Strong fluctuations of the intensity of electromagnetic waves in randomly inhomogeneous media

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(Submitted March 25, 1977)
Zh. Eksp. Teor. Fiz. 73, 481–497 (August 1977)

We consider strong fluctuations of the intensity of an electromagnetic wave behind a random phase screen and in a randomly inhomogeneous medium. The statistical moments of the intensity are represented in the form of ordinary (for the phase screen) and continual integrals of the Feynman type (for the inhomogeneous medium), and the asymptotic forms of these integrals are investigated in the region of strong fluctuations. It is shown that the intensity moments $\langle I^n\rangle$, at not too large values of n, correspond to an exponential distribution for I. The employed method, in which the solution of the problem is represented in the form of a continual integral, makes it possible to investigate the region of applicability of the phase approximation of the Huygens-Kirchhoff method through an inhomogeneous medium. It is shown that this method gives qualitatively correct results but with an error on the order of the principal term.

PACS numbers: 41.10.Hv

1. INTRODUCTION

When electromagnetic waves propagate in randomly inhomogeneous media with large-scale (compared with the wavelength) inhomogeneities of the refractive index, strong field-intensity fluctuations arise in a number of cases. This effect is due to the concentration of the scattered radiation in a narrow angle interval about the initial direction of propagation. Strong intensity fluctuations can arise when radiowaves propagate through the ionosphere, the solar corona, or the interstellar medium, [1] when light propagates in a turbulent atmosphere, [2] when the atmosphere of a planet becomes transparent when the planet occults a natural or artificial radiation source, [3] and in a number of other cases.

A theoretical description of the strong fluctuations in the propagation of a wave in a randomly inhomogeneous medium should be based on methods that go beyond the framework of perturbation theory. The principal equations of strong fluctuations were obtained by various methods in[4–9]. Since the case of interest is that of large-scale inhomogeneities, the depolarization of the radiation can be neglected[10] and we can start with the parabolic equation for the smoothly varying complex amplitude $u(\mathbf{r})$, which is connected with the considered components of the electric field E by the relation $E=u\exp(ikx-i\omega t)$. This equation takes the form

$$\frac{\partial u}{\partial x}=\frac{i}{2k}\frac{\partial^2 u}{\partial \rho^2}+\frac{ik}{2}\tilde{\varepsilon}(x,\rho)u(x,\rho). \qquad (1.1)$$

Here x is the coordinate along the initial direction of the wave propagation, $\rho=(0,y,z)$ is the transverse radius vector, $k=c^{-1}\omega(\bar{\varepsilon})^{1/2}$, $\tilde{\varepsilon}=(\varepsilon-\bar{\varepsilon})/\bar{\varepsilon}$, $\bar{\varepsilon}$ is the mean value of the permittivity, which we shall assume to be constant, and $\tilde{\varepsilon}$ are the relative fluctuations of ε. In the plane $x=0$ we assume the initial distribution of the field to be specified: $u(0,\rho)=u_0(\rho)$. We consider henceforth cases in which $\tilde{\varepsilon}$ differs from zero in the entire layer from the $x=0$ plane to the observation point (this may be, for example, propagation of light along the earth's surface), and the case of a very thin inhomogeneous layer—a "phase screen," with which it is easiest to explain the calculation procedure which is common to both methods.

We consider the statistical moment of the field u

$$\left\langle \prod_{k=1}^{n} u(x,\rho_{2k-1})u^*(x,\rho_{2k}) \right\rangle=\Gamma_{2n}(x,\rho_1,\ldots,\rho_{2n}) \qquad (1.2)$$

(the angle brackets denote averaging over all the possible realizations of $\tilde{\varepsilon}$). In the Markov random process approximation[5] it is assumed that $\tilde{\varepsilon}(x,\rho)$ is a Gaussian field that is δ-correlated in x, so that

$$\langle\tilde{\varepsilon}(x_1,\rho_1)\tilde{\varepsilon}(x_2,\rho_2)\rangle=\delta(x_1-x_2)A(\rho_1-\rho_2),$$
$$A(0)-A(\rho)=D(\rho)=2\pi\iint_{-\infty}^{\infty}\Phi_\varepsilon(0,\varkappa)(1-\cos\varkappa\rho)\,d^2\varkappa, \qquad (1.3)$$

where $\Phi_\varepsilon(\varkappa_x,\varkappa)$ is the three dimensional spectral density of the fluctuations $\tilde{\varepsilon}$. In this approximation Γ_{2n} satisfies the equation[4–9]

$$\frac{\partial\Gamma_{2n}}{\partial x}=\frac{i}{2k}[\Delta_1-\Delta_2+\ldots+\Delta_{2n-1}-\Delta_{2n}]\Gamma_{2n}+\frac{k^2}{8}\sum_{j,l=1}^{2n}(-1)^{j+l}D(\rho_j-\rho_l)\Gamma_{2n}, \qquad (1.4)$$

where $\Delta_l=\partial^2/\partial\rho_l^2$.

We confine ourselves next to the case of a plane incident wave, when the initial condition for Eq. (1.4) can be taken in the form $\Gamma_{2n}(0,\{\rho_l\})=1$. The solution of Eq. (1.4) can be obtained analytically only for $n=1$, and it takes the form

$$\Gamma_2(x,\rho_1,\rho_2)=\exp\{-k^2xD(\rho_1-\rho_2)/4\}. \qquad (1.5)$$

The function $\Gamma_2=\langle u(x,\rho_1)u^*(x,\rho_2)\rangle$ determines the coherence of the second order of the field. From the condition

$${}^1\!/_4k^2xD(\rho_c)=1 \qquad (1.6)$$

we can determine the coherence radius ρ_c for the field.

 0038-5646/77/4602-0252$02.40

For $n \geq 2$ it is impossible to obtain the solution of (1.4) in quadratures. At $n=2$ the equation for the function Γ_4 in the case of a plane wave can be simplified and takes the form

$$\frac{\partial \Gamma_4(x, \mathbf{r}_1, \mathbf{r}_2)}{\partial x} = \frac{i}{k} \frac{\partial^2 \Gamma_4}{\partial \mathbf{r}_1 \partial \mathbf{r}_2} - \frac{k^2}{4} F(\mathbf{r}_1, \mathbf{r}_2) \Gamma_4, \tag{1.7}$$

where

$$F(\mathbf{r}_1, \mathbf{r}_2) = 2D(\mathbf{r}_1) + 2D(\mathbf{r}_2) - D(\mathbf{r}_1+\mathbf{r}_2) - D(\mathbf{r}_1-\mathbf{r}_2), \quad \mathbf{r}_1 = \rho_2-\rho_1 = \rho_3-\rho_4, \quad \mathbf{r}_2 = \rho_4-\rho_1 = \rho_3-\rho_2. \tag{1.8}$$

Equation (1.7) was investigated in[11–13] by numerical methods. The result was a fluctuation-intensity fluctuation that agreed with the experimental data[2] qualitatively.

If we denote by L the longitudinal coordinates of the observation point, then the quantity $k^2 L F(\mathbf{r}_1, \mathbf{r}_2)$ is proportional to the mean square phase shift of the wave in the inhomogeneous layer. If $k^2 L F(\mathbf{r}_1, \mathbf{r}_2) \ll 1$ at $|\mathbf{r}_{1,2}| \sim (L/k)^{1/2}$, then we can assume in (1.7) that $\frac{1}{4}k^2 F \Gamma_4 \approx \frac{1}{4}k^2 F$, and this leads to results that agree with the calculation by the smooth perturbation method (SPM).[14] In the opposite case $k^2 LF \gg 1$ the solution was investigated in[15–17] (see also the review[18]), where the asymptotic form of Γ_4 as $k^2 LF \to \infty$ was obtained.

In this paper we investigate the asymptotic forms of the functions Γ_{2n} in the region of strong fluctuations. Exact solutions for Γ_{2n} are written in the form of Feynman continual integrals (or in their equivalent operator form) and the asymptotic limit of these integrals is investigated as $L \to \infty$.

