Quantum effects and the Friedmann model

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Quantum effects near a singularity lead to multiparticle production and vacuum polarization. The vacuum polarization gives rise to various kinds of nonlinear additions in the four-curvature in the Lagrangian of the gravitational field. In the case of an isotropic cosmological model, their local part can be written in the form $\Delta L_g = A + BR^2 + CR^2 \ln|R|$, $(R = g_{ik}R^{ik})$. It is shown that allowance for ΔL_g in the Friedmann model gives a regular minimum of the scale factor b_0; out of this minimum, there emanate two solution branches with decreasing and increasing value of $|R|$. Because of the change in the sign of the logarithm, the first branch goes over into an oscillatory regime; the second gives a divergence of R as $b \to \infty$. In this case, the solution $b(t)$, which has a regular minimum, approaches the Friedmann behavior in the asymptotic limit $t \to -\infty$, and the singularity is displaced to the opposite end of the time axis. Allowance for particle production or inclusion of viscosity near b_0 may restrict the growth of the curvature in the second solution branch, and the universe may then approach the Friedmann solution in the limit $t \to \infty$ as well.

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1. INTRODUCTION

The inevitability of a singularity in the general cosmological solution of Einstein's equations[1] has stimulated recently investigations of the modifications to general relativity due to quantum effects. The most important of these are multiparticle production and vacuum polarization in the strong variable gravitational field near the singularity (see[2] and the literature quoted there).

Particle production in a field with given metric can be taken into account fairly rigorously.[3] Difficulties arise when one considers the self-consistent problem with the back reaction of the created particles on the metric. As was noted in[4,5], the back reaction can be taken into account approximately by the introduction of an effective viscosity. Vacuum polarization leads to various additions in the four-curvature in the Lagrangian density L_g of the gravitational field.[3,6,7] If only their local part is taken into account, then in the case of a homogeneous and isotropic cosmological model these additions can be represented in the form

$$\Delta L_g = A + BR^2 + CR^2 \ln|R|, \quad (1)$$

where $R = g_{ik}R^{ik}$, and A, B, C are constants. Although this approach has a number of limitations, which were noted in[8,9], it enables one to investigate the possibility of eliminating the singularity in the framework of classical gravitation.

It should be noted that allowance for only second viscosity of the matter in the framework of general relativity for an isotropic model enables one to eliminate the singularity in a certain sense.[10,5] The corresponding solution as $t \to -\infty$ is described by the de Sitter metric with zero energy density ε, and as $t \to \infty$ it tends to the Friedmann solution with all matter generated by dissipation during the expansion process. Independently of the physical justification of this model, it too is incapable of giving a continuous transition from contraction to expansion when $\varepsilon > 0$.

The addition to (1) corresponding to the second term (the first leads to the cosmological λ term) gives a regular minimum of the scalar factor b_0, as was first noted in[11]. But the sign of B which ensures this behavior of the solution leads to a divergence of R with increasing b. Asymptotic approach to the Friedmann solution occurs only when R^2 is added with the opposite sign. The results of [11] made it necessary to consider additions of more general form; this was done in[12,13], in which the possibility was demonstrated of constructing regular cosmological models with a value of R^2 which is regular for all t and which tend to the Friedmann solution as $t \to \infty$. Unlike the additions of type (1), no justification was given for the form of the new additions.

New possibilites are opened up if the effects of polarization and particle production are taken into account simultaneously. The equations of general relativity modified by means of (1) give two solution branches emanating from the point b_0, one with a decreasing and the other with an increasing value of $|R|$.[12] (Symmetric branches lead to divergence of the curvature as $t \to \pm\infty$.) This last applies equally to (1) and to a quadratic addition. In both cases, with increasing b, the solution branches with increasing curvature lead to divergence of the curvature. As is shown below, there is an essential difference for the branch with decreasing $|R|$. At a definite value of R_*, the addition (1) changes sign because of the presence of the logarithm, and this ensures an asymptotic approach to a Friedmann universe. This possibility was noted in [9], in which however the branch with increasing value of $|R|$ was investigated. Thus, allowance for polarization effects gives a regular minimum of the scale factor $b(t)$ and Friedmann asymptotic behavior as $t \to +\infty$ or $-\infty$, the singularity $(R^2 \to \infty)$ being displaced in this case to the opposite end of the time axis.

