Connection between strong-field quantum electrodynamics with short-distance quantum electrodynamics

V. I. Ritus

P. N. Lebedev Physical Institute of the USSR Academy of Sciences
(Submitted April 25, 1977)
Zh. Eksp. Teor. Fiz. **73**, 807–821 (September 1977)

The exact Lagrange function of a constant electromagnetic field is considered as a competitor for the photon propagator in the investigation of questions of principle in quantum electrodynamics. A condensed gauge-invariant method is proposed for the calculation of radiative contributions to the Lagrange function, based on a closed functional expression. For extremely strong fields all radiative effects are concentrated in a scale multiplier of the field variable which is universal for quantum electrodynamics (or the Callan-Symanzik β function). In addition to the expansion terms of the β function in spinor electrodynamics which were determined before, we have obtained the first two terms in its expansion for scalar electrodynamics. A comparison is carried out between the renormalization-invariant charges determined by the photon propagator and the Lagrange function and the bare charge determined by the Z_3-factor. It is proved that their Gell-Mann–Low functions are different, the latter containing α^3 terms which makes the appearance of a common zero in these functions possible; this corresponds to a finite limit charge. Integral transformations of the renormalization-invariant charges are considered, which do not change their boundary and limit values, but subject their Gell-Mann–Low function to a transformation.

PACS numbers: 11.10.Ef, 11.10.Gh, 11.10.Np

I. INTRODUCTION

As is well known, the behavior of quantum electrodynamics at small distances and the problem of its internal self-consistency are usually studied by means of the exact photon propagator, cf. the classical papers of Landau, Abrikosov and Khalatnikov, [1] and Gell-Mann and Low. [2] Based on the renormalizability of quantum electrodynamics (QED), Gell-Mann and Low have shown that in the region of large squared four-momenta the photon propagator becomes a function of a single variable: if the exact propagator is $D=k^{-2}d$, the renormalization-invariant quantity $\alpha d(k^2/m^2, \alpha) \to \Phi_1(\varphi(\alpha)k^2/m^2)$—the contribution of all the radiative corrections reduces to the scale factor $\varphi(\alpha)$ multiplying the dynamical variable k^2/m^2. This means that the form of the charge distribution at small distances from the center does not depend on α.

For the internal consistency of QED it is essential whether the limit of the function $\Phi_1(z)$ is finite or infinite for $z\to\infty$. Indeed, if $\Phi_1(z)$ tends to a constant α_*, this means that for very large k^2 the exact photon propagator becomes a free propagator up to the factor α_*/α, which replaces the fine-structure constant α by its limit α_*. In this case the charge density at small distances is described by the function $(4\pi\alpha_*)^{1/2}\delta(x)$, i.e., corresponds to a finite bare point charge $e_*=(4\pi\alpha_*)^{1/2}$. If the function $\Phi_1(z)$ tends to infinity as $z\to\infty$, the charge distribution at small distances has at the origin a singularity which is worse than the delta-function, and it is hard to attribute a physical meaning to such a singularity. Even more intriguing is the situation when $\Phi_1(z)$ has a pole for a finite z. In this case the theory may "exist" only for vanishing physical charge, cf. the papers by Landau and Pomeranchuk[3] and Fradkin. [4]

The most economical description of the situation is realized in terms of the Gell-Mann–Low function $\psi_1(y)$ which is an invariant characteristic of the form of the function $\Phi_1(z)$, i.e., does not depend on a multiplicative change $z\to tz$ of the argument of the latter. To a finite limit α_* of the function $\Phi_1(z)$ for $z\to\infty$ corresponds a zero of high enough order of ψ_1 at $y=\alpha_*$. In a series of papers[5–8] Johnson, Baker and Willey have shown that a zero of the Gell-Mann–Low function $\psi_1(y)$ is also a zero of the simpler function $f_1(\alpha)$ which is the one-loop contribution to the first expansion coefficient of the reciprocal propagator d^{-1} in powers of $\ln(k^2/m^2)$ for $k^2/m^2\to\infty$. Presently only the first three terms in the expansion of these functions have been obtained by means of perturbation theory.

In addition to the photon propagator, the Lagrange function of the electromagnetic field plays an essential role in quantum electrodynamics. As is well known, in classical (Maxwell) electrodynamics the Lagrange function is quadratic in the field. The radiative quantum effects of the interaction of the electromagnetic field with the field of the charged particles in the vacuum leads to the appearance of essentially nonlinear terms in the Lagrange function. The first radiative correction to the Lagrange function of the constant electromagnetic field has been found by Heisenberg and Euler. [9] A significant contribution to the investigation of this correction was made by Weisskopf[10] and Schwinger, [11] the latter creating an elegant method for the consideration of the problem of vacuum polarization by an external electromagnetic field.

In a previous paper[12] the author has determined the radiative correction to the Lagrange function of a constant electromagnetic field which follows after the Heisenberg-Euler term, and it was shown that on account of renormalization invariance, the exact Lagrange function of the electromagnetic field, taking into account all radiative corrections, becomes a function of one variable in the asymptotic region of strong fields: if $l=\mathscr{L}/\mathscr{L}^{(0)}$ is the ratio of the exact Lagrange function to the Maxwell Lagrange function, then αl^{-1}

 0038-5646/77/4603-0423$02.40

$\rightarrow \Phi_2(\varphi(\alpha)eF/m^2)$ with the same scale multiplier $\varphi(\alpha)$ of the dynamical variable eF/m^2 as for the case of the exact photon propagator. This makes the Lagrange function of the strong field a convenient object for the investigation of questions of principle in QED. In particular, the Lagrange function may serve as a source of information about the universal function $\varphi(\alpha)$ of QED, function which reflects the role of radiative corrections in extreme conditions, when one has to go beyond perturbation theory.

In the present paper it is shown that the functions $\Phi_1(z)$ and $\Phi_2(z)$ are different and cannot be identified by any multiplicative change of their arguments: $\Phi_1(z) \neq \Phi_2(tz)$, $t=\text{const}$. This follows from the fact that the corresponding Gell-Mann–Low functions $\psi_a(y)$, $a=1,2$ differ in the third of their expansions. However, if for $z\rightarrow\infty$ the function $\Phi_1(z)$ tends to the constant α_* then Φ_2 also tends to α_*. Indeed, if for very large k^2 the photon propagator becomes free up to a factor α_*/α then both the field equations and the Lagrange function from which they are derived must differ from the Maxwellian quantities only by a factor α_*/α. Here the Maxwellian character must exist exactly in the region of very strong fields, where, as one can see from a calculation, the distances which are essential for the formation of the Lagrange function are of the order $(eF)^{-1/2} \ll m^{-1}$ and thus small compared to the Compton wave lengths. To this class of quantities which tend to α_* belongs also the bare fine structure constant α_0, considered as a function of α and of the dimensionless regularization parameter $x\rightarrow\infty$ (c.f. Sec, 3). Thus, the problem of whether the constant α_* is or is not finite can be studied in terms of that function $\Phi_a(z)$ or in terms of that Gell-Mann–Low function $\psi_a(y)$ which is more convenient for the purpose. Among the advantages of the Lagrange function one can list the simplicity of its computation, which at each step preserves gauge invariance, and the fact that the corresponding Gell-Mann–Low function $\psi_2(y)$ has a negative third-order term, which makes it easier for this function to have a zero (in distinction from the Gell-Mann–Low function of the photon propagator).

