Boundary conditions in phonon hydrodynamics

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The boundary condition on the interface of a dielectric with a metal is investigated for the hydrodynamic variables δT (the rise of the phonon-gas temperature above the bath temperature T_0) and u (the phonon-gas drift velocity normal to the b interface). It follows from the kinetic equation in the boundary layer that the boundary condition takes the form $(\delta T - \delta \bar{T})/T_0 = -\alpha(u/s)$, where s is the speed of sound, $\delta \bar{T}$ is the rise of the metal temperature over the bath temperature, and the coefficient α can be expressed in terms of the coefficient of phonon reflection from the interface.

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INTRODUCTION: FORMULATION OF PROBLEM

Experiments performed during the 70's have shown that thermal pulses are highly effective means of investigating the properties of phonons and of the electron-phonon interaction. [1] Thermal pulses are frequently excited and detected with the aid of metallic films sputtered on dielectric or semiconductor crystals in which the pulses propagate. To excite a thermal pulse, the film is heated with current or by radiation (laser or microwave), and the pulse is detected by measuring the change it produces in the film resistance.

This raises the question of the "thermal resistance" of the metal–dielectric interface. It is significant that this resistance depends not only on the properties of the interface, but also on the propagation regime of the thermal pulse. A distinction is made between three thermal-pulse propagation regimes: ballistic, hydrodynamic (the so-called second-sound regime), and diffusion (thermal conduction). The pulse propagation regime depends on the relation between three characteristic times, τ_N, τ_R, and L/s. The time τ_N is determined by the normal phonon-phonon collisions, in which the energy and momentum of the phonon system are conserved (N-processes); after a time τ_N, the phonon gas, left to itself, goes over into a state of "thermodynamic equilibrium," i.e., in a state described by a biased Planck distribution. The time τ_R is determined by umklapp processes and by scattering from static defects, in other words, by collisions in which the energy of the phonon system is conserved, but the momentum is not (R-processes). After a time τ_R, the phonon system relaxes to a state with zero total momentum, i.e., to a state with an isotropic distribution function. In pure crystals at low temperatures we have $\tau_R \gg \tau_N$; we are interested only in this case. The third characteristic time L/s is the time of flight of the phonon from the source to the detector (L is the distance between them and s is the speed of sound).

The ballistic regime is realized at $L/s \ll \tau_N$. In this regime all the phonons comprising the thermal pulse propagate independently; the thermal resistance of the interface is therefore determined simply by the coefficient of transmission of an individual phonon (calculated, e.g., by the acoustic mismatch model), averaged in suitable manner over the propagation directions and over the polarizations.

The hydrodynamic regime is realized at $\tau_N \ll L/s \ll \tau_R$. The phonons propagation is described by a system of equation for the phonon-gas temperature $T(\mathbf{r}, t)$ and the phonon-gas drift velocity $\mathbf{u}(\mathbf{r}, t)$. The reflection of the thermal pulse is determined in this regime by the boundary conditions imposed on T and $\mathbf{u}$ at the interface between the dielectric and the metal. It is usually assumed that the metal can be characterized by a certain temperature $\bar{T}$ higher than the helium-bath temperature T_0. Although estimates[1] show that there is no special basis for this assumption, we shall make use of it. It should be assumed none the less that the boundary condition can be formulated in the form $T = \bar{T}$. The purpose of the present paper is to find the correct boundary condition. To avoid misunderstandings, we emphasize from the very outset that we are interested not in small corrections of order $s\tau_N/L$, and that the boundary condition differs substantially from the condition $T = \bar{T}$.

At small deviations from equilibrium $T \equiv T_0$ and $\mathbf{u} \equiv 0$ we obtain for $\delta T(\mathbf{r}, t) = T(\mathbf{r}, t) - T_0$ and $\mathbf{u}(\mathbf{r}, t)$, as is well known (see the review[2]), linear wave equations that describe perturbations propagating at the speed of second sound. Knowledge of the boundary conditions for δT and $\mathbf{u}$ allows us to solve the problem of the reflection of a second-sound pulse from a dielectric–metal interface, as well as the problem of exciting a second-sound pulse in a dielectric by a heat pulse in a metal.

The diffusion regime is realized at $L/s \gg \tau_R$. In this case the phonon propagation is described by the diffusion equation for the phonon-energy density. At small deviations from equilibrium we obtain the heat-conduction equation for $\delta T(\mathbf{r}, t)$. In the diffusion regime, the thermal momenta are more likely to spread than to propagate, and this regime is therefore less interesting. We mention it only in order to emphasize that the boundary condition $T = \bar{T}$ is valid only in the diffusion limit.

As is customary in hydrodynamics, we obtain the boundary conditions by considering the Knudsen surface layer. [3] In this layer, whose thickness is of the order of the mean free path $l_N = s\tau_N$, the collisions with the boundary are just as frequent as the collisions between

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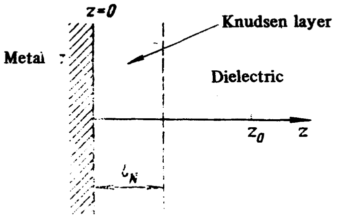

FIG. 1.

the phonons, so that no local equilibrium, i.e., no biased Planck distribution of the phonons, is established. For this reason, the equations of hydrodynamics do not hold in the Knudsen layer.