The solution of the stochastic Eq. (1.1) can be written in the form[19,20]:

$$u(x, \rho) = \exp\left\{\frac{i}{2k}\int_0^x d\xi \frac{\delta^2}{\delta \tau^2(\xi)}\right\} u_0\left(\rho + \int_0^x \tau(\xi)\, d\xi\right) \times \exp\left\{\frac{ik}{2}\int_0^x dx' \tilde{\varepsilon}\left(x', \rho + \int_{x'}^x \tau(\xi)\, d\xi\right)\right\}\Big|_{\tau=0}. \tag{1.9a}$$

It is equivalent to the continual integral

$$u(x, \rho) = \int D\mathbf{v}(\xi)\, u_0\left(\rho + \int_0^x d\xi\, \mathbf{v}(\xi)\right) \times \exp\left\{\frac{ik}{2}\int_0^x dx'\left[\mathbf{v}^2(x') + \tilde{\varepsilon}\left(x', \rho + \int_{x'}^x \mathbf{v}(\xi)\, d\xi\right)\right]\right\}.$$

Here

$$D\mathbf{v}(\xi) = \prod_{\xi=0}^{x} d^2 v(\xi) \Big/ \int \ldots \int \prod_{\xi=0}^{x} d^2 v(\xi) \exp\left\{\frac{ik}{2}\int_0^x v^2(\xi)\, d\xi\right\}. \tag{1.9b}$$

Formula (1.9a) can be easily obtained from (1.1) by the method of[21,22], while (1.9b) is obtained from (1.9a) with the aid of a continual Fourier transformation.

If we assume the fluctuations $\tilde{\varepsilon}$ to be Gaussian and δ-correlated in x, then by substituting (1.9b) in (1.2) and averaging over $\tilde{\varepsilon}$ we can obtain the formula

$$\Gamma_{2n}(x, \{\rho_i\}) = \int \ldots \int D\mathbf{v}_1(\xi) \ldots D\mathbf{v}_{2n}(\xi) \times \exp\left\{\frac{ik}{2}\int_0^x d\xi \sum_{j=1}^{2n} (-1)^{j+1} v_j^2(\xi)\right\} \Gamma_{2n}^{(0)}\left(\left\{\rho + \int_0^x d\xi\, \mathbf{v}(\xi)\right\}\right) \times \exp\left\{-\frac{k^2}{8}\sum_{j,l=1}^{2n} (-1)^{j+l+1} \int_0^x d\xi\, D\left(\rho_j - \rho_l + \int_\xi^x dx' [\mathbf{v}_j(x') - \mathbf{v}_l(x')]\right)\right\} \tag{1.10a}$$

or, in operator form, which is sometimes more convenient for actual calculations

$$\Gamma_{2n}(x, \{\rho_i\}) = \left\{\prod_{l=1} \exp\left[\frac{i}{2k}(-1)^{1+l} \int_0 d\xi \frac{\delta^2}{\delta \boldsymbol{\tau}_l^2(\xi)}\right]\right\} \Gamma_{2n}^{(0)}\left(\left\{\rho + \int_0 \boldsymbol{\tau}(\xi)\, d\xi\right\}\right) \times \exp\left\{-\frac{k^2}{8}\sum_{j,l=1}^{2n} (-1)^{j+l+1} \int_0^x D\left(\rho_j - \rho_l + \int_\xi^x dx' [\boldsymbol{\tau}_j(x') - \boldsymbol{\tau}_l(x')]\right) d\xi\right\}\Big|_{\tau=0}. \tag{1.10b}$$

Formulas (1.10a) and (1.10b) can, of course, be obtained also as solutions of Eq. (1.4), derived from (1.1) under the same assumption: $\tilde{\varepsilon}(\xi, \rho)$ is a Gaussian random function that is δ-correlated in ξ. We note that Γ_{2n} was expressed in the form of a continual integral in[23], where an attempt was made to investigate the behavior of Γ_4 under strong fluctuations, but an incorrect result was obtained.

Before we investigate the asymptotic form of Γ_{2n} for the case of fluctuating parameters of the medium, we consider the simpler of the fluctuations of the field behind a random phase screen. The analogy between these problems has already been noted in[15,16,18]. However, when the solutions for Γ_{2n} are expressed in continual form, this analogy goes over in fact into a method common to both problems for obtaining the asymptotic form of solutions in the region of strong fluctuations.

2. RANDOM PHASE SCREEN

Assume a layer of inhomogeneous medium of thickness Δx so small that when a wave passes through the layer it acquires only a random phase shift

$$S(\rho) = \frac{k}{2}\int_0^{\Delta x} \tilde{\varepsilon}(\xi, \rho)\, d\xi \tag{2.1}$$

and does not change its amplitude. We assume that $\tilde{\varepsilon}(\xi, \rho)$ is a Gaussian field, δ-correlated in ξ and described by formulas (1.3). After passing through the inhomogeneous layer the wave propagates in a homogeneous medium, and its propagation is described by an equation obtained from (1.1) with $\tilde{\varepsilon} = 0$. The solution of this problem is given by the formulas

$$u(x, \rho) = \exp\left\{\frac{ix}{2k}\Delta\right\} \exp\{iS(\rho)\} \tag{2.2a}$$

$$= \frac{k}{2\pi i x}\int d^2 v \exp\left\{\frac{ik}{2x} v^2\right\} \exp\{iS(\rho + \mathbf{v})\}, \tag{2.2b}$$

which are analogous to formulas (1.9a) and (1.9b) given above.

We consider now the function $\Gamma_{2n}(x, \{\rho\})$. Substituting (2.2b) in (1.2) and averaging, we readily obtain

$$\Gamma_{2n}(x,\{\rho\})=\left(\frac{k}{2\pi x}\right)^{2n}\int\ldots\int d^2v_1\ldots d^2v_{2n}\exp\left\{\frac{ik}{2x}\sum_{j=1}^{2n}(-1)^{j+1}v_j^2\right\}$$
$$\times\exp\left\{-\frac{k^2\Delta x}{8}\sum_{j=1}^{2n}\sum_{l=1}^{2n}(-1)^{j+l+1}D(\rho_j-\rho_l+\mathbf{v}_j-\mathbf{v}_l)\right\}. \qquad (2.3)$$

This formula is the analog of (1.10b). Let us examine in greater detail the case $n=2$ for pairwise coinciding observation points $\rho_1=\rho_2=\rho'$, $\rho_3=\rho_4=\rho''$, $\rho'-\rho''=\rho$. Then

$$\Gamma_4(x,\rho',\rho',\rho'',\rho'')=\langle I(x,\rho')I(x,\rho'')\rangle$$

is the covariation of the intensities $I=|u|^2$. If we introduce in (2.3) (at $n=2$) new integration variables

$$\mathbf{v}_1-\mathbf{v}_2=\mathbf{r}_1,\quad \mathbf{v}_1-\mathbf{v}_4=\mathbf{r}_2,\quad \mathbf{v}_1-\mathbf{v}_3=\mathbf{r}_3\quad (\mathbf{v}_1+\mathbf{v}_2)/2=\mathbf{r},$$

then the integration with respect to $\mathbf{r}$ and $\mathbf{r}_3$ can be carried out and we obtain as a result the formula

$$\langle I(x,\rho')I(x,\rho'')\rangle=\left(\frac{k}{2\pi x}\right)^2\iint d^2r_1\,d^2r_2\exp\left\{\frac{ik}{x}\mathbf{r}_1(\mathbf{r}_2-\rho)-\frac{k^2\Delta x}{4}F(\mathbf{r}_1,\mathbf{r}_2)\right\}, \qquad (2.4)$$

where $F(\mathbf{r}_1,\mathbf{r}_2)$ is determined from (1.8). The integral (2.4) was investigated in detail (including by numerical methods) in a number of studies (see the review[18]). We examined its asymptotic form as $x\to\infty$.

The factor $\exp\{-k^2\Delta xF(\mathbf{r}_1,\mathbf{r}_2)/4\}\equiv f_1(\mathbf{r}_1,\mathbf{r}_2)$ becomes equal to unity at $\mathbf{r}_1=0$ and at $\mathbf{r}_2=0$. The equation

$$\tfrac{1}{4}k^2\Delta xF(\mathbf{r}_1,\mathbf{r}_2)=1 \qquad (2.5)$$

determines the boundary of the region outside of which $f_1(\mathbf{r}_1,\mathbf{r}_2)$ is exponentially small.[1)] Since F consists of a linear combination of the functions D, and the equation $\frac{1}{4}k^2\Delta xD(\rho_c)=1$ (see (1.6)) determines the coherence radius of the field, it is clear that ρ_c is also one of the characteristic dimensions of this region.