The structure of the paper and its main results are as follows. In Sec. 2 we briefly expose the method of calculation of the Lagrange function for the electromagnetic field, on the example of electrodynamics of scalar charged particles. This method is described in more detail in the author's paper[12] devoted to the Lagrange function of the electromagnetic field in spinor electrodynamics. However, this time we start from the closed functional relation between the exact action of the electromagnetic field, taking into account all the radiative corrections, and the first approximation to this expression. Moreover, scalar electrodynamics has some specific pecularities compared to spinor electrodynamics, owing to the more complicated structure of the current (which is proportional to the momentum operators Π_α and not to the c-number γ_α-matrix), and owing to the higher divergences, i.e., the terms proportional to k^2 (or reciprocal to the proper time). In spite of this, the method described here is sufficiently flexible and compact for an adequate dealing with these peculiarities. The correction to second order in α to the Lagrange function is obtained, as well as the Z_3-factor in the corresponding approximation and the first two terms in the expansion of the Callan-Symanzik β function.

By means of the renormalization group, we determine in Sec. 3 the behavior of the exact Lagrange function of the electromagnetic field for extremely strong fields and carry out a comparison of various renormalization-invariant charges in the asymptotic region of their respective dynamical variables. In addition to the usual renormalization-invariant charge, defined in terms of the photon propagator, we consider the invariant charge defined in terms of the exact Lagrange function of the constant electromagnetic field, and the bare charge defined in terms of the Z_3-factor. In spinor electrodynamics we have determined for the two latter charges the Gell-Mann–Low functions $\psi_2(y)$ and $\psi_3(y)$ in the α^3 approximation. These two functions turn out to differ from each other and from the Gell-Mann–Low function $\psi_1(y)$ for the first charge:

$$\psi_a(y)=y\left\{\frac{1}{3}\frac{y}{\pi}+\frac{1}{4}\left(\frac{y}{\pi}\right)^2+\begin{bmatrix}0{,}049991\ldots\\-0{,}079785\ldots\\-0{,}238631\ldots\end{bmatrix}\left(\frac{y}{\pi}\right)^3+\ldots\right\},\quad a=1,2,3.$$

The negative third terms enhance the possible existence of zeroes for these functions, zeroes which are necessary for the internal consistency of quantum electrodynamics.

Finally, Sec. 4 contains some integral transformations of the renormalization invariant charges which do not change the limit (α) and the asymptotic (α_*) values of these charges, but subject their Gell-Mann–Low functions to a transformation. It is shown that the integral transformations may render the third term of the expansion of the Gell-Mann–Low function negative.

2. THE LAGRANGE FUNCTION OF THE ELECTROMAGNETIC FIELD

One can indicate two methods of calculation of the Lagrange function of the electromagnetic field. The first consists in integrating the expression for the variation of the action for a change of the potential $A_\mu(x)$ of the external field by $\delta A_\mu(x)$

$$\delta W=\int d^4x\,\delta A_\mu(x)\langle j_\mu(x)\rangle, \tag{1}$$

where $\langle j_\mu(x)\rangle$ is the expectation value of the current density operator, induced in the vacuum by the external field. In scalar QED

$$\langle j_\mu(x)\rangle=e(x|\Pi_\mu G+G\Pi_\mu|x),\qquad \Pi_\mu=p_\mu-eA_\mu. \tag{2}$$

Making use of this expression of the current and $e\delta A_\mu = -\delta\Pi_\mu$, one can write (1) in the form

$$\delta W=i\mathrm{Tr}\,\delta G_0^{-1}G, \tag{3}$$

where $G_0=-i(\Pi^2+m^2)^{-1}$ is the Green's function of the particle in an external field without radiative corrections and Tr is the diagonal sum over space-time coordinates. Thus, in a first approximation, when $G \simeq G_0$ we obtain $W^{(1)} = i\,\mathrm{Tr}\ln G_0$. This expression, as does (3), differs by a sign from the corresponding expression in spinor electrodynamics. Further radiative corrections to the action can be obtained taking into account the next corrections to G in integrating (1) or (3).

It is however more convenient to make use of closed functional expressions for the exact vacuum-to-vacuum amplitude in the presence of the field (cf., e. g., [13])

$$e^{iW}=\exp\left[\frac{1}{2}\int d^4\xi\, d^4\xi'\,\frac{\delta}{\delta A_\alpha(\xi)}D_0(\xi-\xi')\frac{\delta}{\delta A_\alpha(\xi')}\right]e^{iW^{(1)}}, \qquad (4)$$

where D_0 is the vacuum photon propagator. In this relation is expressed the universality of the electromagnetic interaction: the amplitude $\exp(iW^{(1)}[eA])$ which takes into account only the interaction of electrons with an external field, determines the exact amplitude $\exp(iW[eA, \alpha])$, which also takes into account the radiative interactions of the electrons with each other via the quantized electromagnetic field. Both interactions are characterized by the same charge e. In particular, we obtain for $W^{(2)}$

$$W^{(2)}=ie^2\int d^4x\, d^4x'\, D_0(x-x')\,\{(x|\Pi_\alpha G_0|x')(x'|\Pi_\alpha G_0|x)$$
$$+(x|\Pi_\alpha G_0\Pi_\alpha|x')(x'|G_0|x)+4i\delta(x-x')(x|G_0|x')+(x|\Pi_\alpha G_0|x)(x'|\Pi_\alpha G_0|x')\} \qquad (5)$$

For a constant and homogeneous field $F_{\mu\nu}$ the last term in (5) does not contribute to $W^{(2)}$, since the average current induced in the vacuum by such a field vanishes.

It will be convenient to use the Fock-Schwinger[14,11] proper-time representation for the Green's function of a scalar particle in a static field

$$G_0(x,x')\equiv(x|G_0|x')=\frac{-ie^{i\eta}}{(4\pi)^2}\int_0^\infty\frac{ds}{s^2}\exp\left[-im^2s-L(s)+\frac{iz\beta z}{4}\right], \qquad (6)$$

where

$$\beta=eF\,\mathrm{cth}\,eFs,\quad L(s)=\frac{1}{2}\,\mathrm{tr}\ln\left(\frac{\mathrm{sh}\,eFs}{eFs}\right)$$

are a matrix and scalar function of the matrix $F_{\mu\nu}$, respectively, $z = x - x'$, η is the nondiagonal phase of the Green's function, equal to the line integral of the potential $eA_\mu(y)$ along a straight line joining the points x' and x.

