this dependence be the same for both the tricritical and the critical regions.

In analytic notation, Eq. (12′) takes the following form in principal order in $\tilde{u}$ and $\tilde{v}$:

$$d\tilde{u}/dr_c^{-2}=r_c^{6-d}a_1\tilde{u}^2+r_c^{8-2d}a_2\tilde{u}\tilde{v}+r_c^{10-3d}a_3\tilde{v}^2. \tag{17}$$

In complete analogy, by differentiating (11) with respect to r_c^{-2}, we obtain an equation for the sixfold vertex $\tilde{v}$:

$$d\tilde{v}/dr_c^{-2}=r_c^{8-2d}b_1\tilde{v}^2+r_c^{6-d}b_2\tilde{u}\tilde{v}-r_c^{8-d}b_3\tilde{u}^3. \tag{18}$$

In Eqs. (17), (18), the coefficients a_i and b_i depend on the number of components n of the order parameter φ and the dimensionality of the space d. In particular, for the case $d=3$, these coefficients are equal to

$$a_1=\frac{n+8}{96\pi},\quad a_2=\frac{n+4}{240\pi^2},\quad a_3=\frac{n^2+18n+56}{19200\pi^3}\ln\frac{4}{3},$$
$$b_1=\frac{3n+22}{480\pi^2},\quad b_2=\frac{n+14}{32\pi},\quad b_3=\frac{5(n+26)}{192\pi}.$$

The relations (15, (17) and (18) completely specify the equation of state of the system, both near the critical and near the tricritical points. The tricritical point is obtained in the case in which the bare fourfold verte (13) vanishes. By virtue of the relation (16), the vertices $\tilde{u}$ and $\tilde{v}$ can be regarded as functions that depend only on χ (which are obtained simply by the substitution $r_c^{-2}\to\chi^{-1}$).

Equations (17) and (18) represent the set of nonlinear differential equations relative to $\tilde{u}$ and $\tilde{v}$, the solution of which is very difficult in general form. Putting the functions $\tilde{u}(\chi)$ and $\tilde{v}(\chi)$ in (17) and (18) in the form

$$\tilde{u}(\chi)=g(\chi)\chi^{-(4-d)/2},\quad \tilde{v}(\chi)=p(\chi)\chi^{-(3-d)}, \tag{19}$$

we obtain

$$\frac{dg}{d\ln\chi^{-1}}=\psi_1(g,p),\qquad \frac{dp}{d\ln\chi^{-1}}=\psi_2(g,p). \tag{20}$$

The functions $\psi_1(g, p)$ and $\psi_2(g, p)$ in (20) are equal to the following:

$$\psi_1(g,\ p)=-{}^1/_2(4-d)g+a_1g^2+a_2gp+a_3p^2,$$
$$\psi_2(g,\ p)=-(3-d)p+b_1p^2+b_2pg-b_3g^3. \tag{21}$$

The set of equations (20) is analogous to the equations of the renormalization group for complete fourfold and sixfold vertices. The vanishing of the functions $\psi_1(g, p)$ and $\psi_2(g, p)$ determines the fixed points of the initial Hamiltonian. Solving the equations $\psi_1=\psi_2=0$, we find the fixed points

$$1)\ g_t=\tilde{v}_t=0, \tag{22}$$

$$2)\ g_c\approx\frac{4-d}{2a_1},\quad p_c\approx\frac{b_3g_c^3}{b_2g_c+d-3}. \tag{23}$$

The first of these equations corresponds to the tricritical and the second, to the critical point.