These properties of the modified gravitational equa-

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tions are important for the following reason. During the stage of contraction that begins with the Friedmann asymptotic behavior as $t\to-\infty$, creation of gravitons[4] must commense in the neighborhood of b_0. Their back reaction on the metric may halt the growth of $|R|$ in the second solution branch, which could lead to an approach to a Friedmann universe in the limit $t\to+\infty$ as well. Rigorous investigation of this possibility requires solution of the self-consistent problem mentioned above. Allowance for matter production by the introduction of an effective viscosity, as in the conclusion to this paper, only demonstrates the possibility that the growth of the curvature is restricted and the second solution branch goes over into an oscillatory regime.

2. BASIC EQUATIONS AND BOUNDARY CONDITIONS

We represent the addition (1) in the form

$$\Delta L_g=f(\rho)=(\rho^2+\lambda)\ln[(\rho^2+\lambda)/\rho_c^2]. \tag{2}$$

Here, using the characteristic lengths l, we have introduced the dimensionless four-curvature

$$\rho_{ik}=l^2R_{ik},\quad \rho=l^2R, \tag{3}$$

and the constants λ and ρ_c^2 are related by the inequality

$$\rho_c^2\gg\lambda. \tag{4}$$

For $\rho^2\gg\lambda$, the expression (2) is equivalent to the second and third terms in (1). For $\lambda\gg\rho^2$, it is nearly equivalent to the first term.

We restrict ourselves to the model with flat comoving space:

$$dS^2=d\tau^2-b^2(\tau)(dx^2+dy^2+dz^2), \tag{5}$$

where $\tau=ct$. The gravitational equations corresponding to $L_g=\rho+f(\rho)$ are obtained in [12]. For investigation it is convenient to take from this system of equations the one with T_0^0, and replace the others by the equations $T_{i;k}^k=0$, which follow from this system. The order of the chosen equation can be reduced by introducing the new variable

$$y=l^2\dot{b}^2b^2\quad(\dot{b}=db/d\tau), \tag{6}$$

by means of which we obtain for the scalar curvature

$$\rho=-3y'/b^3\quad(y'=dy/db). \tag{7}$$

Then the basic equation takes the form[12]

$$y+\left\{\frac{\partial f}{\partial\rho}\left(y-\frac{by'}{2}\right)-\frac{b^4}{6}f+yb\frac{d}{db}\left(\frac{\partial f}{\partial\rho}\right)\right\}=E, \tag{8}$$

where

$$E=\varkappa l^2b^4\varepsilon/3. \tag{9}$$

Here, $\varkappa$ is Einstein's constant and ε is the energy density. In the case of the hot model, $\varepsilon=3p$, and (9) is a constant. After substituting (2) into (8) and making some simplifications on the basis of the condition (4) (see the conclusion), we obtain

$$y''y\varphi=\frac{(y-E)b^2}{6}+\frac{(y')^2}{4}(\varphi-1)+\frac{2yy'}{b}(\varphi+1),$$
$$\varphi=\ln[(\rho^2+\lambda)/\rho_*^2], \tag{10}$$

where $\rho_*=\rho_c e^{3/2}$.

In accordance with what we said in the Introduction, we shall seek a solution $b(\tau)$ that has a regular minimum at $\tau=0$, i.e.

$$b(\tau)=b_0+\ddot{b}_0\tau^2/2+\dddot{b}_0\tau^3/6+\ldots \tag{11}$$

To the single-valued dependence $y(\tau)$ there corresponds the two-valued dependence

$$y[b(\tau)]=\begin{cases}y_+(b), & \tau\geqslant0,\\ y_-(b), & \tau<0.\end{cases} \tag{12}$$