It is not hard to see that the matrix elements $(x|\Pi_\alpha G_0|x')$ and $(x|G_0\Pi_\alpha|x')$ differ from (6) by the additional factors $1/2(\beta+eF)_{\alpha\beta}z_\beta$, $1/2(\beta-eF)_{\alpha\beta}z_\beta$ under the sign of the proper-time integral, and the matrix element $(x|\Pi_\alpha G_0\Pi_\alpha|x')$ can be represented in the form

$$(x|\Pi_\alpha G_0\Pi_\alpha|x')=-i\delta(x-x')-m^2(x|G_0|x')+{}^1/{}_2e^2zFFz(x|G_0|x'). \qquad (7)$$

Taking these remarks into account and utilizing for D_0 the proper-time representation

$$D_0(z)=\frac{-i}{(4\pi)^2}\int_0^\infty\frac{dt}{t^2}\exp\left(\frac{iz^2}{4t}\right),$$

we obtain from (5) the following expression for $\mathcal{L}^{(2)}$:

$$\mathcal{L}^{(2)}=\frac{-e^2}{(4\pi)^6}\iiint_0^\infty\frac{ds\,ds'\,dt}{(ss't)^2}\exp[-im^2(s+s')-L-L']$$
$$\times\int d^4z\exp\left(\frac{izAz}{4}\right)\left[-\frac{1}{4}z(\beta'\beta-3e^2FF)z-m^2\right]$$
$$+\frac{3e^2}{(4\pi)^4}\iint_0^\infty\frac{ds\,dt}{(st)^2}\exp(-im^2s-L). \qquad (8)$$

Here $A = \beta + \beta' + t^{-1}$ is a symmetric 4×4 matrix and β', L' are obtained from β, L by means of the substitution $s \to s'$.

The integration over z in the first term, and the subsequent integration with respect to t are carried out just as in [12]. In the second term the integral with respect to t is regularized by introducing a lower integration limit t_0, and the integral with respect to s, by means of subtracting a term which does not depend on the field. As a result of this we obtain:

$$\mathcal{L}^{(2)}=\frac{-i\alpha}{64\pi^5}\iint_0^\infty\frac{ds\,ds'\exp[-im^2(s+s')](e^2\eta\varepsilon)^2}{\sin e\eta s\,\mathrm{sh}\,e\varepsilon s\,\sin e\eta s'\,\mathrm{sh}\,e\varepsilon s'}\left(-m^2I_0-\frac{i}{2}I\right)$$
$$-\frac{\alpha^2}{16\pi^2im^2t_0}\mathcal{L}^{(0)}+\frac{3\alpha}{4\pi it_0}\frac{\partial\mathcal{L}_R^{(1)}}{\partial m^2}, \qquad (9)$$

where $\mathcal{L}^{(0)}$ is the Lagrange function of the Maxwell field, $\mathcal{L}_R^{(1)}$ is the nonlinear correction to it found by Schwinger[11]:

$$\mathcal{L}^{(0)}=\frac{\varepsilon^2-\eta^2}{2},$$
$$\mathcal{L}_R^{(1)}=-\frac{1}{16\pi^2}\int_0^\infty\frac{ds}{s}\exp(-im^2s)\left[\frac{e^2\eta\varepsilon}{\sin e\eta s\,\mathrm{sh}\,e\varepsilon s}-\frac{1}{s^2}-\frac{e^2(\eta^2-\varepsilon^2)}{6}\right], \qquad (10)$$

and the functions I_0, I are defined in Eq. (36) of [12], where the parameters a, b, are the same as for the spinor case, and the parameters p, q are different:

$$p=2e^2\eta^2(\mathrm{ctg}\,e\eta s\,\mathrm{ctg}\,e\eta s'+3),$$
$$q=2e^2\varepsilon^2(\mathrm{cth}\,e\varepsilon s\,\mathrm{cth}\,e\varepsilon s'-3). \qquad (11)$$

The integral term in (9) does not vanish when the field is switched on and requires regularization. This is done in the same way as in [12]. As a result $\mathcal{L}^{(2)}$ has the following representation:

$$\mathcal{L}^{(2)}=\mathcal{L}^{(0)}\frac{3\alpha^2}{16\pi^2}\left(\ln\frac{1}{i\gamma m^2s_0}-\frac{11}{9}\ln 2-\frac{10}{9}-\frac{1}{3im^2t_0}\right)$$
$$+\delta m^2\frac{\partial\mathcal{L}_R^{(1)}}{\partial m^2}+\mathcal{L}_R^{(2)}, \qquad (12)$$

where

$$\mathcal{L}_R^{(2)}=\frac{-i\alpha}{32\pi^3}\int_0^\infty ds\left\{\int_0^s ds'\left[K(s,s')-\frac{K_0(s)}{s'}\right]+K_0(s)\left(\ln i\gamma m^2s-\frac{7}{6}\right)\right\}, \qquad (13)$$

$$K(s,s')=\frac{(e^2\eta\varepsilon)^2\exp[-im^2(s+s')]}{\sin e\eta s\,\mathrm{sh}\,e\varepsilon s\,\sin e\eta s'\,\mathrm{sh}\,e\varepsilon s'}\left(-m^2I_0-\frac{i}{2}I\right)$$
$$+\frac{m^2\exp[-im^2(s+s')]}{ss'(s+s')}\left[1+\frac{i}{m^2(s+s')}\right.$$
$$\left.+\frac{e^2(\eta^2-\varepsilon^2)}{6}\left((s+s')^2-ss'+i\frac{11ss'}{m^2(s+s')}\right)\right], \qquad (14)$$

$$K_0(s)=e^{-im^2s}\left(-m^2+\frac{i}{2}\frac{\partial}{\partial s}\right)\left(\frac{e^2\eta\varepsilon}{\sin e\eta s\,\mathrm{sh}\,e\varepsilon s}-\frac{1}{s^2}-\frac{e^2(\eta^2-\varepsilon^2)}{6}\right), \qquad (15)$$

and the radiative correction to the particle mass is

$$\delta m^2=\frac{3\alpha m^2}{4\pi}\left(\ln\frac{1}{i\gamma m^2 s_0}+\frac{7}{6}+\frac{1}{im^2t_0}\right),\quad \ln\gamma=0{,}577\ldots \tag{16}$$

We call attention to the divergences linear in t_0^{-1} in $\mathscr{L}^{(2)}$ and δm^2.