Figure 1 shows plots of (2.5) for the function $D(\mathbf{r})=pC_\varepsilon^2r^{5/3}$, which appears in the problem of wave propagation in a turbulent medium.[5,14] The curves are plotted on the plane $r_1=|\mathbf{r}_1|$, $r_2=|\mathbf{r}_2|$ for two limiting

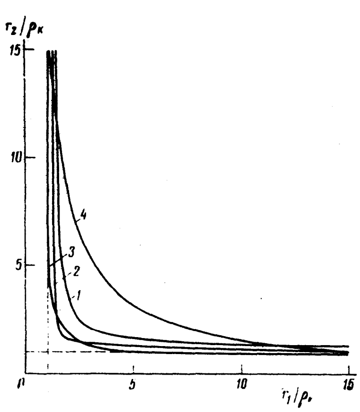

FIG. 1. Plots of $\frac{1}{4}k^2\Delta xF(\mathbf{r}_1,\mathbf{r}_2)=1$ for the cases 1) $D(\mathbf{r})\propto r^{5/3}$, $\mathbf{r}_1\perp\mathbf{r}_2$, 2) $D(\mathbf{r})\propto r^{5/3}$, $\mathbf{r}_1\parallel\mathbf{r}_2$, 3) $D(\mathbf{r})\propto[1-\exp(-r^2/2l^2)]$, $\mathbf{r}_1\perp\mathbf{r}_2$, $\rho_c^2/2l^2=0.1$. Curve 4 corresponds to the Fresnel factor at $r_1r_2=16\rho_c^2$. The dashed lines are the isolines $r_1=\rho_c$, $r_2=\rho_c$.

cases of mutual orientation of the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, namely $\mathbf{r}_1\cdot\mathbf{r}_2=0$ and $\mathbf{r}_1\times\mathbf{r}_2=0$. All the remaining curves lie between those shown in the figure. In the regions $r_1\gg r_2$ and $r_2\gg r_1$ the curves (2.5) approach asymptotically straight lines that are the solutions of the equations $\frac{1}{2}k^2\Delta xD(r_2)=1$ and $\frac{1}{2}k^2\Delta xD(r_1)=1$. The reason is that the combination

$$\Omega(\mathbf{r}_1,\mathbf{r}_2)=D(\mathbf{r}_1+\mathbf{r}_2)+D(\mathbf{r}_1-\mathbf{r}_2)-2D(\mathbf{r}_1) \qquad (2.6)$$

vanishes at $r_1\gg r_2$ and only the first term is left in (2.5) in this region. This form of the region bounded by the curve $\frac{1}{4}k^2\Delta xF=1$ is practically independent of the choice of the correlation function B_ε. If, for example, we plot this region for the function $B_\varepsilon(\mathbf{r})=\sigma_\varepsilon^2\exp\{-r^2/2l^2\}$, then we obtain (at $\mathbf{r}_1\cdot\mathbf{r}_2=0$) curve 3 of Fig. 1, which has the same character as for the power-law function D.

The factor $f_2(\mathbf{r}_1,\mathbf{r}_2)=\exp\{ikx^{-1}\mathbf{r}_1(\mathbf{r}_2-\rho)\}$ has a characteristic scale $\rho_F=(x/k)^{1/2}$ (the radius of the first Fresnel zone). Therefore, if

$$(x/k)^{1/2}\gg\rho_c, \qquad (2.7)$$

then the function f_2 varies little inside the central part of the region bounded by the curve (2.5), i.e., the cutoff factor is here the function f_1. At $r_{1,2}>\pi x/2k\rho_c$, however, the period of the oscillations of the diffraction factor becomes smaller than ρ_c (see Fig. 1), so that f_2 becomes the essential factor on the periphery of this region. The length of that band in Fig. 1 which is significant for the integration is therefore determined by the factor f_2 and is of the order of $x/k\rho_c$. In this connection, a second characteristic scale appears in the problem, namely

$$r_0=x/k\rho_c. \qquad (2.8)$$

It can thus be assumed that when the condition (2.7) is satisfied the essential region for the integral (2.4) is the one adjacent to the hyperplane $\mathbf{r}_1=0$ and $\mathbf{r}_2=0$. The larger the parameter $r_0/\rho_c=x/k\rho_c^2$, the narrower this region. Therefore the integral (2.4) can be broken up into two, one of which extends over the region $r_1\lesssim\rho_c$, and the other over the region $r_2\lesssim\rho_c$. But in the first region of integration we can use the expansion

$$f_1(\mathbf{r}_1,\mathbf{r}_2)=\exp\left\{-\frac{k^2\Delta x}{2}D(\mathbf{r}_1)+\frac{k^2\Delta x}{4}\Omega(\mathbf{r}_2,\mathbf{r}_1)\right\}$$
$$\approx\exp\left\{-\frac{k^2\Delta x}{2}D(\mathbf{r}_1)\right\}\left\{1+\frac{k^2\Delta x}{4}\Omega(\mathbf{r}_2,\mathbf{r}_1)+\ldots\right\}, \qquad (2.9)$$

since the isolines of the function f_1 are here quite close to the isolines of the function $\exp\{\frac{1}{2}k^2\Delta xD(\mathbf{r}_1)\}$ shown dashed in Fig. 1. In exactly the same manner, in the region $r_2\lesssim\rho_c$ we can put

$$f_1(\mathbf{r}_1,\mathbf{r}_2)\approx\exp\left\{-\frac{k^2\Delta x}{2}D(\mathbf{r}_2)\right\}\left\{1+\frac{k^2\Delta x}{4}\Omega(\mathbf{r}_1,\mathbf{r}_2)+\ldots\right\}. \qquad (2.10)$$

Since the factors $\exp\{-\frac{1}{2}k^2\Delta xD(\mathbf{r}_{1,2})\}$ decrease rapidly outside the bands $r_1<\rho_c$ and $r_2<\rho_c$, each of the integrals can now be extended to infinity. As a result we obtain the asymptotic form

$$\langle I(x,\rho')I(x,\rho'')\rangle=\left(\frac{k}{2\pi x}\right)^2\int d^2r_1\int d^2r_2\exp\left\{\frac{ik}{x}\mathbf{r}_1(\mathbf{r}_2-\rho)\right\}$$
$$\times\left\{\exp\left[-\frac{k^2\Delta x}{2}D(\mathbf{r}_1)\right]\left[1+\frac{k^2\Delta x}{4}\Omega(\mathbf{r}_2,\mathbf{r}_1)+\ldots\right]\right.$$
$$\left.+\exp\left[-\frac{k^2\Delta x}{4}D(\mathbf{r}_2)\right]\left[1+\frac{k^2\Delta x}{4}\Omega(\mathbf{r}_1,\mathbf{r}_2)+\ldots\right]\right\}. \qquad (2.11)$$

We note that we took here twice into account the integration region made up of the intersection of the bands $r_1 < \rho_c$ and $r_2 < \rho_c$. The contribution that is taken into account twice can be estimated at

$$\delta\approx\left(\frac{k}{2\pi x}\right)^2\iint d^2r_1\,d^2r_2\exp\left\{-\frac{k^2\Delta x}{2}[D(\mathbf{r}_1)+D(\mathbf{r}_2)]\right\},$$

and turns out to be a quantity of higher order of smallness (with respect to the small parameter $\rho_c/r_0=k\rho_c^2/x$) than the principal included term, and also smaller than some of the succeeding terms. Nonetheless, neglect of this term does not make it possible to obtain still higher order terms of the asymptotic expansion.

If we use the spectral expansion (1.3) of the function D in formula (2.6) for Ω, then we get from (2.11)

$$\langle I(x,\rho')I(x,\rho'')\rangle=1+\exp\left\{-\frac{k^2\Delta x}{2}D(\rho)\right\}$$
$$+\pi k^2\Delta x\int\Phi_\varepsilon(0,\varkappa)\left[1-\cos\frac{\varkappa^2 x}{k}\right]\exp\left\{i\varkappa\rho-\frac{k^2\Delta x}{2}D\left(\frac{\varkappa x}{k}\right)\right\}d^2\varkappa$$
$$+\pi k^2\Delta x\int\Phi_\varepsilon(0,\varkappa)\left[1-\cos\left(\varkappa\rho-\frac{\varkappa^2 x}{k}\right)\right]\exp\left\{-\frac{k^2\Delta x}{2}D\left(\rho-\frac{\varkappa x}{k}\right)\right\}d^2\varkappa\ldots \qquad (2.12)$$

If we put here $\rho\equiv\rho'-\rho''=0$ and recognize that $\varkappa\lesssim r_0^{-1}$, then we obtain for the mean squared value of the relative fluctuations

$$\beta^2=1+\pi x^2\Delta x\int\Phi_\varepsilon(0,\varkappa)\varkappa^4\exp\left\{-\frac{k^2\Delta x}{2}D\left(\frac{\varkappa x}{k}\right)\right\}d^2\varkappa+\ldots \qquad (2.13)$$

Let the fluctuations $\tilde{\varepsilon}$ in the inhomogeneous layer be caused by the turbulence, so that

$$\Phi_\varepsilon(0,\varkappa)=AC_\varepsilon^2\varkappa^{-11/3},\qquad D(\mathbf{r})=\pi pC_\varepsilon^2 r^{5/3},$$

where $A\approx0.033$, $p\approx0.46$ and C_ε^2 is the structural characteristic that enters in the "$\frac{2}{3}$-law" for $\tilde{\varepsilon}$ (see[14]). Then (2.13) leads to the known result[18]

$$\beta^2=1+0.429\beta_0^{-4/5}+\ldots, \qquad (2.14)$$

where $\beta_0^2=0.563C_\varepsilon^2k^{7/6}x^{5/6}\Delta x$ is the mean square, calculated by perturbation theory, of the relative intensity fluctuations. The error δ cited above turns out to be in this case of the order of $(\beta_0^2)^{-12/5}$, i.e., it is small at $\beta_0^2\gg1$ in comparison with the term taken into account in (2.14).