Hence, in accordance with (11), we have the boundary conditions

$$\begin{gathered}y_+(b_0)=y_-(b_0)=0,\\ y_+'(b_0)=y_-'(b_0)=2(lb_0)^2\ddot{b}_0,\\ \lim_{b\to b_0}(y_+^{1/2}y_+'')=-\lim_{b\to b_0}(y_-^{1/2}y_-'')=2(lb_0)^2\dddot{b}_0.\end{gathered} \tag{13}$$

3. QUALITATIVE INVESTIGATION OF THE SOLUTION

We consider the behavior of the solution (10) near the point b_0 of the regular minimum. In accordance with (13),

$$y'\approx a+dx^{1/2}+O(x^{3/2}),\qquad x=b-b_0\ll b_0. \tag{14}$$

At the same time, it follows from (10) that

$$a=y_0'=\pm\left[\frac{2}{3}\frac{E}{(\varphi_0-1)}\right]^{1/2}b_0, \tag{15}$$

where the subscript 0 indicates the values of the quantities at b_0. It can be seen from (15) that the solution can intersect the axis $y=0$ only for $\varphi_0\geqslant1$, and that the upper sign of the root corresponds to a minimum of b. For known first derivative, we have

$$d=\pm\left[\frac{2a(b_0^2+6a)}{\varphi_0}+\frac{18a^2}{b_0}\right]. \tag{16}$$

It follows from this that two solution branches emanate from b_0 on $y=0$ with equal first derivatives. One, with increasing derivative, leads to divergence of ρ as $b\to\infty$.

The second solution (lower sign in (16) leads to a decrease in $|\rho|$ and a change in the sign in Eq. (10) at the point $b=b_*$. The behavior of the solution in the neighborhood of b_* is qualitatively the same as in the neighborhood of b_0 because the factors multiplying the highest derivative vanish in both cases. Hence, setting

$$y\approx y_*+a_*x+d_*x^2+O(x^3),\quad x=b-b_*\ll b_*, \tag{17}$$

we obtain from (10)

$$a_*=\frac{4y_*}{b_*}\pm\left[\frac{16y_*^2}{b_*^2}+\frac{2b_*^2(y_*-E)}{3}\right]^{1/2},$$
$$d_*=\frac{3a_*}{b_*}\pm\left[\frac{9a_*^2}{b_*^2}+A_*\right]^{1/2}, \tag{18}$$
$$A_*=a_*^2\left(\frac{b_*^2}{12y_*}+\frac{a_*}{2y_*b_*}-\frac{14}{b_*^2}\right)+\frac{b_*a_*}{y_*}(y_*-E).$$

It follows from (18) that at the point $y=y_*$, $b=b_*$ there are two solutions which have the same derivative but second derivatives that increase and decrease with increasing x. The second of the solutions corresponds to analytic continuation of the solution that leaves b_0 with decreasing value of $|\rho|$.

As will be shown below, the curvature ρ after passage through the point $b=b_*\gg 1$ continues to decrease rapidly, so that one can set

$$\varphi=-|\ln(\lambda/\rho_*^2)|, \tag{19}$$

where $|\varphi|\gg 1$ by virtue of the condition (4). After the introduction of $E=1$ and the variables

$$\xi=b^2/|24\varphi|^{1/2},\quad y=1+\psi \tag{20}$$

Eq. (10) then takes the form

$$(1+\psi)\frac{d^2\psi}{d\xi^2}=\frac{1}{4}\left(\frac{d\psi}{d\xi}\right)^2-\psi+\frac{(1+\psi)}{2\xi}\frac{d\psi}{d\xi}. \tag{21}$$

We first set $\psi\sim\delta\ll 1/\xi$ (this last is achieved by the inequalities y_*', $(1-y_*)\ll 1/\xi_*$); then (21) after linearization yields

$$\frac{d^2\psi}{d\xi^2}-\frac{1}{2\xi}\frac{d\psi}{d\xi}+\psi=0, \tag{22}$$

whose solution has the form

$$\psi=\xi^{3/4}[\delta_1 J_{3/4}(\xi)+\delta_2 N_{3/4}(\xi)]. \tag{23}$$

Here, δ_1 and δ_2 are constants of integration and $J_{3/4}(\xi)$ and $N_{3/4}(\xi)$ are Bessel and Neumann functions. For $\xi\gg 1$,

$$\psi\sim\xi^{1/4}\left[\delta_1\cos\left(\xi-\frac{5\pi}{8}\right)+\delta_2\sin\left(\xi-\frac{5\pi}{8}\right)\right].$$