The radiative correction of first order in α to the Lagrange function is [11]

$$\mathscr{L}^{(1)}=\mathscr{L}^{(0)}\frac{\alpha}{12\pi}\ln\frac{1}{i\gamma m_0^2 s_0}+\mathscr{L}_R\ (m_0^2), \tag{17}$$

where $\mathscr{L}_R^{(1)}$ is defined by Eq. (10) and the unrenormalized values of the fine structure constant and the mass are denoted by the subscript zero. Thus, up to radiative corrections of order α^2 we have

$$\mathscr{L}=\mathscr{L}^{(0)}+\mathscr{L}^{(1)}+\mathscr{L}^{(2)}=\mathscr{L}_R^{(0)}+\mathscr{L}_R^{(1)}(m^2)+\mathscr{L}_R^{(2)}(m^2), \tag{18}$$

where the expression on the left is defined in terms of the unrenormalized values of the field, charge and mass, and the expression on the right in terms of the renormalized ones. The renormalized mass is related to the bare mass by $m^2=m_0^2+\delta m^2$. The relation $\mathscr{L}_R^{(0)}=\mathscr{L}^{(0)}Z_3^{-1}$ between the renormalized and unremoralized Lagrange functions of the Maxwell field, leading to the field renormalization $\eta=\eta_0 Z_3^{-1/2}$, $\varepsilon=\varepsilon_0 Z_3^{-1/2}$, and the charge renormalization $e=e_0 Z_3^{1/2}$, is achieved with the Z_3-factor

$$Z_3^{-1}=1+\frac{\alpha_0}{12\pi}\ln\frac{1}{i\gamma m^2 s_0}+\frac{\alpha_0^2}{4\pi^2}\left(\ln\frac{1}{i\gamma m^2 s_0}-\frac{11}{12}\ln 2-\frac{13}{24}\right). \tag{19}$$

Owing to the use of the renormalized mass, only logarithmic divergences have remained in Z_3^{-1}. Whereas the coefficient of the first logarithm is well established, [15,16] the existing calculations of the coefficient in front of the second logarithm[15,16] do not agree with each other. The coefficient we have determined agrees with the one obtained by Zofia Bialynicka-Birula. [17]

Making use of the expression for Z_3 and Eq. (31) of Sec. 3, we obtain the following expansion:

$$\beta(\alpha)=\frac{1}{12}\frac{\alpha}{\pi}+\frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2+\ldots \tag{20}$$

for the Callan-Symanzik function of scalar QED.

The correction $\mathscr{L}_R^{(2)}$ has the following asymptotic properties:

$$\mathscr{L}_R^{(2)}=\frac{\alpha^3}{\pi m^4}\left[\frac{275}{2592}(\eta^2-\varepsilon^2)^2+\frac{4}{81}(\eta\varepsilon)^2\right],\quad \frac{e\eta}{m^2},\ \frac{e\varepsilon}{m^2}\ll 1, \tag{21}$$

$$\mathscr{L}_R^{(2)}=\frac{\alpha^2\eta^2}{8\pi^2}\left(\ln\frac{2e\eta}{\gamma\pi m^2}+a_2\right),\quad \frac{e\eta}{m^2}\gg 1,\quad \frac{e\varepsilon}{m^2}\lesssim 1, \tag{22}$$

Equation (21) does not exhibit an imaginary part which is exponentially small compared to the real part. In the case of a weak electric field the imaginary part of the Lagrange function(18) equals

$$\operatorname{Im}\mathscr{L}_R=\operatorname{Im}(\mathscr{L}_R^{(1)}+\mathscr{L}_R^{(2)})=\frac{\alpha\varepsilon^2\exp(-\pi m^2/e\varepsilon)}{4\pi^2}(1+\pi\alpha+\ldots),\quad e\varepsilon/m^2\ll 1. \tag{23}$$

We note that in spinor electrodynamics the corresponding limit is twice as large on account of the doubled number of states for a pair with vanishing orbital angular momentum projection.

Whereas the behavior of the real part of the Lagrange function in a weak field determines the charge and mass renormalization, the behavior of its imaginary part for a weak electric field determines the radiative correction δm^2 to the square of the bare mass, namely it fixes the constant 7/6 in the expression (16). If one chooses another constant b in place of 7/6 in the expression for δm^2, then b would appear in place of 7/6 also in the last term of (13). Then in the parentheses of Eq. (23) for the imaginary part of the Lagrange function of the weak field there would appear the additional term $(3\alpha m^2/4e\varepsilon)\times(b-7/6)$ which, then translated into the exponent of the exponential function would change the parameter m^2 by $-(3\alpha m^2/4\pi)(b-7/6)$, which is finite for $\varepsilon\to 0$. This would mean that m is not the mass of a real particle, since according to the physical principle of renormalization the observed mass already contains all the radiative correction. Thus, the parameter m which is related to the bare mass m_0 by $m_0^2=m_0^2+\delta m^2$ can be interpreted as the mass of a real particle only for $b=7/6$. The expression for δm^5 was verified by means of the mass operator and the position of the pole of the modified Green's function of the particle in the vacuum. Thus, the renormalization of the field strength, charge and mass is uniquely determined by the behavior of the exact Lagrange function in the weak field limit, i.e., by a boundary condition: its real part must be Maxwellian, and the imaginary part must be equal to

$$(e\varepsilon)^2 f(\alpha,\ \eta/\varepsilon)\exp(-\pi m^2/e\varepsilon).$$

The coefficient f characterizes the spin of the polarized charged field and is inessential for the mass renormalization.

We note that in the calculation of the exact photon propagator in the vacuum the charge and mass renormalizations can also be defined only by a boundary condition on the behavior of the propagator: $\operatorname{Re}d\to 1$ for $k^2\to 0$ and $\operatorname{Im}d\to-(1+4m^2k^{-2})^{1/2}g_1(\alpha)$ or $\operatorname{Im}d\to(1+4m^2k^{-2})^{3/2}g_2(\alpha)$ for $k^2\to-4m^2$, respectively for the spinor and scalar charged fields.

As is evident from Eq. (18), the Lagrange function is a renormalization-invariant quantity, i.e., it does not change when one replaces in it the unrenormalized parameters by the renormalized ones. This property stems from the *a priori* invariance of the amplitude $\exp i(W^{(0)}+W)$ for the vacuum to vacumm transition with respect to this substitution, and together with the fundamental properties of the Lagrange function it makes it into a convenient object for the investigations of questions of principle in quantum electrodynamics.

3. A COMPARISON OF THE VARIOUS RENORMALIZED CHARGES IN THE ASYMPTOTIC REGION

We consider three quantities which are invariant with respect to the renormalization group in QED (spinor or scalar).