The procedure of finding the boundary conditions for the hydrodynamic variables T and $\mathbf{u}$ consists of the following. We separate near the boundary a region $0<z<z_0$, and z is the distance to the boundary (see Fig. 1). We solve in this region the kinetic equation for the phonon distribution function $f(\mathbf{r}, t;\mathbf{q})$ with account taken only of N processes and of the interaction with the boundary. The characteristic spatial scale of this problem is l_N. One should therefore expect at $z \gg l_N$ the obtained distribution function to be a biased Planck function with certain T and $\mathbf{u}$. Assuming that this description becomes matched to the hydrodynamic description at $z \sim z_0$, we arrive at the conclusion that the connection obtained between T and u by solving the kinetic equation is in fact the sought boundary condition for the hydrodynamic equations.

1. KINETIC EQUATION AND QUASIHYDRODYNAMIC VARIABLES

Under the conditions of interest to us, the kinetic equation takes the form

$$\partial f/\partial t+\mathbf{v}_{\mathbf{q}}\nabla f=\hat{S}f, \tag{1.1}$$

where $\mathbf{v}_{\mathbf{q}}$ is the group velocity of a phonon with momentum $\mathbf{q}$, and $\hat{S}f$ is the collision term describing only N-processes. The explicit form of $\hat{S}f$ is very complicated and depends on the concrete type of the normal collisions (three-phonon or four-phonon, etc.). In most cases the experiments do not yield so detailed a discrimination of the N-processes. We therefore approximate the collision term $\hat{S}f$ in the same spirit as in plasma theory for collisions between particles. [4]

The employed approximation consists in the following. Owing to the normal collisions, any distribution $f(\mathbf{q})$ relaxes to a certain biased Planck distribution

$$\hat{P}f(\mathbf{q})=\{\exp[(\omega_{\mathbf{q}}-\mathbf{uq})/T]-1\}^{-1}, \tag{1.2}$$

and we obtained T and $\mathbf{u}$ from the condition that the total energy and momentum of the phonons are the same before and after relaxation. We denote the energy and momentum of the distribution f as follows:

$$\varepsilon[f]=\int\frac{d\mathbf{q}}{(2\pi)^3}\omega_{\mathbf{q}}f(\mathbf{q}).\quad \mathbf{p}[f]=\int\frac{d\mathbf{q}}{(2\pi)^3}\mathbf{q}f(\mathbf{q}). \tag{1.3}$$

The energy and momentum conservation in the relaxation means then that

$$\begin{aligned}&\varepsilon[f]=\varepsilon[\hat{P}f]\equiv\varepsilon(T,\mathbf{u}),\\&\mathbf{p}[f]=\mathbf{p}[\hat{P}f]\equiv\mathbf{p}(T,\mathbf{u}).\end{aligned} \tag{1.4}$$

Equations (1.4) constitute the system for the determination of T and $\mathbf{u}$ from the specified distribution $f(\mathbf{q})$.

It is natural to approximate

$$\hat{S}f\to-\tau^{-1}(f-\hat{P}f), \tag{1.5}$$

where τ is a certain relaxation time that can be identified with τ_N. It can be shown that in this approximation τ cannot depend on $\mathbf{q}$. In fact, the rate of change of the energy and momentum in N processes is equal to zero, i.e.,

$$(\partial/\partial t)_N\varepsilon[f]=\varepsilon[\hat{S}f]=0.\qquad(\partial/\partial t)_N\mathbf{p}[f]=\mathbf{p}[\hat{S}f]=0. \tag{1.6}$$

It is obvious that under the approximation (1.5) the conditions (1.6) will be satisfied for any function $f(\mathbf{q})$ only when τ does not depend on $\mathbf{q}$.

If the distribution f depends on $\mathbf{r}$ and t, then the temperature T and the drift velocity $\mathbf{u}$ obtained from (1.4) also depend on $\mathbf{r}$ and t. We shall call these $T(\mathbf{r}, t)$ and $\mathbf{u}(\mathbf{r}, t)$ quasihydrodynamic variables (the reason for this designation will be made clear below).

Using (1.5), we write down the kinetic equation in the form

$$\frac{\partial f}{\partial t}+\mathbf{v}_{\mathbf{q}}\nabla f+\frac{f}{\tau}=\frac{1}{\tau}\hat{P}f. \tag{1.7}$$

It is necessary to add to this equation the boundary condition in the interface between the dielectric and the metal. It is assumed henceforth that the phonons are reflected elastically and that the probability of reflection from the state $\mathbf{q}'$ to the state $\mathbf{q}$ depends only on the directions of these momenta, which are specified by the unit vectors $\mathbf{e}'$ and $\mathbf{e}$. Then the boundary condition on the distribution function takes the form

$$f(\mathbf{r}_0,t;\mathbf{q})|_{\mathbf{vn}>0}=f_0(\mathbf{r}_0,t;\mathbf{q})+\int_{-} do'r(\mathbf{e}'\to\mathbf{e})f(\mathbf{r}_0,t;\mathbf{q}')|_{\mathbf{v}'\mathbf{n}<0}. \tag{1.8}$$