We consider now the intensity correlation function, which according to (2.12) consists of three terms:

$$B_I(x,\rho)=\langle I(x,\rho')I(x,\rho'')\rangle-1=B_I^{(1)}(x,\rho)+B_I^{(2)}(x,\rho)+B_I^{(3)}(x,\rho).$$

The largest of them is

$$B_I^{(1)}(x,\rho)=|\Gamma_2(x,\rho)|^2=\exp\{-{}^1/_2k^2\Delta xD(\rho)\}.$$

Calculation of the terms $B_I^{(2)}$ and $B_I^{(3)}$ yields

$$B_I^{(2)}(x,\rho)=\begin{cases}(\beta^2-1)/2 & \text{if } \rho\ll r_0 \qquad (2.15a)\\ -x^2\Delta x\Delta^2D(\rho)/4 & \text{if } \rho\gg r_0 \qquad (2.15b)\end{cases}$$

$$B_I^{(3)}(x,\rho)=(\beta^2-1)/2 \quad \text{if} \quad \rho\ll\rho_c, \qquad (2.15c)$$
$$B_I^{(3)}(x,\rho)=\frac{\pi^2}{2}k^2\Delta x\Phi_\varepsilon\left(0,\frac{k\rho}{x}\right)\rho^2$$
$$\times\int_0^\infty\exp\left\{-\frac{k^2\Delta x}{2}D\left(\frac{\varkappa x}{k}\right)\right\}\varkappa^3d\varkappa \quad \text{if} \quad \rho_c\ll\rho\ll r_0. \qquad (2.15d)$$

The scale of the function $B_I^{(3)}(x,\rho)$ is the coherence radius ρ_c. The function $B_I(x,\rho)$ is shown schematically in Fig. 2.

The foregoing arguments can be readily generalized to include the higher moments Γ_{2n}. We confine ourselves here to an investigation of $\langle I^n\rangle=\Gamma_{2n}(x,0)$. Formula (2.3) takes in this case the form

$$\langle I^n\rangle=\left(\frac{k}{2\pi x}\right)^{2n}\int\ldots\int d^2v_1\ldots d^2v_{2n}$$
$$\times\exp\left\{\frac{ik}{2x}\sum_{j=1}^{2n}(-1)^{j-1}v_j^2-F(\mathbf{v}_1,\ldots,\mathbf{v}_{2n})\right\}, \qquad (2.16)$$

where

$$F=\frac{1}{8}k^2\Delta x\sum_{j=1}^{2n}\sum_{l=1}^{2n}(-1)^{j+l-1}D(\mathbf{v}_j-\mathbf{v}_l). \qquad (2.17)$$

The random phase shifts $S(\mathbf{v}_j)$ defined by formula (2.1) are connected with the function F by the relation

$$F=\frac{1}{2}\left\langle\left[\sum_{j=1}^{2n}S(\mathbf{v}_j)(-1)^{j-1}\right]^2\right\rangle\geqslant0.$$

It is clear therefore that if all the odd points $\mathbf{v}_{2l+1}$ coincide pairwise with some even points, then the positive and negative phase shifts cancel each other and F vanishes. It therefore becomes obvious that at $(x/k)^{1/2}\gg\rho_c$ the main contribution to $\langle I^n\rangle$ is given by those regions where such a cancellation takes place. It is easy to calculate that the number of these regions in $n!$. Then, replacing (2.16) by the integral, multiplied by $n!$, over only one of these regions A_1, in which

$$|\mathbf{v}_1-\mathbf{v}_2|\sim|\mathbf{v}_3-\mathbf{v}_4|$$
$$\ldots\sim|\mathbf{v}_{2n-1}-\mathbf{v}_{2n}|<\rho_c,$$

we obtain

$$\langle I^n\rangle\approx n!\left(\frac{k}{2\pi x}\right)^{2n}\int_{A_1}\ldots\int d^2v_1$$
$$\ldots d^2v_{2n}\exp\left\{\frac{ik}{2x}\sum_{j=1}^{2n}(-1)^{j-1}v_j^2-F\right\}. \qquad (2.18)$$

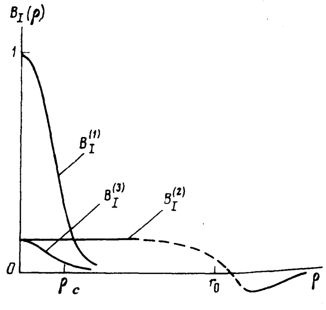

FIG. 2. Schematic form of the correlation function $B_I(x,\rho)$ of the intensity fluctuations.

The decrease of the integrand with respect to each of the variables $\mathbf{v}_1-\mathbf{v}_2$, $\mathbf{v}_3-\mathbf{v}_4$, etc. is ensured by the corresponding term

$$\tfrac{1}{8}k^2\Delta x[D(\mathbf{v}_1-\mathbf{v}_2)+D(\mathbf{v}_2-\mathbf{v}_1)],\quad \tfrac{1}{8}k^2\Delta x[D(\mathbf{v}_3-\mathbf{v}_4)+D(\mathbf{v}_4-\mathbf{v}_3)]$$

etc. from (2.17). These terms should be retained in the argument of the exponential, and the exponential in the remaining terms, just as in the case $n=2$ considered above, should be expanded in a series:

$$\langle I^n\rangle\approx n!\left(\frac{k}{2\pi x}\right)^{2n}\int\limits_{A_1}\cdots\int d^2v_1\ldots d^2v_{2n}\exp\left\{\frac{ik}{2x}\sum_{m=1}^{2n}(-1)^{m-1}v_m^2\right.$$
$$\left.-\frac{k^2\Delta x}{4}\sum_{m=1}^{n}D(\mathbf{v}_{2m-1}-\mathbf{v}_{2m})\right\}\left\{1+\frac{k^2\Delta x}{8}{\sum_{j=1}^{2n}}'{\sum_{l=1}^{2n}}'(-1)^{j+l}D(\mathbf{v}_j-\mathbf{v}_l)+\ldots\right\}. \tag{2.19}$$

The primes of the summation signs denote that the sums do not include the terms that have gone into the exponential. The integration in (2.19) can be extended over all of space, inasmuch as outside the region A_1 the integrand is negligibly small. Taking this fact into account, the multiple integrals in (2.19) can be calculated exactly and we obtain for $\langle I^n\rangle$ the formula

$$\langle I^n\rangle=n![1+n(n-1)(\beta^2-1)/4]+\ldots, \tag{2.20}$$

in which β^2 is defined by (2.13). We shall discuss this formula somewhat later, after we consider the propagation of waves in a randomly inhomogeneous medium, since the results obtained in both cases are analogous.