1) The invariant charge $\alpha d_R(x, \alpha)$ defined by the exact propagator of the photon $k^{-2}d$ and depending on the fine structure constant α and the ratio $x = k^2/m^2$ of the square of the photon momentum to the square of the electron mass. [2]

2) The invariant charge $\alpha\, l_R(x, y, \alpha)^{-1}$ defined by the ratio $l_R = \mathscr{L}_R/\mathscr{L}_R^{(0)}$ of the Lagrange function $\mathscr{L}_R$ of a strong constant electromagnetic field to the Lagrange function $\mathscr{L}_R^{(0)}$ of the Maxwell field, and depending on α and the magnetic field η and the electric field ε in the frame in which they are parallel, and appropriately made dimensionless: $x = e\eta/m^2$, $y = e\varepsilon/m^2$. [12]

3) The bare coupling constant $\alpha_0 = \alpha Z_3(x, \alpha)^{-1}$, depending on α and the cutoff parameter $x = (im^2 s_0)^{-1}$ for which we use the proper times s_0, or more precisely, the spacelike interval is_0. [12]

As a consequence of the renormalizability of quantum electrodynamics, in the limit $x \to \infty$ all these quantities are described by their asymptotic (subscript ∞) functions which depend only on one variable $z = x\varphi(\alpha)$:

$$\alpha d_{R\infty}(x, \alpha) = \Phi_1(z), \quad \alpha l_{R\infty}^{-1}(x, \alpha) = \Phi_2(z), \quad \alpha Z_3(x, \alpha)^{-1} = \Phi_3(z), \tag{24}$$

where $\varphi(\alpha)$ is a universal function in quantum electrodynamics which occurs in the solution of the equation $\alpha_0 = \alpha Z_3(x, \alpha)^{-1}$ with respect to the cutoff parameter: $x^{-1} \equiv im^2 s_0 = \varphi(\alpha)\,\chi(\alpha_0)$. The factorization of the right-hand side into a product of functions of α and α_0 is a consequence of the same renormalizability.

Indeed, one can write the relation (18) in the form

$$\alpha_0^{-1} l\left(\frac{eF}{m^2}, \alpha_0, im^2 s_0\right) = \alpha^{-1} l_R\left(\frac{eF}{m^2}, \alpha\right), \tag{25}$$

if in the left-hand side one effects a mass renormalization and makes use of the invariance of the product $e_0F_0 = eF$. When one of the field parameters tends to infinity, e.g., $x = e\eta/m^2 \to \infty$, the asymptotic expression for l in the second approximation, $l = 1 + l^{(1)} + l^{(2)}$ does not depend either on m^2 or the second parameter $e\varepsilon$. We assume that this important property is valid for l in any order in α_0. Then $\lim l = l_\infty(ie\eta s_0, \alpha_0)$. Substituting into l_∞ the quantity $is_0 = m^2\varphi(\alpha, \alpha_0)$ obtained from the relation $\alpha = \alpha_0 Z_3(im^2 s_0, \alpha_0)$ we obtain according to (25)

$$\alpha_0^{-1} l_\infty\left(\frac{e\eta}{m^2}\varphi(\alpha, \alpha_0), \alpha_0\right) = \alpha^{-1} l_{R\infty}\left(\frac{e\eta}{m^2}, \alpha\right). \tag{26}$$

Since the right-hand side does not depend on α_0, the left-hand side must also be independent of α_0. This can happen only if $\varphi(\alpha, \alpha_0)$ factorizes into a product of functions of α and α_0, respectively:

$$im^2 s_0 = \varphi(\alpha, \alpha_0) = \varphi(\alpha)\chi(\alpha_0). \tag{27}$$

After α_0 disappears from it the left-hand side of (26) becomes a function only of $(e\eta/m^2)\varphi(\alpha)$, and thus, from Eqs. (26) and (27) we obtain for $\alpha l_{R\infty}^{-1}$ and $\alpha_0 = \alpha Z_3^{-1}$ the expressions (24).

Since, as we assume, the boundary conditions imposed on the exact Lagrange function and on the exact propagator determine the same coupling constant—the fine structure constant—the relation between α and α_0 is unique for the same regularization method and does not depend on whether it has been determined through the Lagrange function or through a calculation of the photon propagator. Consequently, when calculating $\alpha d_{R\infty}$ we obtain the expression (24) with the same function $\varphi(\alpha)$.

It is clear that the function $\varphi(\alpha)$ is defined up to a multiplicative constant, on which the invariant charges (24) do not depend, but the functions $\Phi_a(z)$ do. This constant is a parameter of the renormalization group. Since

$$x\varphi(\alpha) = \Phi_1^{-1}[\alpha d_{R\infty}(x, \alpha)] \tag{28}$$

etc., for $a = 2, 3$, then the functions $\Phi_a^{-1}(y)$ which are the inverses of $\Phi_a(z)$ are defined up to the same multiplicative constant. But the logarithmic derivatives

$$[\ln\varphi(\alpha)]' = 1/\alpha\beta(\alpha), \quad [\ln\Phi_a^{-1}(y)]' = 1/\psi_a(y), \tag{29}$$

which determine the Callan-Symanzik β function and the Gell-Mann–Low functions ψ_a are uniquely defined and are renormalization-invariant characteristics of the representation (24).

Equations (28), (29) imply the functional relations

$$\alpha\beta(\alpha)\frac{\partial}{\partial\alpha}[\alpha d_{R\infty}(x, \alpha)] = x\frac{\partial}{\partial x}[\alpha d_{R\infty}(x, \alpha)] = \psi_1[\alpha d_{R\infty}(x, \alpha)] \tag{30}$$

etc., for $a = 2, 3$. The first of these is the Callan-Symanzik equation[19,20] satisfied by the functions $\alpha d_{R\infty}$, $\alpha l_{R\infty}^{-1}$, αZ_3^{-1}. The second relation is a differential form of the Gell-Mann–Low equation. [2] A consequence is the functional relation between the β and ψ_a functions. [20,21]

If in (27) one considers α a function of $x = (im^2 s_0)^{-1}$ and α_0 then differentiation yields

$$\alpha\beta(\alpha) = -x\left(\frac{\partial\alpha}{\partial x}\right)_{\alpha_0} = -\frac{\alpha}{Z_3}x\left(\frac{\partial Z_3}{\partial x}\right)_{\alpha_0}. \tag{31}$$

In general we obtain from (28) in a similar manner

$$\alpha\beta(\alpha) = -x\left(\frac{\partial\alpha}{\partial x}\right)_{\alpha_{a\infty}}, \tag{32}$$

where $\alpha_{a\infty}$ is the a-th renormalization-invariant charge in the asymptotic x region.

Thus, if the physical meaning of the parameters α, m is fixed by the renormalization conditions, any renormalization-invariant charge and the Z_3 factor obtained through any regularization method (momentum cutoff, proper-time cutoff, etc.) contain universal information on quantum electrodynamics in the form of the same function $\alpha\beta(\alpha)$. We use for α, m the fine structure constant and the physical electron mass. This physical meaning of α, m is guaranteed by definite boundary conditions for the exact Lagrange function for $F \to 0$, or for the exact photon propagator for $k^2 \to 0$ and

$k^2 \to -4m^2$, cf. Sec. 2. To another physical meaning of of α, m would correspond to a different function $\alpha\beta(\alpha)$. In general one can say about the Callan-Symanzik β-function and the Gell-Mann–Low ψ_a-functions that their form is determined by the physical meaning of their argument.