Here $\mathbf{r}_0$ is a point on the interface, $\mathbf{n}$ is a unit vector normal to this point and directed into the interior of the dielectric, $r(\mathbf{e}'\to\mathbf{e})$ is the phonon reflection coefficient, and the sign (−) under the integral sign means that the integration is over the hemisphere on which $\mathbf{v}'\cdot\mathbf{n}<0$. Next, f_0 is the distribution function of the phonons emitted from the metal; it is assumed specified and is completely determined by the temperature $\bar{T}(\mathbf{r}_0, t)$ of the metal:

$$f_0(\mathbf{q})=f_{\bar{T}}(q)\bar{t}(\mathbf{e}), \tag{1.9}$$

where $f_{\bar{T}}(q)$ is an equilibrium distribution with temperature $\bar{T}$, and

$$\bar{t}(\mathbf{e})=\int_{+} do'\bar{t}(\mathbf{e}'\to\mathbf{e}). \tag{1.10}$$

Here $\tilde{t}(\mathbf{e}'\to\mathbf{e})$ is the probability that a phonon incident on the interface from the metal in a direction $\mathbf{e}'$ will be emitted into the dielectric in the direction $\mathbf{e}$. The sign (+) under the integral sign denotes integration over the hemisphere on which $\mathbf{v}'\cdot\mathbf{n}>0$.

From the detailed balancing principle we obtain for the scattering by the wall

$$\tilde{t}(\mathbf{e}'\to\mathbf{e})=t(-\mathbf{e}\to-\mathbf{e}'), \tag{1.11}$$

$$r(\mathbf{e}'\to\mathbf{e})=r(-\mathbf{e}\to-\mathbf{e}'), \tag{1.12}$$

where t is the probability of the transition of the phonon from the dielectric to the metal. Using (1.11) and (1.12), we have

$$\tilde{t}(\mathbf{e})=\int_{-} do' t(-\mathbf{e}\to\mathbf{e}')=1-\int_{+} do' r(-\mathbf{e}\to\mathbf{e}')=1-\int_{-} do' r(\mathbf{e}'\to\mathbf{e}). \tag{1.13}$$

We see now that the boundary condition is automatically satisfied by an equilibrium distribution $f_{\tilde{T}}(q)$ with the metal temperature.

We shall show that in the approximation (1.5) the kinetic equation (1.7) and the boundary condition (1.8) can be made into a closed system for the quasihydrodynamic variables T and $\mathbf{u}$, or in other words that the system can be reduced to a system of two equations for $T(\mathbf{r}, t)$ and $\mathbf{u}(\mathbf{r}, t)$. We consider for this purpose the right-hand side of (1.7) as a specified source; obviously, it is expressed in terms of T and $\mathbf{u}$. The left-hand side of (1.7) describes the dynamic motion of the phonons, which is accompanied by their annihilation, but without scattering. This particle motion follows the dynamic trajectories, and f can be easily expressed in terms of the specified sources. For trajectories arriving at the interface, $\tau^{-1}\hat{P}f$ is the only source, and therefore the distribution of the phonons moving towards the interface, $f(v_n<0)$, can be easily expressed in terms of the quasihydrodynamic variables. For trajectories emerging from interface, there is an additional source—the boundary itself, i.e., $f(v_n>0)$ on the boundary. It is clear from (1.8), however, that this source is expressed in terms of $f(v_n<0)$, i.e., in final analysis via T and $\mathbf{u}$. Thus, all the sources at $f(v_n>0)$ are expressed in terms of T and $\mathbf{u}$. Consequently, the distribution of the phonons that move from the interface is also expressed in terms of the quasihydrodynamic variables. We now multiply $f(v_n>0)$ and $f(v_n<0)$ by $\omega_{\mathbf{q}}$ and $\mathbf{q}$, integrate each distribution over the corresponding half of the phase space, and add the resultant equations. Using (1.4), we make the system closed with respect to the quasihydrodynamic variables.

We can now explain the meaning of the term "quasihydrodynamic." As a result of the described procedure we have obtained a closed system of two equations for $T(\mathbf{r}, t)$ and $\mathbf{u}(\mathbf{r}, t)$—here is where the analogy with hydrodynamics comes in. In contrast to hydrodynamics, however, this is a system not of differential but of integral equations; the corresponding nonlocality reflects the finite character of the mean free path l_N. This accounts for the prefix "quasi."

2. THE KINETIC EQUATION IN THE KNUDSEN LAYER

We shall carry out explicitly the program described above for the particular case of a stationary problem with a flat interface. The solution of this problem suffices to determine the boundary conditions of the hydrodynamic equations. We shall assume, in addition, that $\mathbf{u}\parallel\mathbf{n}\parallel z$ in the hydrodynamic equations. For simplicity we assume that there is only one phonon branch with an isotropic spectrum without dispersion, and that the reflection is axially symmetric about z. It is then clear from symmetry considerations that the distribution functions f depend only on z, q, and $\cos\theta$, where θ is the angle between q and z, and the quasihydrodynamic variables T and $\mathbf{u}$ depend only on z, with $\mathbf{u}\parallel z$. The distance z will be measured in units of $s\tau$, and $u\equiv u_z$ in units of s.