It is not difficult to see that the solution (22) is stable only in the region $g \ll p$. Neglecting all terms containing g in this region in Eqs. (21), we obtain

$$p=\frac{v_0(3-d)}{(3-d-b_1v_0)\chi^{d-3}+b_1v_0},$$
$$\tilde{v}=\frac{v_0(3-d)}{3-d+b_1v_0(\chi^{3-d}-1)}\xrightarrow{d\to3}\frac{v_0}{1+b_1v_0\ln\chi}. \tag{24}$$

In the case $g \gtrsim p$, the solution (23) turns out to be stable. We emphasize that p_c in this case (and, consequently $\tilde{v}_c$ also) is of the order of ε^2 (in the case $d \to 3$). In other words, the term proportional to φ_0^5 in Eq. (15), appears only in second order in ε, which agrees with the result obtained earlier by Avdeeva.[11]

4. We return to the equation of state (15). We immediately consider the case $d=3$. Substituting $\tilde{u}(\chi)$ in the form (19) in Eq. (15), and taking (16) into account, we obtain

$$h=\chi^{-1-\beta/\gamma}\left[m-\frac{1}{3}g(\chi)m^3-\frac{4}{5!}\tilde{v}(\chi)m^5\right]. \tag{25}$$

The new variable

$$\varphi_0=\chi^{-\beta/\gamma}m, \tag{26}$$

is introduced in Eq. (25), and $g(\chi)$ and $\tilde{v}(\chi)$ are the solutions of Eqs. (20). The expression (25) is entirely analogous to the parametric equation of state proposed by Migdal,[8] with only the difference that the coefficients in (26) are singular functions of χ. We emphasize that (25) is valid both near the critical and near the tricritical points.

The transition phenomena are determined by solutions of Eqs. (20) for $g(\chi)$ and $\tilde{v}(\chi)$, in contrast with the work of Riedel and Wegner,[12] where these phenomena were studied on the basis of phenomenological interpolation equations. The susceptibility in (25) must be regarded as a function of the parameter m. Generally speaking, this function can be found directly from the expression for the susceptibility (14). However, it is simpler for us to do it by starting out from Eqs. (25) and (26). Actually, substituting Eqs. (25) and (26) in the determination of the susceptibility $\chi^{-1}=\partial h/\partial\varphi_0$, we find:

$$\chi'\chi^{-1}=\left(gm+\frac{\tilde{v}}{3!}m^3\right)\left[-1+\left(1+\frac{\beta}{\gamma}\right)\left(\frac{g}{3}m^2+\frac{4\tilde{v}}{5!}m^4\right)+\frac{m^2}{3}\frac{dg}{d\ln\chi^{-1}}+\frac{4m^4}{5!}\frac{d\tilde{v}}{d\ln\chi^{-1}}\right]^{-1}. \tag{27}$$

In Eq. (27), the derivative of the susceptibility with respect to the variable m is denoted by χ'.

In the asymptotic critical region, we must replace the quantities g and $\tilde{v}$ by their values at the critical fixed point, g_c and $\tilde{v}_c$. For the critical exponent of the susceptibility γ_c, we obtain the following expression in the usual fashion[8]:

$$\gamma_c=\frac{6g_c+\tilde{v}_cm_0^2}{5g_c+\tilde{v}_cm_0^2}\approx\frac{8}{7}, \tag{28}$$

where m_c is the first root of the equation

$$-1+\frac{5}{12}g_cm^2+\frac{1}{4!}\tilde{v}_cm^4=0.$$

In the tricritical region, neglecting the term in (15) that is proportional to m^3 and replacing $\tilde{v}(\chi)$ by the expression (24), we find

$$h=\chi^{-5/4}\left(m-\frac{4}{5!b_1}m^5\ln^{-1}\chi\right). \tag{25'}$$

We make the change of variable

$$\theta=m\ln^{-1/4}\chi.$$

Then the expressions (26), (25′) and (27) take the form

$$\varphi_0=\chi^{-1/4}\theta\ln^{1/4}\chi, \tag{29}$$

$$h=\chi^{-5/4}\Phi(\theta)\ln^{5/4}\chi, \tag{30}$$

$$\chi^{-1}\frac{\partial\chi}{\partial\theta}=\frac{\Phi'(\theta)-1}{{}^5/_4\Phi(\theta)-{}^1/_4\theta-{}^1/_4\ln^{-1}\chi[\Phi(\theta)-\theta]}, \tag{31}$$

where $\Phi(\theta)=\theta-4\theta^5/5!\,b_1$. Equation (31) differs from the corresponding equation of the work of Migdal[8] by the term proportional to $\ln^{-1}\chi$ in the denominator of the right side.