The explicit forms of these functions are known only within the framework of perturbation theory. According to perturbation theory the functions $d_{R\infty}^{-1}$, $l_{R\infty}^{-1}$, Z_3 can be expanded in double series in powers of α and $\ln x$:

$$l_{R\infty}(x,\alpha)=1+\frac{\alpha}{\pi}(a_{10}+a_{11}\ln x)+\sum_{n=2}^{\infty}\left(\frac{\alpha}{\pi}\right)^n\sum_{k=0}^{n-1}a_{nk}\ln^k x. \tag{33}$$

According to the Callan-Symanzik equation (30) the coefficients a_{nk} can be expressed in terms of a_{n0} and the coefficients of the power series[1] for $\beta(\alpha)$

$$\beta(\alpha)=\sum_{k=1}^{\infty}\beta_k\left(\frac{\alpha}{\pi}\right)^k=\frac{1}{3}\frac{\alpha}{\pi}+\frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2-\frac{121}{288}\left(\frac{\alpha}{\pi}\right)^3+\ldots, \tag{34}$$

via the recurrence relations

$$ka_{nk}=-\beta_n\delta_{k1}+\sum_{i=k}^{n-1}(i-1)a_{ik-1}\beta_{n-i}, \quad k\geqslant 1. \tag{35}$$

It follows from the last relations that in terms of the order α^n, $n\geqslant 2$, the highest power of the logarithm is $n-1$. The coefficients of the highest powers of the logarithms in (33) are determined only by the coefficients β_1, β_2:

$$a_{11}=-\beta_1, \quad a_{nn-1}=-\beta_2\beta_1^{n-2}/(n-1), \tag{36}$$

and the coefficients a_{nn-r}, $2\leqslant r\leqslant n-1$ of the other logarithms are determined by the coefficients β_1, $\beta_2, \ldots,$ β_{r+1} and the constants a_{20}, $a_{30}, \ldots, a_{r0}$. The constants a_{n0} reflect the individual traits of the functions $\Phi_a(z)$ and their one-parameter arbitrariness due to the renormalization group.

The series (33) for the functions $d_{R\infty}^{-1}$, $l_{R\infty}$, Z_3 can differ only in the constants a_{n0}. On the other hand, under a group transformation $\varphi(\alpha)\to t^{-1}\varphi_a(\alpha)$, $\Psi(z)\to\Phi_a(tz)$ the constants a_{n0} transform in the following manner:

$$\begin{gathered}a_{10}\to a_{10}-\beta_1\ln t, \quad a_{20}\to a_{20}-\beta_2\ln t,\\ a_{30}\to a_{30}-(\beta_3-\beta_1a_{20})\ln t-{}^1/{}_2\beta_1\beta_2\ln^2 t,\\ a_{40}\to a_{40}-(\beta_4-\beta_2a_{20}-2\beta_1a_{30})\ln t\\ -({}^1/{}_2\beta_2^2+\beta_1\beta_3-\beta_1^2a_{20})\ln^2 t-{}^1/{}_3\beta_1^2\beta_2\ln^3 t,\ldots\end{gathered} \tag{37}$$

Such a transformation does not change the form of $\Phi_a(z)$ described by the invariant function $\psi_a(y)$. Therefore the differences in the form of the functions $\Phi_a(z)$ which do not reduce to a multiplicative change in the argument are reflected in the differences in their Gell-Mann–Low functions. Expanding the latter in series

$$\psi_a(y)=y\sum_{n=1}^{\infty}\psi_n^{(a)}(y/\pi)^n \tag{38}$$

and using (30) to relate the coefficients $\psi_n^{(a)}$ with β_n and $a_{n0}^{(a)}$, we obtain (the index a in the coefficient ψ_n and constants a_{n0} is understood)

$$\begin{gathered}\psi_1=\beta_1, \quad \psi_2=\beta_2, \quad \psi_3=\beta_3+\beta_2a_{10}-\beta_1a_{20},\\ \psi_4=\beta_4+2\beta_3a_{10}+\beta_2a_{10}^2-2\beta_1(a_{30}+a_{10}a_{20}),\\ \psi_5=\beta_5+3\beta_4a_{10}+\beta_3(3a_{10}^2+a_{20})+\beta_2(a_{10}^3-a_{30})\\ -\beta_1(6a_{10}a_{30}+3a_{10}^2a_{20}+2a_{20}^2+3a_{40}),\ldots.\end{gathered} \tag{39}$$

The constants a_{n0} entered into the right-hand sides of the relations (39) in combinations which are invariant with respect to the transformations (37). It is just by such combinations that the functions $\psi_a(y)$ differ from the universal function $y\beta(y)$ and from each other, and the difference can start only from terms of third order, $n\geqq 3$, and in nth order is determined by the combination of β_k and a_{k0} of order $k\leqq n-1$.

Table I lists the constants a_{10}, a_{20}, and the coefficients ψ_3 determined from (39) and (34) for the functions $d_{R\infty}^{-1}$, $l_{R\infty}$, and Z_3 of spinor electrodynamics. Here $\ln\gamma = 0.577$ is the Euler constant and $\zeta(x)$ is the Riemann zeta function. The data for the $d_{R\infty}^{-1}$ function have been obtained in[23–26, 7] and for $l_{R\infty}$ and Z_3 in the present paper, based on Ref.[12] cf. also the preliminary publication.[27] The constant a_{20} for $l_{R\infty}$ is defined by the following expression containing a double integral:

$$a_{20}=-\frac{5}{12}\ln\pi-\frac{5}{4}+\frac{1}{\pi^2}\zeta'(2)+\frac{1}{2}\int_0^{1/2}d\xi\int_0^{\infty}du\ln uQ'(u,\xi), \tag{40}$$

where the function $Q(u, \xi)$ equals

$$\begin{gathered}Q(u,\xi)=\frac{5}{3}+\frac{2[\operatorname{ch}u(1-2\xi)-uw/\operatorname{sh}u]}{(w-1)\operatorname{sh}u\xi\operatorname{sh}u(1-\xi)}\left(1-\frac{\ln w}{w-1}\right)\\ +\frac{1}{\xi(1-\xi)}\left[\frac{2w}{\operatorname{sh}^2u}-\frac{2}{u^2}-\frac{\operatorname{cth}z}{z}-\frac{1}{\operatorname{sh}^2z}+\frac{2}{z^2}\right],\\ w=\frac{\xi(1-\xi)u\operatorname{sh}u}{\operatorname{sh}u\xi\operatorname{sh}u(1-\xi)}, \quad z=u(1-\xi),\end{gathered} \tag{41}$$

and the derivative is taken with respect to u. This integral was done numerically:

$$\int_0^{1/2}d\xi\int_0^{\infty}du\ln uQ'(u,\xi)=\int_0^{\infty}du\left[\frac{5}{6(1+u)}-\frac{1}{u}\int_0^{1/2}d\xi\,Q(u,\xi)\right]=1.886784\ldots.$$