3. RANDOMLY INHOMOGENEOUS MEDIUM

We consider here the asymptotic form of the higher moments Γ_{2n} of a field propagating in a randomly homogeneous medium. The solution of this problem is given formally by expressions (1.10a) and (1.10b). They differ from the formulas considered above for a phase screen only in that the ordinary integration is replaced by continual integration. We consider first the quantity $\langle I(x,\rho')I(x,\rho'')\rangle$ which is obtained from Γ_4 by pairwise merging of the observation points $\rho_1=\rho_2=\rho'$, $\rho_3=\rho_4=\rho''$. For a plane wave ($\Gamma_{2n}^{(0)}=1$), using (1.10b) and introducing new variables, we obtain ($\rho=\rho'-\rho''$)

$$\langle I'I''\rangle=\exp\left\{\frac{i}{k}\int_0^x d\xi\frac{\delta^2}{\delta\mathbf{r}_1\delta\mathbf{r}_2}\right\}\exp\left\{-\frac{k^2}{4}\int_0^x dx'\left[2D\left(\rho+\int_{x'}^x\mathbf{r}_1\,d\xi\right)\right.\right.$$
$$\left.\left.\left.+2D\left(\int_{x'}^x\mathbf{r}_2\,d\xi\right)-D\left(\rho+\int_{x'}^x(\mathbf{r}_1+\mathbf{r}_2)\,d\xi\right)-D\left(\rho+\int_{x'}^x(\mathbf{r}_1-\mathbf{r}_2)\,d\xi\right)\right]\right\}\right|_{\mathbf{r}_\alpha=0}. \tag{3.1}$$

Formula (3.1) can be written also in a form that follows from (1.10a), but we shall write it in operator form. We put

$$R_1=\rho+\int_{x'}^x\mathbf{r}_1(\xi)\,d\xi,\quad R_2=\int_{x'}^x\mathbf{r}_2(\xi)\,d\xi.$$

The functional in the exponential

$$\Psi=\frac{k^2}{4}\int_0^x dx'[2D(\mathbf{R}_1)+2D(\mathbf{R}_2)-D(\mathbf{R}_1+\mathbf{R}_2)-D(\mathbf{R}_1-\mathbf{R}_2)],$$

is non-negative and vanishes on trajectories $\mathbf{r}_1$ and $\mathbf{r}_2$ such that $\mathbf{R}_1=0$ or $\mathbf{R}_2=0$. A substantial contribution to the continual integral is made by the trajectories lying in the region whose boundary is determined by the condition $\Psi=1$. However, for trajectories having $\mathbf{R}_2$ large in comparison with $\mathbf{R}_1$ we have

$$2D(\mathbf{R}_2)-D(\mathbf{R}_2+\mathbf{R}_1)-D(\mathbf{R}_2-\mathbf{R}_1)\to 0$$

and the condition $\Psi=1$ goes over into

$$\frac{k^2}{2}\int_0^x dx' D(\mathbf{R}_1)=1.$$

For these trajectories we therefore have

$$\exp\{-\Psi\}\approx\exp\left\{-\frac{k^2}{2}\int_0^x dx' D(\mathbf{R}_1)\right\}$$
$$\times\left\{1-\frac{k^2}{4}\int_0^x dx'[D(\mathbf{R}_2+\mathbf{R}_1)+D(\mathbf{R}_2-\mathbf{R}_1)-2D(\mathbf{R}_2)]+\ldots\right\}, \tag{3.2}$$

and the integration can be extended over the entire region, similar to what was done for the phase screen. Exactly the same formula can be written also for the second region, where $|\mathbf{R}_1|\gg|\mathbf{R}_2|$.

If $x/k\rho_c^2\gg 1$ then, just as in the case of the phase screen, the diffraction operator

$$\hat{L}=\exp\left\{\frac{i}{k}\int_0^x d\xi\frac{\delta^2}{\delta\mathbf{r}_1(\xi)\delta\mathbf{r}_2(\xi)}\right\} \tag{3.3}$$

is effective only on the periphery of the integration region, where we have either $|\mathbf{R}_1|\gg|\mathbf{R}_2|$ or $|\mathbf{R}_2|\gg|\mathbf{R}_1|$.

We substitute the expansion of $\exp\{-\Psi\}$ in the corresponding region in (3.1). We then have

$$\langle I'I''\rangle=\hat{L}\exp\left\{-\frac{k^2}{2}\int_0^x dx' D\left(\rho+\int_{x'}^x\mathbf{r}_1(\xi)\,d\xi\right)\right\}$$
$$\times\left\{1+\frac{k^2}{4}\int_0^x dx'\left[D\left(\rho+\int_{x'}^x(\mathbf{r}_1+\mathbf{r}_2)\,d\xi\right)\right.\right.$$
$$\left.\left.\left.+D\left(\rho+\int_{x'}^x(\mathbf{r}_1-\mathbf{r}_2)\,d\xi\right)-2D\left(\int_{x'}^x\mathbf{r}_2\,d\xi\right)\right]+\ldots\right\}\right|_{\mathbf{r}_1=\mathbf{r}_2=0}$$
$$+\hat{L}\exp\left\{-\frac{k^2}{2}\int_0^x dx' D\left(\int_{x'}^x\mathbf{r}_2(\xi)\,d\xi\right)\right\}\left\{1+\frac{k^2}{4}\int_0^x dx'\left[D\left(\rho+\int_{x'}^x(\mathbf{r}_1+\mathbf{r}_2)\,d\xi\right)\right.\right.$$
$$\left.\left.\left.+D\left(\rho+\int_{x'}^x(\mathbf{r}_1-\mathbf{r}_2)\,d\xi\right)-2D\left(\rho+\int_{x'}^x\mathbf{r}_1\,d\xi\right)\right]+\ldots\right\}\right|_{\mathbf{r}_1=\mathbf{r}_2=0}. \tag{3.4}$$

If we use spectral expansions for the functions D in the pre-exponential factors, then we can apply the operator L to (3.4), after which $\langle I'I''\rangle$ takes the form

$$\langle I'I''\rangle-1=B_I(x,\rho)=B_I^{(1)}(x,\rho)+B_I^{(2)}(x,\rho)+B_I^{(3)}(x,\rho), \tag{3.5}$$

where

$$B_I^{(1)}(x,\rho)=\exp\{-\tfrac{1}{2}k^2xD(\rho)\}, \tag{3.6}$$

$$B_I^{(2)}(x,\rho)=\pi k^2\int_0^x dx'\iint d^2\varkappa\,\Phi_\varepsilon(0,\varkappa)\left\{\exp\left[i\varkappa\rho-\frac{k^2x'}{2}D\left(\frac{\varkappa}{k}(x-x')\right)\right.\right.$$

$$-\frac{k^2}{2}\int_{x'}^{x} D\left(\frac{\varkappa}{k}(x-x'')\right)dx''\Big]\Big[1-\cos\left(\frac{\varkappa^2}{k}(x-x')\right)\Big]\Big\}, \qquad (3.7)$$

$$B_I^{(3)}(x,\rho)=\pi k^2\int_0^x dx'\iint d^2\varkappa\Phi_\varepsilon(0,\varkappa)\Big\{\exp\Big[-\frac{k^2x'}{2}D\Big(\rho-\frac{\varkappa}{k}(x-x')\Big)$$

$$-\frac{k^2}{2}\int_{x'}^{x} D\Big(\rho-\frac{\varkappa}{k}(x-x'')\Big)dx''\Big]\Big[1-\cos\Big(\rho\varkappa-\frac{\varkappa^2}{k}(x-x')\Big)\Big]\Big\}. \qquad (3.8)$$

Putting here $\rho=0$ and taking into account the first term in the expansion of $1-\cos\varkappa^2(x-x')/k$, we obtain for $\beta^2(x)=B_I(x,0)$ a formula analogous to (2.13):

$$\beta^2(x)=1+\pi\int_0^x dx'(x-x')^2\iint d^2\varkappa\,\varkappa^4\Phi_\varepsilon(0,\varkappa)$$

$$\times\exp\Big[-\frac{k^2x'}{2}D\Big(\frac{\varkappa}{k}(x-x')\Big)-\frac{k^2}{2}\int_{x'} D\Big(\frac{\varkappa}{k}(x-x'')\Big)dx''\Big]+\ldots \qquad (3.9)$$

Expression (3.9) remains in force also in the case when Φ_ε and D vary slowly along x.[24] In this case it is easy to change over from (3.9) to (2.13) if it is assumed that $\Phi_\varepsilon=0$ outside the layer $0\leqslant x'\leqslant\Delta x\ll x$.