Introducing the notation

$$\varkappa=\frac{\beta_t\gamma_t}{\beta_t+\gamma_t}=\frac{1}{5},\qquad\theta_0{}^4=4!b_1,\qquad y=\ln\chi,$$

it is not difficult to find the solution of Eq. (31):

$$\frac{\theta^4}{\theta_0{}^4}=-e^yy^{-\varkappa}\int\limits_{y(\theta=0)}^{y}e^{-t}t^{\varkappa}\,dt=-e^yy^{-\varkappa}\gamma(y_0,y)\approx1-e^{y-y_0}\left(\frac{y_0}{y}\right)^{\varkappa}, \tag{32}$$

where $y_0\equiv y(\theta=0)$. In the last equation of the expression (32), we used the expansion of the incomplete gamma function in $y^{-1}=\ln^{-1}\chi\ll1$, which is always valid close to the phase transition points.

Returning again to the susceptibility, we find, by taking into account the smallness of the parameter $\varkappa$:

$$\chi(r,\theta)=\chi(r,0)\left(1-\frac{\theta^4}{\theta_0{}^4}\right)\left[1+\varkappa\frac{\ln(1-\theta^4/\theta_0{}^4)}{\ln\chi(r,0)}\right], \tag{33}$$

where $r=(T-T_t)/T_t$ is the departure from the tricritical temperature, $\chi(r,0)=b/r$, b is a constant. This equation determines the behavior of the susceptibility near the tricritical point in the nonzero ordering field.

We proceed to the calculation of the singular part of the free energy. We write down the free energy F in the form

$$F=\int\limits_{h}^{h_k}\varphi_0\,dh+F(h_k,T). \tag{34}$$

The function $F(h_k, T)$ does not have a singularity as $r\to0$. The singular part of the free energy F_s is determined by the value of the integral in (34) at the lower limit. The free energy then is of the form

$$F=-\frac{5}{4}\int\limits_{y}^{y_k} e^{-3y/2}y^{1/2}\theta(y)\left\{(1-\varkappa y^{-1})\Phi[\theta(y)]-\frac{4}{5}\frac{d\Phi[\theta(y)]}{dy}\right\}dy+F(h_k,T). \tag{35}$$

Substituting the found solution (32) in place of $\theta(y)$ and changing over under the integral sign to the variable

$$z=e^{y-y_0}(y_0/y)^{\varkappa},$$

we obtain

$$F=-\frac{\theta_0^2}{4}\exp\left(-\frac{3}{2}y_0\right)y_0^{5\varkappa/2}\int\limits_{z}^{z_k}\frac{dz}{(1-z)^{1/2}z^{5/2}}\left[\frac{y(z)}{y_0}\right]^{\varkappa}+F(h_k,T). \tag{36}$$

Using the smallness of $\varkappa$, we write

$$\left[\frac{y(z)}{y_0}\right]^{\varkappa}\approx 1+\varkappa\frac{\ln z}{y_0}=1+\frac{\varkappa}{y_0}\lim_{\varepsilon\to 0}\frac{z^{\varepsilon}-1}{\varepsilon}. \tag{37}$$

Since the basic part of the free energy F_s is determined by the value of the integral in (36) at the lower limit, we can replace the upper limit by zero. Here F_s is expressed in terms of the hypergeometric function:

$$F_s=\frac{\theta_0^2}{4}\exp\left(-\frac{3}{2}y_0\right)y_0^{5\varkappa/2}\left\{-\frac{2}{3}z^{-3/2}(1-z)^{1/2}{}_2F_1\left(-1,1;-\frac{1}{2};z\right)\right.$$
$$+\frac{\varkappa}{y_0}(1-z)^{1/2}z^{-3/2}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\frac{2z^{\varepsilon}}{2\varepsilon-3}{}_2F_1\left(\frac{1}{2},-\frac{3}{2}+\varepsilon;-\frac{1}{2}+\varepsilon;z\right)\right.$$
$$\left.\left.+\frac{2}{3}{}_2F_1\left(\frac{1}{2},-\frac{3}{2};-\frac{1}{2};z\right)\right]\right\}. \tag{38}$$