The kinetic equation (1.7) now takes the form

$$x\frac{\partial f}{\partial z}+f=\hat{P}f,\quad x=\cos\theta. \tag{2.1}$$

For phonons traveling towards the interface we have from (2.1)

$$f(z;q,x)\,|_{x<0}=-\int_{z}^{\infty}\frac{dz'}{x}\exp\left[-\frac{z-z'}{x}\right]\hat{P}f(z';q,x). \tag{2.2}$$

For phonons traveling away from the boundary,

$$f(z;q,x)\,|_{x>0}=f(0;q,x)e^{-z/x}+\int_{0}^{z}\frac{dz'}{x}\exp\left[-\frac{z-z'}{x}\right]\hat{P}f(z';q,x). \tag{2.3}$$

The boundary condition (1.8) becomes

$$f(0;q,x)\,|_{x>0}=f_0(q,x)+\int_{-1}^{0}dx'r(x,x')f(0;q,x'), \tag{2.4}$$

$$r(x,x')=\int_{0}^{2\pi}d\varphi' r(\mathbf{e}'\to\mathbf{e}),\quad \mathbf{e}=(\theta,\varphi),\quad \mathbf{e}'=(\theta',\varphi'), \tag{2.5}$$

$$f_0(q,x)=f_{\tilde{T}}(q)\left[1-\int_{-1}^{0}dx'r(x',x)\right]. \tag{2.6}$$

If we substitute (2.2) at $z=0$ in (2.4) and then (2.4) in (2.3), then $f(x>0)$ will be expressed in terms of T and u. The expression for $f(x<0)$ in terms of T and $\mathbf{u}$ is obtained directly from (2.2). To make the equations closed in terms of the variables T and $\mathbf{u}$, we multiply $f(x>0)$ and $f(x<0)$ by qx^l $(l=0, 1)$, integrate with respect to q and with respect to x, and add the results, using Eq. (1.4) for the left-hand sides. The equation with $l=0$ corresponds then to conservation of the energy ε, while $l=1$ corresponds to conservation of the normal component of the momentum p.

The integration of the second term of (2.3) gives rise to the integral

$$\frac{1}{(2\pi)^3}\int_0^\infty dq\,q^2\int_{+} do\,qx^{l-1}e^{-z/x}\left\{\exp\left[\frac{q}{T}(1-ux)\right]-1\right\}^{-1}\equiv\frac{\pi^2}{60}T^4\Phi_l(u|z), \tag{2.7}$$

$$\Phi_l(u|z)=\int_0^1 dx\,x^{l-1}e^{-z/x}(1-ux)^{-4}. \tag{2.8}$$

The result of the integration of the right-hand side of (2.2) after replacing x by $-x$ can be reduced to the integrals $\Phi_l(-u|z)$. It is easy to verify that the left-hand sides of the equations, i.e., $\varepsilon(T,u)$ and $p(T,u)$, can also be expressed in terms of $\Phi_l(u|0)$.

Upon integration of the first term in (23), we obtain in lieu of Φ_l more complicated integrals of the type

$$\Psi_{ll'}(u|z,z')=\int_0^1 dx\int_{-1}^0 dx'\, r(x',x)x^{l-1}(-x')^{l'-1}e^{-z/x}e^{z'/x'}(1-ux')^{-4}. \quad (2.9)$$

The two terms in (2.4) are expressed in terms of $\Psi_{ll'}$ with $l=0$ and 1, $l'=1$, $u=0$ and with $l=0$ and 1 and $l'=0$.

Using the functions Φ and Ψ we can write down a system of equations for the quasihydrodynamic variables in the form

$$\int_0^z dz'\,T'^4\Phi_l(u'|z-z')+(-1)^l\int_z^\infty dz'\,T'^4\Phi_l(-u'|z'-z)$$

$$+\int_0^\infty dz'T'^4\Psi_{l0}(u'|z,z')+\bar{T}^4[\Phi_{l+1}(0|z)-\Psi_{l1}(0|z,0)]$$

$$=T^4[\Phi_{l+1}(u|0)+(-1)^l\Phi_{l+1}(-u|0)]. \quad (2.10)$$

For the sake of brevity we have put here

$$u\equiv u(z),\quad u'\equiv u(z'),\quad T\equiv T(z),\quad T'\equiv T(z').$$

Expression (10) constitutes a system of two ($l=0,1$) nonlinear integral equations for the determination of the two functions $u(z)$ and $T(z)$. Once these functions are obtained, we can use (2.2) and (2.3) to reconstruct the distribution function $f(z;\, q, x)$. As already mentioned, one should expect the functions $T(z)$ and $u(z)$ to vary slowly as $z\to\infty$. Then we can take $\hat{P}f$ outside the integral sign in (2.2) and (2.3) as $z\to\infty$. Integrating with respect to z', we verify that $f(z=\infty)=\hat{P}f(z=\infty)$. This means that far from the wall the distribution is of the Planck type and that the quasihydrodynamic variables $u(z)$ and $T(z)$ go over as $z\to\infty$ into real hydrodynamic variables.

The functions (2.8) have the following obvious properties:

$$\int_z^\infty dz'\,\Phi_l(u|z')=\Phi_{l+1}(u|z). \quad (2.11)$$

$$\frac{d}{dz}\Phi_l(u|z)=-\Phi_{l-1}(u|z). \quad (2.12)$$

The functions (2.9) have analogous properties. The value of the index l changes if Ψ is integrated or differentiated with respect to z, and the value of l' changes if the integration or differentiation is with respect to z'.