Thus, all three functions $\psi_a(y)$ are different, and in distinction from $\psi_1(y)$, the functions $\psi_2(y)$ and $\psi_3(y)$ contain negative third-order terms. The latter enhances the possibility of these functions vanishing for $y\sim 1$. As is well known, the existence of a sufficiently strong zero at $y=\alpha_*$ in the Gell-Mann–Low function means that the corresponding invariant charge becomes α_* in the limit $x\to\infty$.[2] For internal consistency of QED the finiteness of the limit $\alpha_*=\lim\alpha_0(x,\alpha)$, $x\to\infty$ of the bare charge, with respect to which one expands the initial perturbation series, is necessary. If this fact has physical significance, then the limit α_* must not depend on the cutoff method which determines the val-

Table I.

	x	a_{10}	a_{20}	ψ_3
$d_{R\infty}^{-1}$	k^2/m^2	${}^5/{}_9$	${}^5/{}_{24}-\zeta(3)$	${}^1/{}_3\zeta(3)-{}^{101}/{}_{288}=0.049991\ldots$
$l_{R\infty}$	$e\eta/\gamma\pi m^2$	$-2\pi^{-2}\zeta'(2)$	$-0.878572\ldots$	$-0.079785\ldots$
Z_3	$(i\gamma m^2 s_0)^{-1}$	0	${}^5/{}_{12}\ln 2-{}^5/{}_6$	$-{}^{41}/{}_{288}-{}^5/{}_{36}\ln 2=-0.238631\ldots$

ues of the constants a_{n0} in the representation (33) for Z_3, i. e., the form of the function $\Phi_3(z)$. A change in the cutoff method allows one to make the function $\Phi_3(z)$ agree either with the function $\Phi_1(z)$ or with $\Phi_2(z)$, and consequently the limits of these functions for $x\to\infty$ must also be finite and equal to α_*. The finiteness of all limiting values is possible if $\beta(\alpha)=0$ for $\alpha=1/137$ (cf. (30) and [21]), or for $\alpha=\alpha_*$ which is more likely, or, if there exists yet another reason for the vanishing of all $\psi_a(y)$ for the same $y=\alpha_*\sim 1$.

4. INTEGRAL TRANSFORMATIONS OF THE RENORMALIZATION-INVARIANT CHARGES

In the Introduction and in Sec. 3 we have produced general arguments in favor of the fact that all the renormalization-invariant charges have the same limit, if that limit exists at least for one of them. It would be desirable to give a mathematical proof of this assertion.[2)] In this section we can only indicate the existence of some renormalization-invariant charges which differ in the asymptotic region, i. e., have different Gell-Mann–Low functions, but have manifestly identical asymptotic limits.

We consider the charge $eq(m^2r^2, \alpha)$ situated within a sphere of radius r surrounding a point source of the field with physical charge e:

$$\frac{eq(m^2r^2,\alpha)}{4\pi r}=e\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\mathbf{x}}\frac{d(\mathbf{k}^2/m^2,\alpha)}{\mathbf{k}^2} \tag{42}$$

or

$$q(t,\alpha)=\frac{2}{\pi}\int_0^\infty\frac{dz}{z}\sin z\, d\left(\frac{z^2}{t},\alpha\right),\quad t=m^2r^2. \tag{43}$$

It follows from this integral transformation that $\alpha q(t,\alpha)$ is a renormalization-invariant quantity which for $t=\infty$ and $t=0$ has values coinciding with $\alpha d(0,\alpha)=\alpha$ and $\alpha d(\infty,\alpha)=\alpha_*$. Using the Källén-Lehmann representation[28,29]

$$d(x,\alpha)=1+x\int_4^\infty\frac{dz\,\rho(z,\alpha)}{z(z+x-i\varepsilon)},\quad \rho(z,\alpha)=-\frac{1}{\pi}\operatorname{Im} d(-z,\alpha), \tag{44}$$

we obtain for $q(t,\alpha)$ the following expression:

$$q(t,\alpha)=1+\int_4^\infty\frac{dz}{z}\rho(z,\alpha)\exp[-(zt)^{1/2}]. \tag{45}$$

The first term of the expansion of the spectral function $\rho(z,\alpha)=\rho^{(1)}(z)+\rho^{(2)}(z)+\ldots$ was found by Schwinger,[30] the second term was calculated by Källén and Sabry[24]:

$$\rho^{(1)}(z)=\frac{\alpha}{3\pi}\left(1+\frac{2}{z}\right)\left(1-\frac{4}{z}\right)^{1/2}, \tag{46}$$

$$\begin{aligned}\rho^{(2)}(z)=&-\left(\frac{\alpha}{\pi}\right)^2\Big\{\left(\frac{13}{108}+\frac{7}{216}\tau^{-1}+\frac{1}{9}\tau^{-2}\right)(1-\tau^{-1})^{1/2}\\&+\left(-\frac{22}{9}+\frac{1}{3}\tau^{-1}+\frac{5}{8}\tau^{-2}+\frac{1}{9}\tau^{-3}\right)\ln(\tau^{1/2}+(\tau-1)^{1/2})\\&+\left(\frac{2}{3}+\frac{1}{3}\tau^{-1}\right)(1-\tau^{-1})^{1/2}\ln[8\tau^{1/2}(\tau-1)]\\&+\left(-\frac{2}{3}+\frac{1}{6}\tau^{-2}\right)\int_\tau^\infty dy\left[\frac{3y-1}{y(y-1)}\ln(y^{1/2}+(y-1)^{1/2})-\frac{\ln[8y^{1/2}(y-1)]}{(y(y-1))^{1/2}}\right]\Big\},\\&z=4\tau.\end{aligned} \tag{47}$$

With the help of this spectral function we find, according to (45) the following expansion for q in the asymptotic region $t=m^2r^2\to 0$ or $x\equiv(\gamma mr)^{-2}\to\infty$:

$$q_\infty(t,\alpha)^{-1}=1+\frac{\alpha}{\pi}\left[\frac{5}{9}-\frac{1}{3}\ln x\right]+\left(\frac{\alpha}{\pi}\right)^2\left[\frac{5}{24}-\frac{\pi^2}{27}-\zeta(3)-\frac{1}{4}\ln x\right]+\ldots. \tag{48}$$

This expansion has the common structure (33) of the asymptotic behavior of invariant charges. The coefficient ψ_3 extracted from this expansion and Eq. (39) is

$$\psi_3={}^1/_3\zeta(3)-{}^{101}/_{288}+\pi^2/81=0{,}171838.$$

Thus, the invariant charges αd and αq have different Gell-Mann–Low functions but the same limits for $x\to\infty$, if those limits exist.