The correlation function $B_I(x,\rho)$ is investigated in analogy with the phase-screen case considered above. The principal term $B_I^{(1)}$ in (3.5) is the square of the modulus of the second-order coherence function. In the second term of (3.7), owing to the presence of the factor $\exp\{-\frac{1}{2}k^2x'D(\varkappa(x-x')/k)\}$, the argument of the cosine is small in the greater part of the integration with respect to x', so that

$$B_I^{(2)}(x,\rho)=\frac{\pi}{2}\int_0^x (x-x')^2dx'\iint d^2\varkappa\,\varkappa^4\Phi_\varepsilon(0,\varkappa)\exp\Big\{i\varkappa\rho$$

$$-\frac{k^2x'}{2}D\Big(\frac{\varkappa(x-x')}{k}\Big)-\frac{k^2}{2}\int_{x'}^{x} D\Big(\frac{\varkappa(x-x'')}{k}\Big)dx''\Big\}. \qquad (3.10)$$

Inasmuch as $\varkappa\lesssim k\rho_c/x=1/r_0$ in (3.10), we have $\rho\ll r_0\exp(i\varkappa\rho)\approx1$. It follows therefore that relation (2.15a) is satisfied in this case. On the other hand, if $\rho\gg r_0$, then we need retain in the exponential in (3.10) only the term $i\varkappa\cdot\rho$, which leads to a formula analogous to (2.15b)

$$B_I^{(2)}(x,\rho)\approx-{}^1/_{12}x^3\Delta_\rho^2D(\rho),\quad \rho\gg r_0. \qquad (3.11)$$

At $\rho\ll\rho_c$ the term $B_I^{(3)}$ is equal to $(\beta^2-1)/2$, which agrees with (2.15c). In the distance region $\rho_c\ll\rho\ll r_0$ we easily obtain the expression

$$B_I^{(3)}(x,\rho)\approx\frac{\pi k^2}{2}\rho_j\rho_l\int_0^x dx'\,\Phi_\varepsilon\Big(0,\frac{k\rho}{x-x'}\Big)\iint d^2\varkappa\,\varkappa_j\varkappa_l$$

$$\times\exp\Big\{-\frac{k^2x'}{2}D\Big(\frac{\varkappa(x-x')}{k}\Big)-\frac{k^2}{2}\int_{x'}^{x} D\Big(\frac{x''-x'}{x-x'}\rho-\frac{\varkappa(x-x'')}{k}\Big)dx''\Big\}, \qquad (3.12)$$

which is similar to (2.15d).

If we consider the case of a turbulent medium, then we get from (3.9)[15–17]

$$\beta^2(x)=1+0.861(\beta_0^2)^{-2/5}+\ldots, \qquad (3.13)$$

where we have used the symbol $\beta_0^2=0.307C_\varepsilon^2k^{7/6}x^{11/6}$ for the mean square of the intensity fluctuations,[14] calculated by the method of smooth perturbations. For the terms of the correlation function $B_I(x,\rho)$ we obtain in this case formulas that coincide with the corresponding results of[16,17].

We consider now the higher moments $\langle I^n\rangle=\Gamma_{2n}(x,0)$. It is clear from (1.10a) that at $\rho_j=0$ the main contribution to the integral is made by those trajectories for which the functional

$$\frac{k^2}{8}\sum_{j,l=1}^{2n}(-1)^{j+l+1}\int_0^x dx'\,D\Big(\int_{x'}^x d\xi(\mathbf{v}_j(\xi)-\mathbf{v}_l(\xi))\Big)$$

vanishes. We can proceed here in analogy with the phase screen. Separating one of the $n!$ essential integration regions and retaining in the argument of the exponential of (1.10b) only terms of the type

$$D\Big(\int_{x'}^x(\tau_{2j+1}-\tau_{2j})d\xi\Big),$$

which ensures rapid decrease of the integrand of the functional, we obtain

$$\langle I^n\rangle\approx n!\exp\Big\{\frac{i}{2k}\sum_{l=1}^{2n}(-1)^{l+1}\int_0^x d\xi\frac{\delta^2}{\delta\tau_l^2(\xi)}\Big\}$$

$$\times\exp\Big\{-\frac{k^2}{4}\sum_{j=1}^{n}\int_0^x D\Big(\int_{x'}^x(\tau_{2j-1}(\xi)-\tau_{2j}(\xi))d\xi\Big)dx'\Big\}$$

$$\times\Big\{1-\frac{k^2}{8}\sum_{j,l=1}^{2n}{}'(-1)^{j+l+1}\int_0^x D\Big(\int_{x'}^x(\tau_j(\xi)-\tau_l(\xi))d\xi\Big)dx'+\ldots\Big\}\Big|_{\tau_\alpha=0}. \qquad (3.14)$$

The prime on the summation sign means that the sum does not include terms retained in the exponential. We introduce new variables

$$\tau_{2j-1}-\tau_{2j}=\mathbf{r}_j,\quad \tau_{2j-1}+\tau_{2j}=2\mathbf{R}_j.$$

Then

$$\frac{\delta^2}{\delta\tau_{2j-1}^2}-\frac{\delta^2}{\delta\tau_{2j}^2}=2\frac{\delta^2}{\delta\mathbf{r}_j\delta\mathbf{R}_j}.$$

In terms of these variables, the first term in (3.14) takes the form

$$J_1=n!\exp\Big\{\frac{i}{k}\sum_{j=1}^n\int_0^x d\xi\frac{\delta^2}{\delta\mathbf{r}_j\delta\mathbf{R}_j}\Big\}$$

$$\times\exp\Big\{-\frac{k^2}{4}\sum_{j=1}^n\int_0^x dx'\,D\Big(\int_{x'}^x\mathbf{r}_j(\xi)d\xi\Big)\Big\}\Big|_{\mathbf{r}=0,\,\mathbf{R}=0}=n!. \qquad (3.15)$$

If we use in the second term of (3.14) the spectral expansion (1.3) of the functions D in the pre-exponential factor, then after simple transformation we can rewrite this term in the form

$$J_2=-n!\,\pi k^2\sum_{i,s=1}^n\int_0^x dx'\iint d^2\varkappa\,\Phi_\varepsilon(0,\varkappa)\exp\Big\{\frac{i}{k}\sum_{j=1}^n\int_0^x d\xi\frac{\delta^2}{\delta\mathbf{r}_j\delta\mathbf{R}_j}\Big\}$$

$$\times\exp\Big\{-\frac{k^2}{4}\sum_{j=1}^n\int_0^x dx''\,D\Big(\int_{x''}^x\mathbf{r}_j(\xi)d\xi\Big)\Big\}$$

$$\times\sin\Big(\frac{\varkappa}{2}\int_{x'}^x\mathbf{r}_s(\xi)d\xi\Big)\sin\Big(\frac{\varkappa}{2}\int_{x'}^x\mathbf{r}_i(\xi)d\xi\Big)\times$$

$$\times \exp\left\{i\varkappa \int_0^x \theta(\xi'-x')[\mathbf{R}_s(\xi')-\mathbf{R}_l(\xi')]d\xi'\right\}\Big|_{\mathbf{R}_\alpha-\mathbf{r}_\alpha=0}. \quad (3.16)$$

Applying the operators, we ultimately obtain

$$J_2=-n!\,\pi k^2 \sum_{l,s=1}^{n} \int_0^x dx' \iint d^2\varkappa\, \Phi_\epsilon(0,\varkappa)$$

$$\times \exp\left\{-\frac{k^2}{4}\sum_{j=1}^{n}\int_0^x dx'' D\left(\int_{x''}^{x} d\xi \frac{\varkappa}{k}\theta(\xi-x')(\delta_{sj}-\delta_{lj})\right)\right\}$$

$$\times \sin\left(\frac{\varkappa^2}{2k}\int_{x'}^{x}\theta(\xi-x')(\delta_{ls}-1)d\xi\right)\sin\left(\frac{\varkappa^2}{2k}\int_{x'}^{x}\theta(\xi-x')(1-\delta_{ls})d\xi\right),$$

where δ_{sj} is the Kronecker symbol. Summation over s and l yields

$$J_2=n!n(n-1)\pi k^2\int_0^x dx'\iint d^2\varkappa\Phi_\epsilon(0,\varkappa)\sin^2\frac{\varkappa^2(x-x')}{2k}\cdot$$

$$\times \exp\left\{-\frac{k^2x'}{2}D\left(\frac{\varkappa(x-x')}{k}\right)-\frac{k^2}{2}\int_{x'}^{x}D\left(\frac{\varkappa(x-x'')}{k}\right)dx''\right\}. \quad (3.17)$$

Expanding $\sin^2[\varkappa^2(x-x')/2k]$ in a series and comparing the obtained formula with (3.1), we have

$$J_2=n!n(n-1)(\beta^2(x)-1)/4.$$

Taking (3.15) into account, we find that in the case when waves propagate in a randomly inhomogeneous medium we also have the expansion

$$\langle I^n\rangle=n![1+n(n-1)(\beta^2(x)-1)/4+\ldots], \quad (3.18)$$

which coincides with (2.20) for a phase screen, but naturally $\beta^2(x)$ is determined by a different formula in each individual case.