Using (2.11) and analogous relations for Ψ, we can easily verify that the system (2.10) (at $l=0,1$) is satisfied by the functions

$$T(z)=\bar{T},\quad u(z)=0. \quad (2.13)$$

a fact that corresponds to an equilibrium distribution with the same temperature as the metal.

Differentiating (2.10) at $l=1$ with the aid of (2.12) and analogous relations for Ψ, and then using (2.10) with $l=0$, we readily verify the existence of the following integral:

$$\frac{d}{dz}\{T^4[\Phi_2(u|0)-\Phi_2(-u|0)]\}=0. \quad (2.14)$$

3. SMALL DEVIATIONS FROM THERMODYNAMIC EQUILIBRIUM

We consider the case when the deviations of the phonon gas from equilibrium with the helium bath are small. This means that $u(z)\ll 1$ and

$$T(z)=T_0(1+\sigma(z)),\quad \bar{T}=T_0(1+\bar{\sigma}), \quad (3.1)$$

where $\sigma(z)\ll 1$ and $\bar{\sigma}\ll 1$.

Expanding (2.8) and (2.9) in powers of u, we get

$$\Phi_l(u|z)=E_{l+1}(z)+4uE_{l+2}(z), \quad (3.2)$$

$$\Psi_{ll'}(u|z,z')=K_{ll'}(z,z')-4uK_{ll'+1}(z,z'); \quad (3.3)$$

$$E_l(z)=\Phi_{l-1}(0|z),\quad K_{ll'}(z,z')=\Psi_{ll'}(0|z,z'), \quad (3.4)$$

E_l are integral-exponential functions.[5] It is easily seen that the functions E and K have the same properties with respect to integration and differentiation as the functions Φ and Ψ.

Retaining the terms of lowest order in (2.14), we have

$$u(z)=\text{const}\equiv u. \quad (3.5)$$

We now linearize (2.10) and use (3.5). At $l=0$ we obtain the following integral equation:

$$\sigma(z)=\frac{1}{2}\int_0^\infty dz'\,\sigma(z')[E_1(|z-z'|)+K_{10}(z,z')]$$

$$-\frac{1}{2}u[E_3(z)+K_{12}(z,0)]+\frac{1}{2}\bar{\sigma}[E_2(z)-K_{11}(z,0)]. \quad (3.6)$$

At $l=1$ we obtain one more equation:

$$\int_0^\infty dz'\,\sigma(z')[E_2(|z-z'|)\,\text{sign}(z-z')+K_{20}(z,z')]$$

$$=u[E_4(z)+K_{22}(z,0)]-\bar{\sigma}[E_3(z)-K_{21}(z,0)]. \quad (3.7)$$

We seek for $\sigma(z)$ a solution satisfying (3.6) and (3.7) and having the form

$$\sigma(z)=\bar{\sigma}+\Delta\sigma(z). \quad (3.8)$$

Using the rules for integration of E and K we readily see that for $\Delta\sigma(z)$ we obtain the same equations as for $\sigma(z)$, but with $\bar{\sigma}=0$.

We consider first two cases, when the system (3.6) and (3.7) admits of an analytic solution. From a consideration of these cases it becomes clear that the solution of the equations for $\Delta\sigma(z)$ has as $z\to\infty$ a finite limit $\Delta\sigma(\infty)$, which is obviously proportional to u, so that the following relation holds

$$\sigma(\infty)=\bar{\sigma}-\alpha u(\infty), \quad (3.9)$$

where α is a number determined when the equations are solved. Relation (3.9) is the sought boundary condition.

Diffuse scattering

In the case of diffuse scattering we have

$$r(x', x)=2r|x'| \tag{3.10}$$

and the non-difference part of the kernel in (3.6) and (3.7) can be factored

$$K_{ll'}(z, z')=2rE_{l+1}(z)E_{l'+2}(z'). \tag{3.11}$$

Equation (3.6) takes the following form:

$$\Delta\sigma(z)=\frac{1}{2}\int_0^\infty dz'\,\Delta\sigma(z')E_1(|z-z'|)-\frac{1}{2}uE_3(z)+r\left[(\Delta\sigma)_2-\frac{1}{3}u\right]E_2(z), \tag{3.12}$$

where

$$(\Delta\sigma)_l=\int_0^\infty dz\,\Delta\sigma(z)E_l(z). \tag{3.13}$$

This is the Milne inhomogeneous equation. Its solution can be expressed in terms of the Hopf function $q(z)$[6]:

$$\Delta\sigma(z)=A[z+q(z)]-uq(z)+2r[(\Delta\sigma)_2-{}^1/_3u]. \tag{3.14}$$

The first term with the arbitrary constant A is in fact the solution of the homogeneous Milne equation. The function $q(z)$ increases monotonically from $q(0)=1/\sqrt{3}=0.5773\ldots$ to $q(\infty)=0.7104\ldots$ Substituting (3.14) in (3.13) at $l=1$ and using the equality[6]:

$$\int_0^\infty dz\,q(z)E_2(z)=\frac{1}{3}, \tag{3.15}$$

we get

$$(\Delta\sigma)_2=\frac{1}{1-r}\left[\frac{2}{3}A-\frac{1}{3}u(1+r)\right]. \tag{3.16}$$