In accordance with (23), the solution $y(b)$ oscillates around the Friedmann solution $y=1$ as the frequency $\omega(b)$ increases (see Fig. 1). The amplitude of the oscillations increases as $\sim b^{1/2}$. We have $y'\sim b^{3/2}$, which in accordance with (7) gives $\rho^2\sim b^{-3}$. This last justifies the assumption (19). With increasing amplitude, the nonlinear terms omitted in (22) become important. It is however easy to see that the growth of the amplitude cannot be unbounded. Because (19) holds, $\varphi<0$, which in accordance with (15) prevents the solution intersecting the $y=0$ axis. The form of the nonlinear oscillations with constant amplitude follows from (21) after neglect for $\xi\gg 1$ of the last term:

$$y=1+\psi\approx[1+a\sin(\xi-\xi_1)]^2,\quad |a|<1. \tag{24}$$

The law of increase of $\omega(b)$ remains the same as in the linear oscillations, and $\rho\sim b^{-2}$. In accordance with (6), the corresponding value of $b(\tau)$ for $\tau/l\gg 1$ has the form

$$b(\tau)\approx\left(\frac{2\tau}{l}\right)^{1/2}\left\{1-\frac{al(6\varphi)^{1/2}}{2\tau}\cos[(\tau-\tau_1)/l(6\varphi)^{1/2}]\right\}, \tag{25}$$

where the first term describes the evolution of the Friedman model.

An oscillatory solution also holds for $\lambda=0$. At the point $y'=0$, only y''' has a discontinuity, which is allowed since (10) is an equation of second order.

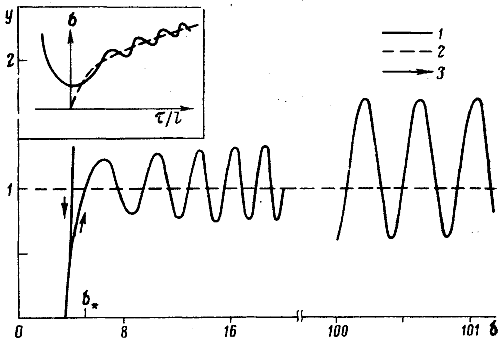

FIG. 1. Solution of Eq. (10) with boundary conditions specified at b_*. 1) Unboundedly increasing branch ($|\rho|\to\infty$ as $b\to\infty$) and oscillating branch; 2) Friedmann solution; 3) direction of time evolution (above, the dependence $b(t)$ corresponding to 1).

4. EXAMPLE OF NUMERICAL CALCULATION

Numerical solution of (10) requires a preliminary investigation of the neighborhoods of the points b_0 and b_*, since the proximity of the second solution leads to instability of the method of finite differences with specification of boundary conditions at one point. If one gets away from these points by means of the above formulas, the Runge–Kutta method can be used to obtain all solution branches. In Fig. 1, we give the results of calculation for $\lambda=1\cdot 10^{-3}$. The boundary conditions were chosen at the point $b_*=5$, $y_*'=0.4$. For $b>b_*$, the calculation proceeded along the direction of evolution of the model; for $b<b_*$, in the opposite direction. After approach to the point $y=0$, $b=b_0$ in accordance with Eqs. (14)–(16) the values of y_0' and y_0'' were determined for the second solution branch with $y(b\to\infty)\to\infty$. After passage through the point b_* an oscillatory regime commences with the above law of increase of the frequency

$\omega(b)$ and slow increase of the amplitude a, which tends to saturation at $b \sim 100$.