Formula (3.18) yields the first two terms of the asymptotic expansion of $\langle I^n\rangle$ as $\beta_0^2\to\infty$. Since $\beta^2\to 1$ as $\beta_0^2\to\infty$, the second term in (3.18) is small in comparison with the first at sufficiently large β_0^2. Only in the case when

$$(\beta^2(x)-1)n(n-1)/4\ll 1, \quad (3.19)$$

does expression (3.18) have any meaning. However, at fixed β_0^2 there are always numbers n for which the condition (3.19) is violated. Therefore formula (3.18) is valid only for not too large values of n. If we disregard the limitation (3.19), then from the known (3.18) we can reconstruct the probability density for I:

$$W(I)=\theta(I)\exp(-I)[1+(\beta^2-1)(1-2I+I^2/2)/2+\ldots]. \quad (3.20)$$

It would follow from this formula that as $\beta^2-1\to 0$ the probability density for I tends to an exponential, which corresponds to a Gaussian distribution for a random wave field. By virtue of the condition (3.19), however, this is not so, and only the lower moments $\langle I^n\rangle$ can be adequately described by the distribution (3.20). It must also be noted that the asymptotic form of (3.18) may be reached as $\beta_0^2\to\infty$ quite slowly (for example, for a power-law structure function $D(\rho)\propto\rho^{5/3}$ the essential role is played by the difference between the growth rate of the functions $\rho^{5/3}$ and ρ^2).

The higher moments $\langle I^n\rangle$ were investigated also in[25], where the obtained probability distribution differed from (3.20). It should be noted, however, that the higher moments $\langle I^n\rangle$ with $n>2$ were not calculated in[25], and an approximation formula, the justification of which is not quite clear, was proposed for them.

4. THE HUYGENS-KIRCHHOFF METHOD

By expressing the statistical moments of the field in the form of continual integrals, we can investigate the applicability of the so called Huygens–Kirchhoff method, which is presently used in a number of studies for numerical estimates[26] as well as to obtain analytic expressions.[27]

The Huygens–Kirchhoff method proposed in[28] is based on writing down the solution of (1.1) in the form

$$u(x,\rho)=\iint d^2\rho'\, G(x,\rho;0,\rho')u_0(\rho'), \quad (4.1)$$

where $u_0(\rho')$ is the distribution of the field in the initial plane $x=0$, and G is the stochastic Green's function for Eq. (1.1). If there are no fluctuations of the dielectric constant ($\tilde\varepsilon=0$), then Green's function takes the form

$$G|_{\tilde\varepsilon=0}=g(x,\rho;x',\rho')=\frac{k}{2\pi i(x-x')}\exp\left\{\frac{ik(\rho-\rho')^2}{2(x-x')}\right\},\quad x>x'. \quad (4.2)$$

The approximation used in[28] consists of expressing the Green's function in a randomly inhomogeneous medium in the form

$$G(x,\rho;x',\rho')=\frac{k}{2\pi i(x-x')}\exp\left\{\frac{ik(\rho-\rho')^2}{2(x-x')}+\Psi_1(x,\rho;x',\rho')\right\}, \quad (4.3)$$

where $\Psi_1(x,\rho;x',\rho')$ is the shift of the complex phase of the spherical wave propagating from the point (x',ρ') to the point (x,ρ), in the first-order approximation in $\tilde\varepsilon$. For randomly inhomogeneous media, the statistical characteristics of Ψ_1 were investigated in[29]. We note that at the present time there is no basis for this approximation.

The Huygens–Kirchhoff method was used in[26] to study the statistical characteristics of wave beams in a turbulent medium. For the quantity Γ_4, which is described in the case of a plane wave by Eq. (1.7), this method yields the expression

$$\Gamma_4(x,\mathbf{r}_1,\mathbf{r}_2)=\left(\frac{k}{2\pi x}\right)^4\iint d^2\rho_1 d^2\rho_2\exp\left\{\frac{ik}{x}(\mathbf{r}_1-\rho_1)(\mathbf{r}_2-\rho_2)\right.$$

$$\left.-\frac{k^2}{4}\int_0^x d\xi\, F\left(\rho_1\left(1-\frac{\xi}{x}\right)+\mathbf{r}_1\frac{\xi}{x};\rho_2\left(1-\frac{\xi}{x}\right)+\mathbf{r}_2\frac{\xi}{x}\right)\right\}, \quad (4.4)$$

where $F(\mathbf{R}_1;\mathbf{R}_2)$ is defined in (1.8).

Let us see how expression (4.4) can be derived from the exact solution of the problem. The exact solution of (1.7) follows from (1.10a) and is of the form

$$\Gamma_4(x,\mathbf{r}_1,\mathbf{r}_2)=\iint D\mathbf{v}_1(\xi)D\mathbf{v}_2(\xi)\exp\left\{ik\int_0^x d\xi\,\mathbf{v}_1(\xi)\mathbf{v}_2(\xi)\right.$$

$$-\frac{k^2}{4}\int_0^x d\xi F\left(\mathbf{r}_1+\int_0^x \boldsymbol{\tau}_1(\eta)d\eta-\int_0^\xi \boldsymbol{\tau}_1(\eta)d\eta;\ \mathbf{r}_2+\int_0^x \boldsymbol{\tau}_2(\eta)d\eta-\int_0^\xi \boldsymbol{\tau}_2(\eta)d\eta\right)\Big\}, \tag{4.5}$$

which is equivalent, when the shift operator is used, to the form

$$\Gamma_4=\iint D\boldsymbol{\tau}_1 D\boldsymbol{\tau}_2 \exp\Big\{ik\int_0^x d\xi\, \boldsymbol{\tau}_1\boldsymbol{\tau}_2+\int_0^x d\xi(\boldsymbol{\tau}_1\nabla_{r_1}+\boldsymbol{\tau}_2\nabla_{r_2})\Big\}$$
$$\times\exp\Big\{-\frac{k^2}{4}\int_0^x d\xi F\Big(\mathbf{r}_1-\int_0^\xi \boldsymbol{\tau}_1 d\eta;\ \mathbf{r}_2-\int_0^\xi \boldsymbol{\tau}_2 d\eta\Big)\Big\}. \tag{4.6}$$

The following identity is obvious:

$$\exp\Big\{-\frac{k^2}{4}\int_0^x d\xi F\Big(\mathbf{r}_1-\int_0^\xi \boldsymbol{\tau}_1 d\eta;\ \mathbf{r}_2-\int_0^\xi \boldsymbol{\tau}_2 d\eta\Big)\Big\}$$
$$=(2\pi)^{-4}\iint d^2\varkappa\, d^2q\iint d^2\rho_1 d^2\rho_2\exp\Big\{i\boldsymbol{\varkappa}(\mathbf{r}_1-\boldsymbol{\rho}_1)+i\mathbf{q}(\mathbf{r}_2-\boldsymbol{\rho}_2)$$
$$-\frac{k^2}{4}\int_0^x d\xi F\Big(\boldsymbol{\rho}_1-\int_0^\xi \boldsymbol{\tau}_1 d\eta;\ \boldsymbol{\rho}_2-\int_0^\xi \boldsymbol{\tau}_2 d\eta\Big)\Big\}. \tag{4.7}$$

If we now substitute (4.7) in (4.6) and apply the shift operator, then we get

$$\Gamma_4=(2\pi)^{-4}\iint D\boldsymbol{\tau}_1 D\boldsymbol{\tau}_2\exp\Big\{ik\int_0^x d\xi\,\boldsymbol{\tau}_1\boldsymbol{\tau}_2\Big\}\iint d^2\varkappa\, d^2q\exp\Big\{i\int_0^x d\xi(\boldsymbol{\varkappa}\boldsymbol{\tau}_1+\mathbf{q}\boldsymbol{\tau}_2)\Big\}$$
$$\times\iint d^2\rho_1 d^2\rho_2\exp\Big\{i\boldsymbol{\varkappa}(\mathbf{r}_1-\boldsymbol{\rho}_1)+i\mathbf{q}(\mathbf{r}_2-\boldsymbol{\rho}_2)$$
$$-\frac{k^2}{4}\int_0^x d\xi F\Big(\boldsymbol{\rho}_1-\int_0^\xi \boldsymbol{\tau}_1 d\eta,\ \boldsymbol{\rho}_2-\int_0^\xi \boldsymbol{\tau}_2 d\eta\Big)\Big\}. \tag{4.8}$$