We investigate in detail the case of a zero ordering field. At $r>0$, i.e., above the tricritical point, $h=0$ corresponds to $z=1$. The principal part of the free energy is equal to

$$F_s=-\frac{\theta_0^2}{4}\exp\left(-\frac{3}{2}y_0\right)y_0^{5\varkappa/2}\frac{4\pi\varkappa}{3y_0}. \tag{39}$$

At $r<0$, i.e., below the tricritical point, $z=-\frac{1}{4}$ and F_s is equal to

$$F_s=-\frac{\theta_0^2}{4}\exp\left(-\frac{3}{2}y_0\right)y_0^{5\varkappa/2}\left(\frac{4\sqrt{5}}{3}-\frac{a}{y_0}\varkappa\right), \tag{40}$$

where

$$a=\frac{4\sqrt{5}}{3}\left(\frac{1}{3}+\ln 4\right)-\frac{8}{3}\operatorname{arsh}\frac{1}{2}\approx 3.66.$$

We now find the principal part of the specific heat C_μ of the system He^3–He^4 at constant chemical potential:

$$C_\mu=-T\frac{\partial^2 F_s}{\partial T^2}=-\frac{T}{T_t^2}\frac{\partial^2 F_s}{\partial r^2}.$$

As a result, we get

$$C_{\mu 1}=\frac{T}{T_t^2}\frac{\pi\varkappa}{4b^{3/2}}\theta_0^2 r^{-1/2}\ln^{-1/2}\frac{b}{r}\quad\text{at}\quad r>0, \tag{41}$$

$$C_{\mu 2}=\frac{T}{T_t^2}\frac{\theta_0^2}{b^{3/2}}|r|^{-1/2}\left\{\frac{\sqrt{5}}{4}\ln^{1/2}\frac{b}{|r|}-\left(\frac{\sqrt{5}}{3}+\frac{3\varkappa a}{16}\right)\ln^{-1/2}\frac{b}{|r|}\right\}\quad\text{at}\quad r<0. \tag{42}$$

Thus, there are logarithmic corrections to the specific heat both above and below the tricritical point, and they are different: at $r>0$, the singularity is weaker than square root, while at $r<0$, it is stronger.

The expressions (41) and (42) differ both from the results of Ref. 13, in which the logarithmic corrections appear only for an interaction constant that differs from zero, and from the results of Ref. 14, in which the logarithmic corrections to the specific heat at $r>0$ are lacking.

We note that, without account of the logarithmic terms in (29), and (30) (which correspond to the approximation $\varkappa=0$ in Eq. (36)), the principal part of the specific heat vanishes above the transition point. This result agrees with the result of the self-consistent field theory.[15]

The ratio of the coefficients at the singularities of the specific heat below and above the tricritical point is universal and is equal to $\sqrt{5}/\varkappa\pi\approx 3.56$. It can be shown that the singularity of the quantity $\partial x/\partial\mu$ (x is the He^3 concentration) in the He^3–He^4 mixture has the same form as does the singularity of the specific heat. Thanks to the existence of a non-vanishing principal part of $\partial x/\partial\mu$ above T_t, the line of phase transitions $T_c(x)$ has a node at the tricritical point, in correspondence to the experimental data.

The experimental values of the tricritical indices agree with the theoretical; however, the accuracy of the experiments is insufficient for calculation of the logarithmic corrections to the behavior of the thermodynamic quantities.[16] Comparison of theory with experiment is made more difficult by one fact. It is shown that the size of the tricritical region is quite insignificant on the side of the ordered phase in the He^3–He^4 mixture.[16] This does not allow us to determine the coefficients in singularities at $T<T_t$.

In conclusion, the authors thank M. A. Anisimov, A. T. Berestov and A. A. Migdal for useful discussions.

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Translated by R. T. Beyer