Note that for $E=\text{const}$ the direction of evolution of the model shown in Fig. 1 can be reversed by virtue of the invariance of the equations under the substitution $t \to -t$.

5. CONCLUSION

Thus, allowance for the polarization correction (2) does not eliminate the singularity in the Friedmann model. Although $R^2 \to \infty$ as $t \to \infty$, at a finite t we already have $R^2 > l_p^{-4}$ (l_p is the Planck length), which indicates that we approach the essentially quantum region, in which the classical treatment of the metric is invalid.

Let us now consider approximately the influence of matter creation. Because of the conformal invariance of the corresponding wave equations for an isotropic universe, production of the majority of particles is forbidden.[2] An exception is gravitons, whose mutiple production is possible when $R^2 \neq 0$.[4] In the case of the hot model, this process becomes effective when the nonlinear addition (2) becomes effective,[14] which according to[3] makes it possible to formulate rigorously the problem of particle production from quantum fluctuations in an external variable classical field. Below, we shall, following[4], describe the corresponding increase in ε by introducing an effective viscosity with coefficient

$$\mu=\alpha(\rho)\varepsilon.$$

The equation for E follows from $T^k_{i;k}u^i=0$ with allowance for (6) and (9):

$$\frac{dE}{db}=\frac{\beta(\rho)E\sqrt{y}}{b^4},\quad \beta(\rho)=9\alpha(\rho)/l. \tag{26}$$

Note first of all that the system (10), (26) has the singular solution

$$E=kb^4,\quad y=mb^4,\quad \rho=-12m, \tag{27}$$

where the constants are related by

$$m=k+f(\rho)/6,\quad \beta\sqrt{m}=4.$$

The solution (27) describes a steady-state cosmological model in which the effects of viscous dissipation play the role of the hypothetical C field. The corresponding rate of matter production has the limiting value. A slower rate of increase of E leads to a divergence of ρ with increasing b; a greater rate, to decrease in the curvature $|\rho|$ and change of sign of the logarithm in (10). Let us now consider the cosmological model given in Fig. 1. Allowance for (26) during the stage of contraction that begins with the Friedmann asymptotic behavior does not change the essential behavior of the solution, leading only to an increase in the value of b_0. We therefore set

$$\beta(\rho)=\begin{cases}\text{const} & \rho\geqslant\rho_0,\\ 0 & \rho<\rho_0,\end{cases}$$

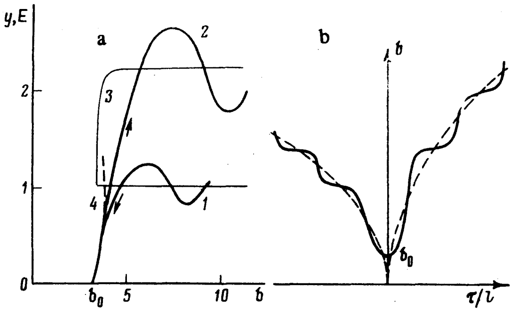

FIG. 2. a) Cosmological model with allowance for effects of polarization and matter creation: 1) $y(b)$ during the stage of collapse, 2) the expansion stage; 3) the behavior of $E(b)$; 4) unboundedly increasing solution with $E=1$. b) The dependence $b(t)$ corresponding to 1 and 2; the dashed curve shows the Friedmann solutions with $E=1$ and 2.2.

which gives a strong increase in E after the point b_0 has been passed. For the example considered in Sec. 4, the rate of increase of the energy was given in the form $dE/db \approx 12\exp[10(b_0-b)]$. An increase in E to ~ 2.2 in the interval $\Delta b \sim 0.2$ halts the growth of the curvature for the second solution branch (see Fig. 2) and carries it into an oscillatory regime similar to the contraction stage.