We determine A by using (3.7). Putting $z=0$ in this equation and taking (3.11) into account, we get

$$(\Delta\sigma)_2(1-r)=-{}^1/_3u(1+r). \tag{3.17}$$

Comparing (3.17) and (3.16) we have

$$A=0. \tag{3.18}$$

We can now obtain from (3.14) the value of $\Delta\sigma(\infty)$ and the coefficient α in (3.9):

$$\alpha=q(\infty)+{}^4/_3r(1-r)^{-1}. \tag{3.19}$$

Almost total specular reflection

For this reflection we have

$$r(x', x)=r(x)\delta(x+x'), \tag{3.20}$$

where $r(x)$ is close to unity. Then

$$K_{ll'}(z, z')=E_{l+l'}(z+z')-S_{l+l'}(z+z'), \tag{3.21}$$

where S_l is a small quantity:

$$S_l(z)=\int_0^1 dx[1-r(x)]x^{l-2}e^{-z/x}. \tag{3.22}$$

In the zeroth order, putting $S_l=0$, we get from (3.6) and (3.7)

$$\Delta\sigma(z)=\frac{1}{2}\int_0^\infty dz'\,\Delta\sigma(z')[E_1(|z-z'|)+E_1(z+z')]-uE_3(z), \tag{3.23}$$

$$\int_0^\infty dz'\,\Delta\sigma(z')[E_2(|z-z'|)\operatorname{sign}(z-z')+E_2(z+z')]=2uE_4(z). \tag{3.24}$$

Putting $z=0$ in (3.24), we get $u=0$, which is an obvious result for total reflection. To solve now (3.23), we continue in even fashion the function $\Delta\sigma(z)$ into the region $z<0$. Now (3.23) with $u=0$ takes the form

$$\Delta\sigma(z)=\frac{1}{2}\int_{-\infty}^{+\infty} dz'\,\Delta\sigma(z')E_1(|z-z'|),\quad -\infty<z<+\infty. \tag{3.25}$$

This equation is solved by a Fourier transformation. The Fourier transform of the function E_1 is of the form $(2/k)\tan^{-1}k$, so that the solutions of the homogeneous equation (3.25) are determined by the zeroes of the quantity $k-k^{-1}\tan^{-1}k$. This quantity has only one double root $k=0$. Therefore the general solution of the homogeneous equation (3.25) is

$$\Delta\sigma(z)=Az+B,\quad z>0. \tag{3.26}$$

Substituting this solution in (3.24) with $u=0$ we find that $A=0$, and B remains indeterminate.

Proceeding to the next order, we represent

$$\Delta\sigma(z)=\Delta\sigma(\infty)+\varphi(z),\quad \Delta\sigma(\infty)\equiv B, \tag{3.27}$$

where $\varphi(z)$ is small and tends to zero as $z\to\infty$. We then have for $\varphi(z)$ the equation

$$\varphi(z)=\frac{1}{2}\int_0^\infty dz'\,\varphi(z')[E_1(|z-z'|)+E_1(z+z')]-f(z), \tag{3.28}$$

where

$$f(z)={}^1/_2\Delta\sigma(\infty)S_2(z)+uE_3(z). \tag{3.29}$$

This equation is also solved via Fourier transformation; it suffices to continue in even fashion the unknown function $\varphi(z)$ and the inhomogeneity $f(z)$ into the region $z<0$. For a solution of the inhomogeneous equation to exist, it is necessary that the Fourier transform of $f(z)$ have a zero at $k=0$ (this zero is automatically of order not lower than the second, since $f(z)$ is even). In other words, we must have

$$\int_0^\infty dz\,f(z)=0. \tag{3.30}$$

This yields

$$\alpha={}^2/_3 S_3(0)^{-1}. \tag{3.31}$$

From an examination of the two particular cases—diffuse reflection and almost total specular reflection—we see that the role of Eq. (3.7) reduces to a separation of that solution of Eq. (6), which is bounded at infinity, i.e., to a determination of the constant A in the form $A=0$. This is not surprising since, in using the integral (2.14), we have actually used one of the two equations of (2.10). Being interested only in the solutions $\Delta\sigma(z)$ with finite limit $\Delta\sigma(\infty)$, we can consider only one equation (3.6) and obtain from it a stationary expression for $\Delta\sigma(\infty)$ (see the Appendix). This stationary expression (A.17) with the simplest trial function leads to the following value of the coefficient of interest to us:

$$\alpha={}^3/_2\{({}^1/_4+K_{32})+({}^1/_3+K_{22})\,({}^1/_3+K_{31})\,({}^1/_2-K_{21})^{-1}\}, \tag{3.32}$$

where

$$K_{ll'}\equiv K_{ll'}(0,0)=\int_0^1 dx\int_{-1}^0 dx'\,r(x',x)\,x^{l-1}(-x')^{l'-1}. \tag{3.33}$$

We compare now the ensuing results with the cases when the solution is obtained directly.

For diffuse scattering we obtain from (3.32)

$$\alpha={}^{17}/_{24}+{}^4/_3 r(1-r)^{-1}. \tag{3.34}$$

For a comparison with the exact result (3.19) we note that:

$${}^{17}/_{24}=0.7083\ldots,\quad q(\infty)=0.7104\ldots$$

i.e., the error of the stationary expression is of the order of one per cent.