It is easy to see that the Källén-Lehmann representation (44) can also be written in the form of a Laplace integral transform:

$$d(x,\alpha)=1+x\int_0^\infty dt\, e^{-xt}\bar\rho(t,\alpha)=x\int_0^\infty dt\, e^{-xt}q_1(t,\alpha), \tag{49}$$

where

$$q_1(t,\alpha)=1+\bar\rho(t,\alpha)=1+\int_4^\infty\frac{dz}{z}\rho(z,\alpha)e^{-zt}. \tag{50}$$

It is obvious that $\alpha q_1(t,\alpha)$ is a renormalization-invariant quantity which for $t=\infty$ and $t=0$ takes respectively the values α and α_*. A calculation of $q_1(t,\alpha)$ with the help of (50) and the spectral function above leads to the following result in the asymptotic region $t\to 0$ or $x=(\gamma t)^{-1}\to\infty$:

$$\begin{aligned}q_{1\infty}(t,\alpha)^{-1}=&1+\frac{\alpha}{\pi}\left[\frac{5}{9}-\frac{1}{3}\ln x\right]\\&+\left(\frac{\alpha}{\pi}\right)^2\left[\frac{5}{24}+\frac{\pi^2}{54}-\zeta(3)-\frac{1}{4}\ln x\right]+\ldots.\end{aligned} \tag{51}$$

Thus, to the invariant charge αq_1 corresponds the Gell-Mann–Low function with negative coefficient

$$\psi_3={}^1/_3\zeta(3)-{}^{101}/_{288}-\pi^2/162=-0.010932\ldots$$

Equations (45) and (50) admit the generalization

$$q_\nu(t,\alpha)=1+\int_4^\infty\frac{dz}{z}\rho(z,\alpha)e^{-(zt)^\nu}, \tag{52}$$

where ν is a positive parameter. It is obvious that αq_ν is a renormalization-invariant quantity which at the points $t=\infty$ and $t=0$ takes on the values α and α_*. It coincides with αq for $\nu=1/2$ and with αq_1 for $\nu=1$. In the asymptotic region $t\to 0$ or $x=(\gamma^{1/\nu}t)^{-1}\to\infty$ the function $q_\nu(t,\alpha)$ behaves in the following manner:

$$\begin{aligned}q_{\nu\infty}(t,\alpha)^{-1}=&1+\frac{\alpha}{\pi}\left[\frac{5}{9}-\frac{1}{3}\ln x\right]\\&+\left(\frac{\alpha}{\pi}\right)^2\left[\frac{5}{24}-\zeta(3)+\frac{\pi^2}{27}\left(1-\frac{1}{2\nu^2}\right)-\frac{1}{4}\ln x\right]+\ldots.\end{aligned} \tag{53}$$

Thus, the invariant charge αq_ν has a Gell-Mann–Low function with third expansion coefficient

$$\psi_3=\frac{1}{3}\zeta(3)-\frac{101}{288}-\frac{\pi^2}{81}\left(1-\frac{1}{2\nu^2}\right), \quad (54)$$

depending on ν. As ν varies from 0 to ∞ the coefficient ψ_3 decreases monotonically from $+\infty$ to $(1/3)\zeta(3) - 101/288 - \pi^2/81 = -0.071856\ldots$ We call attention to the fact that the integral transforms considered here do not allow one to make ψ_3 into a large negative number. Consequently, if the Gell-Mann–Low function has a positive zero α_* this zero cannot be small. Nevertheless, a richer family of transformations could decrease ψ_3 further.

The author is grateful to V. O. Papanyan for checking the calculations of the last section.

1) At present in spinor electrodynamics the first three terms in the expansion of the β-function are known.[22] The terms are listed in Eq. (34). For scalar electrodynamics $\beta_1=\frac{1}{12}$ and $\beta_2=\frac{1}{4}$ as follows from Eqs. (19), (31) or (22), (36).

2) Furthermore, the equality of the limits of the d and Z_3^{-1} functions follows from the spectral representations.

[1] L. D. Landau, A. A. Abrikosov, and I. M. Khalatnikov, Dokl. Akad. Nauk SSSR **95**, 773, 1177; **96**, 261 (1954).
[2] M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).
[3] L. D. Landau and I. Ya. Pomeranchuk, Dokl. Akad. Nauk SSSR **102**, 489 (1955).
[4] E. S. Fradkin, Zh. Eksp. Teor. Fiz. **28**, 750 (1955) [Sov. Phys. JETP **1**, 606 (1956)].
[5] K. Johnson, M. Baker, and R. Willey, Phys. Rev. **136**, B1111 (1964).
[6] K. Johnson, R. Willey, and M. Baker, Phys. Rev. **163**, 1699 (1967).
[7] M. Baker and K. Johnson, Phys. Rev. **183**, 1292 (1969); **D3**, 2516, 2541 (1971).
[8] K. Johnson and M. Baker, Phys. Rev. **D8**, 1110 (1973).
[9] W. Heisenberg and H. Euler. Z. Physik **98**, 714 (1936).
[10] V. Weisskopf, Kgl. Dan. Vid. Sesk. Mat.-Fys. Medd. **14**, No. 6 (1936).
[11] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[12] V. I. Ritus, Zh. Eksp. Teor. Fiz. **69**, 1517 (1975) [Sov. Phys. JETP **42**, 774 (1975)].
[13] H. M. Fried, Functional Methods and Models in Quantum Field Theory, MIT Press, Cambridge, MA, 1972; I. Bialynicki-Birula and Z. Bialynicka-Birula, Quantum Electrodynamics, Pergamon Press, Oxford, 1975.
[14] V. A. Fock, Sow. Phys. **12**, 404 (1937).
[15] M. Neumann and W. H. Furry, Phys. Rev. **76**, 1677 (1949).
[16] L. P. Gor'kov and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **31**, 1062 (1956) [Sov. Phys. JETP **4**, 777 (1957)].
[17] Z. Bialynicka-Birula, Bull. Acad. Polon. Sci. **13**, 369 (1965).
[18] I.-J. Kim and C. R. Hagen, Phys. Rev. **D2**, 1511 (1970).
[19] C. Callan Jr., Phys. Rev. **D2**, 1541 (1970).
[20] K. Symanzik, Commun. Math. Phys. **18**, 227 (1970); **23** 49 (1971).
[21] S. Adler, Phys. Rev. **D5**, 3021 (1972).
[22] E. de Rafael and J. L. Rosner, Ann. Phys. (N.Y.) **82**, 369 (1974).
[23] R. Jost and J. M. Luttinger, Helv. Phys. Acta, **23**, 201 (1950).
[24] G. Källén and A. Sabry, Kd. Dan. Vid. Selsk. Mat. Fys. Medd. **29**, No. 17 (1955).
[25] C. R. Hagen and M. A. Samuel, Phys. Rev. Lett. **20**, 1405 (1968).
[26] B. E. Lautrup and E. de Rafael, Phys. Rev. **174**, 1835 (1968).
[27] V. I. Ritus, Phys. Lett. **65B**, 355 (1976).
[28] G. Källén, Helv. Phys. Acta **25**, 417 (1952).
[29] H. Lehmann, Nuovo Cimento **11**, 342 (1954).
[30] J. Schwinger, Phys. Rev. **75**, 651 (1949).

Translated by M. E. Mayer