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APPENDIX

Substitution of (2) into Eq. (8) gives

$$y''y\left[\Phi+1+\frac{2\rho^2}{\rho^2+\lambda}\right]=\frac{b^2}{6}\left(y-E-\frac{\lambda}{6}b^4\Phi\right)+\frac{(y')^2}{4}(\Phi+2)+\frac{2yy'}{b}\left[\Phi+1+\frac{3\rho^2}{\rho^2+\lambda}\right],\quad \Phi=\ln(\rho^2+\lambda)/\rho_c^2. \tag{A.1}$$

The term $\lambda b^4\Phi/6$ for small b is small by virtue of the smallness of λ. As $b\to\infty$, when $\rho^2\to 0$, $\Phi\to \ln\lambda/\rho_c^2$, which is equivalent to the presence of a λ term. This last may be compensated by the choice of A in (1). Hence, over the whole interval of variation of b this term can be ignored.

If $\rho \gg \lambda$, then $\rho^2/(\rho^2+\lambda)\approx 1$. If $\lambda>\rho^2$, this equation is invalid since $|\Phi|\gg 1$ by virtue of (4), which determines the value of the brackets on the left-hand side of (A.1) and the last term on the right-hand side. This last enables one to retain the given equation for the complete interval of variation of ρ. With allowance for these assumptions, Eq. (10) follows from (A.1) after renotation of ρ_c.

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"Instantons" of higher order

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Solutions have been obtained for the Yang-Mills equations for the gauge group $SU(2)$ in a Euclidean space having a topological characteristic larger than one.

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1. Classical solutions of the Yang-Mills (YM) equations in Euclidean four-space have been the object of numerous theoretical investigations in recent months. The Euclidean signature of the metric allows one to reduce the order of the YM equations and to represent them in the form of duality relations. The solutions of these equations, "instantons," describe in the quasiclassical approximation tunneling transitions between various vacuum states of the YM field in normal (pseudoeuclidean) Minkowski space.[1] More precisely, the instantons are saddle points in the calculation of various Green's functions by means of functional integration methods.

The solution of the YM equations found by Belavin *et al.*[2] in a Euclidean four-space having spherical symmetry (under $O(4)$) (the instanton) turned out to be invariant under the group $O(5)$[3] on account of the conformal invariance of the YM equations. In the action for the free YM field

$$S=\frac{1}{16\pi e^2}\int\langle F_{ij}F_{kl}\rangle g^{ik}g^{jl}g^{1/2}\,d^4x, \tag{1}$$

the metric tensor enters only in the combination

$$(g^{ik}g^{jl}-g^{il}g^{jk})\,g^{1/2}, \tag{2}$$

which is invariant under the substitution

$$g_{ij}\to\lambda(x)\,g_{ij}. \tag{3}$$

Therefore, any solution is a solution not on a particular Riemannian manifold, but on a class of conformally equivalent such manifolds. Since stereographic projection establishes a conformal mapping between the 4-sphere and Euclidean four-space, the solutions of the YM equations in Euclidean space, when mapped by the stereographic projection onto the sphere, will be solutions of the YM equations on the 4-sphere (S^4) and vice versa. The instanton turned out to be a field "constant" (according to the definition introduced by one of us[4]) on the sphere S^4, and is therefore invariant with respect to its symmetry group $O(5)$.

According to the classification of Belavin *et al.*[2] the instanton has topological characteristic 1; at the same time these authors have predicted from topological considerations that there exist instantons with higher topological characteristics, globally defined by the action integral (1) over the whole four-space: for solutions of higher topological type I_n the action integral must be an integral multiple of the integral for the instanton I_1, with the integer n (the multiplicity) characterizing the topological type of the field (the degree of the mapping of S^4 onto $SU(2)\simeq S^3$ which determines the gauge).[1)] But in mapping of the higher degree the symmetry is necessarily lowered, since for such mappings there appear in four-space submanifolds where the solutions branch, which destroy the spherical symmetry. However, if instead of the Euclidean space one considers the solutions on the sphere S^4, one can choose the branching manifold (which has dimension $d-2$, if d is the dimension of space) as a sphere S^2, so that out of the symmetry group $O(5)$ one retains the sufficiently high symmetry $O(3)\otimes O(2)$, allowing one to find the solution.

2. On the sphere S^4 we introduce a coordinate system which explicitly reflects this symmetry, defining

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