In the case of almost total specular reflection, we can neglect the first terms in the curly brackets of (3.32), and assume $r(x)=1$ in the calculation of $K_{22}=K_{31}$. We then obtain

$$\alpha={}^2/_3({}^1/_2-K_{21})^{-1}. \tag{3.35}$$

Noting that

$$\frac{1}{2}-K_{21}(0,0)=\frac{1}{2}-\int_0^1 dx\,r(x)x=S_3(0), \tag{3.36}$$

we see that the stationary α coincides with the "exact" one. These two examples give all grounds for hoping that the stationary expression (3.32) is accurate enough also in other cases.

4. REFLECTION AND EXCITATION OF SECOND SOUND

Returning to dimensional quantities, we write down the boundary condition on the interface between the dielectric and metal in the form

$$(\delta T-\delta\bar{T})/T_0=-\alpha(u/s). \tag{4.1}$$

Here T_0 is the temperature of the helium bath, $\delta\bar{T}$ is the excess of the metal temperature over the helium bath, δT is the excess of the hydrodynamic temperature of the phonons over the temperature of the bath, u is the projection of the hydrodynamic velocity of the phonon on the normal to the interface (directed into the interior of the dielectric), and s is the speed of sound. The coefficient $\alpha>0$ can be calculated from formula (3.32) if the probability of the phonon reflection from the interface is known. In the acoustic matching model[7] we can obtain for the integrals $K_{ll'}$ explicit expressions in terms of the acoustic characteristics of the metal and the dielectric; they are very cumbersome, however, and will not be written out here.

A boundary condition in the form (4.1) at $\delta\bar{T}=0$ was obtained from semi-quantitative considerations by Sussmann and Thellung.[8] It is impossible, however, to obtain a value of α with any degree of confidence with the aid of their arguments.

It is seen from (3.19) and (3.31) that in the case of almost total reflection of the phonons, whether it be specular or diffuse, we have $\alpha\sim(1-r)^{-1}\gg 1$. As $r\to 1$, the boundary condition (4.1) is transformed into $u=0$, as it should. At moderate and weak reflection we have $\alpha\sim 1$. It must be emphasized that there are now phonon-reflection conditions such that $\alpha\ll 1$. This means that in the second-sound regime there is no situation in which the boundary condition $\delta T=\delta\bar{T}$ is valid. In other words, in a Knudsen layer of thickness l_N there is always concentrated a temperature jump equal to $T_0\alpha(u/s)$. This jump is larger the stronger the phonon reflection.

With the aid of (4.1), at $\delta\bar{T}=0$, it is possible to find the reflection coefficient of second sound for normal incidence. It turns out to be

$$R=(\alpha-1/\sqrt{3})\,(\alpha+1/\sqrt{3})^{-1}. \tag{4.2}$$

If the first sound is not reflected from the interface, then we obtain from (3.19) $\alpha=q(\infty)$ at $r=0$. According to (4.2), the second sound is reflected none the less, with $R\approx(0.1)$ (in terms of energy).

We consider now the excitation of a thermal pulse. It is usually assumed[9,10] that the energy flux density from the metal into the dielectric is

$$Q=Q(\bar{T})-Q(T_0), \tag{4.3}$$

where (in our notation, in dimensional units)

$$Q(T)=\frac{\pi^2k^4}{60\hbar^3}\frac{T^4}{s^2}({}^1/_2-K_{21}). \tag{4.4}$$

Let us derive (4.3), to understand the conditions under which this relation is valid. Using (1.9) and (1.13) we can easily verify that the flux carried by the phonons from the metal into dielectric is

$$\int_+\frac{d\mathbf{q}}{(2\pi)^3}(sq)(s\varkappa)f_0(q)=Q(\bar{T}). \tag{4.5}$$

The substracted flux $Q(T_0)$ can be understood in two ways: either as the flux from the dielectric into the metal, or as the flux in the dielectric, incident on the interface, minus the flux reflected from the interface. In either case the subtracted flux is produced by phonons incident on the interface from the dielectric. The assumption that this flux is determined by formula (4.4) with $T=T_0$ is equivalent to the assumption that the phonons incident on the interface from the dielectric are in equilibrium. But in the hydrodynamic regime the injected phonons perturb the distributions of the phonons that move towards the interface, so that the subtracted flux should be of a different form.

With the aid of (4.1) it is easy to find the energy flux injected into the dielectric from the heated film, in the hydrodynamic regime:

$$Q_{\text{hydr}}=\frac{\pi^2 k^4}{90\hbar^3}\frac{T_0^3}{s^2}4\delta\bar{T}\left(\alpha+\frac{1}{\sqrt{3}}\right)^{-1}. \tag{4.6}$$

Assuming the difference between $\bar{T}$ and T_0 in (4.3) to be small, we obtain the flux in the ballistic regime:

$$Q_{\text{ball}}=\frac{\pi^2 k^4}{60\hbar^3}\frac{T_0^3}{s^2}4\delta\bar{T}\left(\frac{1}{2}-K_{21}\right). \tag{4.7}$$

In diffuse reflection, when there is an exact solution in the hydrodynamic regime, we get

$$Q_{\text{ball}}/Q_{\text{hydr}}=r+{}^3/_4(1-r)\left[q(0)+q(\infty)\right]\leqslant 1. \tag{4.8}$$

In the hydrodynamic regime, the injected phonons drag with them the equilibrium phonons and by the same token decrease the reverse flux from the dielectric into the metal. Relation (4.8) differs little from unity: it ranges from 0.9658.... at $r=0$ at $r=0$ to 1 at $r=1$. This circumstance, however, may be caused by the chosen approximation of the collision term. At almost total specular reflection, as seen from (3.36), we have $Q_{\text{ball}}/Q_{\text{hydr}}=1$. Thus, this ratio depends on the details of the reflection, although perhaps not very strongly.