We make the change of the integration variables $\boldsymbol{\tau}_1\to\boldsymbol{\tau}_1+\mathbf{q}/k$, $\boldsymbol{\tau}_2\to\boldsymbol{\tau}_2+\boldsymbol{\varkappa}/k$ in the continual integral with respect to $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$, and then make the change $\boldsymbol{\varkappa}\to\boldsymbol{\varkappa}+(\mathbf{r}_2-\boldsymbol{\rho})k/x$, $\mathbf{q}\to\mathbf{q}+(\mathbf{r}_1-\boldsymbol{\rho}_1)k/x$ in the integral with respect to $\boldsymbol{\varkappa}$ and $\mathbf{q}$. We then obtain

$$\Gamma_4(x,\mathbf{r}_1,\mathbf{r}_2)=(2\pi)^{-4}\iint d^2\rho_1 d^2\rho_2\exp\Big\{\frac{ik}{x}(\mathbf{r}_1-\boldsymbol{\rho}_1)(\mathbf{r}_2-\boldsymbol{\rho}_2)\Big\}$$
$$\times\iint d^2q\iint d^2\varkappa\iint D\boldsymbol{\tau}_1 D\boldsymbol{\tau}_2\exp\Big\{ik\int_0^x d\xi(\boldsymbol{\tau}_1\boldsymbol{\tau}_2-\boldsymbol{\varkappa}\mathbf{q}/k^2)\Big\}$$
$$\times\exp\Big\{-\frac{k^2}{4}\int_0^x d\xi F\Big(\boldsymbol{\rho}_1\Big(1-\frac{\xi}{x}\Big)+\mathbf{r}_1\frac{\xi}{x}+\mathbf{q}\frac{\xi}{k}-\int_0^\xi\boldsymbol{\tau}_1 d\eta;$$
$$\boldsymbol{\rho}_2\Big(1-\frac{\xi}{x}\Big)+\mathbf{r}_2\frac{\xi}{x}+\boldsymbol{\varkappa}\frac{\xi}{k}-\int_0^\xi\boldsymbol{\tau}_2 d\eta\Big)\Big\}. \tag{4.9}$$

The integrals with respect to $\boldsymbol{\tau}_1(\xi)$, $\boldsymbol{\tau}_2(\xi)$ and $\boldsymbol{\varkappa},\mathbf{q}$ can be represented in operator form

$$\Gamma_4(x,\mathbf{r}_1,\mathbf{r}_2)=\Big(\frac{k}{2\pi x}\Big)^2\iint d^2\rho_1 d^2\rho_2\exp\Big\{\frac{ik}{x}(\mathbf{r}_1-\boldsymbol{\rho}_1)(\mathbf{r}_2-\boldsymbol{\rho}_2)\Big\}$$
$$\times\hat{L}_{\rm diffr}(\boldsymbol{\varkappa},\mathbf{q},\boldsymbol{\tau}_1,\boldsymbol{\tau}_2)\exp\Big\{-\frac{k^2}{4}\int_0^x d\xi F\Big(\boldsymbol{\rho}_1\Big(1-\frac{\xi}{x}\Big)+\mathbf{r}_1\frac{\xi}{x}+\mathbf{q}\frac{\xi}{k}$$
$$-\int_0^\xi\boldsymbol{\tau}_1 d\eta;\ \boldsymbol{\rho}_2\Big(1-\frac{\xi}{x}\Big)+\mathbf{r}_2\frac{\xi}{x}+\boldsymbol{\varkappa}\frac{\xi}{k}-\int_0^\xi\boldsymbol{\tau}_2 d\eta\Big)\Big\}\Big|_{\boldsymbol{\varkappa},\mathbf{q}=0;\ \boldsymbol{\tau}_1,\boldsymbol{\tau}_2=0}, \tag{4.10}$$

where we have introduced the diffraction operator

$$\hat{L}_{\rm diffr}=\exp\Big\{\frac{i}{k}\int_0^x d\xi\Big(\frac{\delta^2}{\delta\boldsymbol{\tau}_1\delta\boldsymbol{\tau}_2}-\frac{k^2}{x^2}\frac{\partial^2}{\partial\boldsymbol{\varkappa}\partial\mathbf{q}}\Big)\Big\}.$$

Comparing (4.10) with (4.4) we see that (4.10) goes over into (4.4) only if $\hat{L}_{\rm diffr}=1$, i.e., when the effect of the operator is negligible. But if the parameter $\beta_0^2\ll 1$, then the exponential in (4.10) can expand in a Taylor series in β_0^2 and the first terms of the expansion could be calculated. It is easy to verify here that the correct result corresponds to complete allowance for the operator $\hat{L}_{\rm diffr}$.

Thus, the considered approximation (4.4) does not hold in the region of weak fluctuations, where it produces an error of the same order of magnitude as the principal term. On the other hand, as seen from the foregoing section, the diffraction operator $\hat{L}_{\rm diffr}$ determines also the law that governs the approach of β^2 to its asymptotic value unity as $\beta_0^2\to\infty$. As to the approximation (4.4), it yields here, too, a qualitatively correct result, but with an error of the same order as the principal term. The approximation (4.4) might be useful for the intermediate region $\beta_0^2\approx 1$, where neither perturbation theory nor the asymptotic solutions are correct. However, since at both $\beta_0^2\ll 1$ and $\beta_0^2\gg 1$ the approximation (4.4) results in an error that is not small, there are no grounds for assuming that its accuracy will be satisfactory in the region $\beta_0^2\sim 1$.

[1)]We consider only the case of a "strong" phase screen, when Eq. (2.5) has a solution.

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Translated by J. G. Adashko

Alfven and magnetosonic vortices in a plasma

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(Submitted January 17, 1977)
Zh. Eksp. Teor. Fiz. 73, 498–505 (August 1977)

It is shown that Alfven and magnetosonic waves can exist in a plasma in the form of two- and three-dimensional vortices. The dispersion spreading of such vortices is impeded by nonlinear effects. Magnetosonic waves form a toroidal vortex that travels along the magnetic field with Alfven velocity. Nonlinear Alfven waves form along the magnetic field an axially symmetrical waveguide. Inside of which the plasma executes vortical oscillations in the azimuthal direction. MHD vortices of this kind are observed in the earth's magnetosphere.

PACS numbers: 52.35.Bj, 52.35.Dm, 52.35.Mw, 94.30.Fk

1. INTRODUCTION

It is known that various types of vortices can propagate in an incompressible liquid. A quantitative description of this phenomenon, however, encounters considerable difficulties because there is no small expansion paramter.[1] An examination of such vortices in a low-pressure plasma ($\beta \equiv 8\pi p/H_0^2 \ll 1$, where p is the plasma pressure and H_0 is the unperturbed magnetic field), undertaken in the present paper, is facilitated by the presence of small parameters. This makes it possible to reduce the magnetohydrodynamics equations to simpler equations of non-one-dimensional solitons with the aid of perturbation theory, by assuming the dispersion and nonlinear terms to be small quantities of the same order (an expansion of the Korteweg–de Vries type). The importance of solutions in the form of non-one-dimensional solitons lies primarily in the fact that in contrast to simple wave packets they do not spread out when they propagate in the plasma, as a result of which they accumulate much energy and can therefore be relatively easily observed. The amplitude and dimensions of such formations are related by simple equations, which make it possible to distinguish them in experiment from perturbations of other types.

There are known solutions of the plasma equations in the form of one-dimensional magnetosonic[2] and Alfven[3] solitons. The energy of the one-dimensional solitons, however, is very large, and they can be produced only by large plasma perturbations, such as solar flares. In addition, it can be shown that in the cases considered in the present paper the one-dimensional solitons are stable. Taking this circumstance into account, we are interested in solitons with maximum possible dimensionality. There are a number of known solutions of the plasma equations in the form of three-dimensional solitons.[4,5] All the cited studies, however, were confined to potential oscillations. Yet many observations in the magnetosphere, which is still the best object for the study of waves in a plasma, point to the existence of non-one-dimensional solitons of the Alfven and of the magnetosonic type. They constitute vortices that travel along the magnetic field with a velocity close to the Alfven velocity.

Recently the interest in Alfven and magnetosonic perturbations of plasma has increased because they can be easily made to build up in tokomaks of the future, where the condition $\beta > m/M$ must be satisfied if controlled nuclear fusion (CNF) is to be realized. At low amplitudes these perturbations have a rather large localization region and can therefore be easily stabilized by shear of the force lines. Allowance for the nonlinearity leads to self-focusing—to a decrease of the characteristic dimensions of the perturbations. The influence of the shear of the force line is therefore decreased and the perturbations can increase to amplitudes that are dangerous for plasma containment.

MHD vortices might also be observed in a solid-state plasma, where MHD wave propagation is possible. We note that the existence of one-dimensional MHD solitons traveling along the magnetic field is impossible because of the absence of the nonlinearity which is needed to compensate for the dispersion spreading of the wave

 0038-5646/77/4602-0260$02.40