In conclusion, we discuss two serious model-related assumptions made in this paper, viz., isotropy of the phonon spectrum and the τ-approximation for the N-processes. The last approximation is the most serious and may cast doubts on the quantitative results obtained for the coefficient α in the boundary condition. It must be emphasized in this connection that in the τ-approximation α does not depend on τ, but is determined only by the coefficient of reflection of the phonon from the interface. This means that if we forgo the τ-approximation, then α will not depend on the total rate of the N-processes, but will depend only on the relative rates of the N-processes for phonons with different momenta. On the other hand, this circumstance gives grounds for hoping that the behavior observed by us of α as a function of the reflection coefficient has not only a qualitative but also a semiquantitative meaning. A comparison of the results of the exact solution with the variational estimates must not be taken, of course, as an estimate of the accuracy of the results for α, but only as an estimate of the accuracy of the variational method within the framework of the assumed model.

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APPENDIX

Assume the following equation to be given:

$$\sigma(z)=\frac{1}{2}\int_0^\infty dz'\,\sigma(z')\left[G(|z-z'|)+R(z,z')\right]+f(z). \tag{A.1}$$

The kernel G describes the motion in unbounded space, and the kernel R describes the reflection. As $z\to\infty$, all the functions decrease rapidly enough; the difference between the kernels G and R is that R decreases even in the case when z and z' increase at a constant difference $|z-z'|$. It is assumed that the following condition is satisfied

$$\int_0^\infty dz\,G(z)=1. \tag{A.2}$$

This condition is necessary for the existence of a solution for which a limit $\sigma(\infty)$ exists. If we are interested only in the limiting value $\sigma(\infty)$, then it can be calculated approximately with the aid of the stationary expression. To find the stationary expression it is necessary to represent $\sigma(\infty)$ in the form of a linear functional of the solution; we can then use the Schwinger variational principle.[11,12]

We introduce

$$G_n(z)=\int_z^\infty dz'\,G_{n-1}(z'),\quad G_0(z)\equiv G(z), \tag{A.3}$$

and analogously $f_n(z)$. The kernel R will define functions $R_{nn'}(z,z')$ in which the indices n and n' indicate the number of integrations with respect to z and z', respectively. We consider now the function

$$F(z)=\frac{1}{2}\int_0^\infty dz'\,\sigma(z')\left[G_2(|z-z'|)+R_{20}(z,z')\right]. \tag{A.4}$$

Differentiating, we get

$$F'(z)=-\frac{1}{2}\int_0^\infty dz'\,\sigma(z')\left[G_1(|z-z'|)\operatorname{sign}(z-z')+R_{10}(z,z')\right]. \tag{A.5}$$

From (A.5) it is seen that

$$F'(\infty)=0. \tag{A.6}$$

Differentiating (A.5) once more and using (A.1), we have

$$F''(z)=-f(z). \tag{A.7}$$

Integrating (A.7) twice with allowance for (A.6), we obtain

$$F(\infty)-F(0)=f_2(0),$$

or, substituting $z=0$ and $z=\infty$ in (A.4),

$$\sigma(\infty) G_3(0) = f_2(0) + J,$$

$$J = \frac{1}{2}\int_0^\infty dz'\, \sigma(z') [G_2(z') + R_{20}(0, z')].$$

The obtained relations yield the sought expression for $\sigma(\infty)$ in the form of an integral of the solution. Following the known rules, [11,12] we construct a stationary expression for the integral U:

$$[J] = A_1 A_2 / B,$$

where

$$A_1 = \frac{1}{2}\int_0^\infty dz'\, \sigma(z') [G_2(z') + R_{20}(0. z')],$$

$$A_2 = \int_0^\infty dz\, \tilde{\sigma}(z) f(z),$$

$$B = \int_0^\infty dz\, \tilde{\sigma}(z) \left\{\sigma(z) - \frac{1}{2}\int_0^\infty dz'\, \sigma(z') [G(|z-z'|) + R(z. z')]\right\}.$$

We use the simplest approximation

$$\sigma(z) = 1, \quad \tilde{\sigma}(z) = 1.$$

Then

$$A_1 = {}^1\!/_2 [G_3(0) + R_{21}(0, 0)],$$

$$A_2 = f_1(0), \quad B = {}^1\!/_2 [G_2(0) - R_{11}(0, 0)]$$

and

$$\sigma(\infty) = \frac{f_2(0)}{G_3(0)} + \frac{f_1(0)}{G_3(0)} \frac{G_3(0) + R_{21}(0, 0)}{G_2(0) - R_{11}(0, 0)}.$$

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