General relativistic self-similar solutions with a spherical shock wave

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It is shown that the class of self-similar spherically symmetric solutions of Einstein's equations includes solutions with expanding and collapsing shock waves. The asymptotic behavior of these solutions and the topology of the spacelike sections are investigated. Attention is drawn to solutions with expanding shock waves in which the variation of the gas velocity after the passage of the shock wave has an oscillatory nature.

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INTRODUCTION

In the present paper, we investigate solutions of Einstein's equations and the equations of hydrodynamics that describe the propagation of a spherical shock wave in an ultrarelativistic gas[1] with equation of state $p=k\varepsilon$ (p is the pressure and ε the energy density, $0<k<1$). It is natural to seek solutions with shock waves, as in classical gas dynamics,[2] in the class of self-similar spherically symmetric solutions. This class of solutions in the general theory of relativity was considered in[3–8]; the metric for these solutions has the form

$$ds^2=\exp\{\nu(ct/R)\}c^2dt^2-\exp\{\lambda(ct/R)\}dR^2-R^2(d\theta^2+\sin^2\theta\, d\varphi^2). \quad (1)$$

In[3–5] it is shown that Einstein's equations for self-similar spherically symmetric metrics reduce to a system of three ordinary differential equations. However, this system was not investigated in[3–8] (the investigation is complicated by the fact that the system does not reduce to a two-dimensional one).

In the present paper, we represent the metric of self-similar solutions in a conformally static form (this representation of the metric was not considered in[3–8]). In Sec. 2, we investigate in detail the system of Einstein's equations and the equations of hydrodynamics by the methods of the qualitative theory of differential equations (applied earlier in[9,10]). In Sec. 4, we find solutions with expanding shock wave (solution of the self-similar problem of an explosion in general relativity) and solutions with a collapsing shock wave that arises during the collapse of matter in the presence of pressure.

These problems have a number of properties in common with the analogous problems in classical gas dynamics (see[2,11–15,19]). 1) The system of Einstein's equations has surfaces $V_\pm$ across which the solutions cannot be continued; these separate the solutions with subsonic and supersonic flow of the gas. The impossibility of continuing the solutions leads to the formation of shock waves. 2) The explosion problem has a class of solutions corresponding to integral curves filling the two-dimensional separatrix Z of a certain isolated singular point. 3) The solutions with collapsing shock wave correspond to integral curves passing through a certain interval I_1+I_2 of singular points on the surface V_- discussed above.

However, besides these common properties, there are a number of important properties characteristic of these problems. 1) In the explosion problem there is a further class of solutions corresponding to integral curves that fill a two-dimensional separatrix passing through an interval I_1 of singular points on the surface V_-. In these solutions, the spacelike sections have the same topology as in the Kruskal solution.[1] 2) In the explosion problem, there are solutions in which after the passage of the shock wave the coordinate R (see Eq. (1)) along the integral curves of the motion of the gas varies nonmonotonically and goes through an arbitrary finite number of oscillations. 3) The behavior of the solutions depends strongly on whether the self-similar variable $\zeta=\zeta(ct/R)$ is spacelike (i.e. $\zeta_{,i}\zeta_{,j}g^{ij}<0$) or timelike. 4) In the solutions with collapsing shock wave, the spacelike sections have the same topology as in the Kruskal solution.

§1. BASIC EQUATIONS

1. For spherically symmetric metrics, Einstein's equations $R_{ij}-\frac{1}{2}g_{ij}R=\alpha T_{ij}$ with hydrodynamic energy–momentum tensor of the matter

$$T_{ij}=(p+\varepsilon)u_iu_j-pg_{ij},\quad p=k\varepsilon,\quad 0\leqslant k\leqslant 1, \quad (1.1)$$

are integrable only[1] for $k=0$ ($\alpha=8\pi k/c^4$, c is the velocity of light, and k is the gravitational constant). For $0<k\leqslant 1$, metrics that depend on only one coordinate (on t or on R) have been investigated in detail.[16] In this paper we consider for $0<k<1$ spherically symmetric solutions of the conformally static form

$$ds^2=l^2e^{2\tau}(\sigma e^{\nu(r)}d\tau^2-\sigma e^{\lambda(r)}dr^2-r^2d\Omega^2), \quad (1.2)$$

where $d\Omega^2=d\theta^2+\sin^2\theta d\varphi^2$, $\sigma=\pm1$, l is a constant with dimensions of length, and the variables τ and r are dimensionless. Besides (1.2), one can consider solutions with metric $d\Omega^2$ of constant negative curvature ($d\Omega^2=d\theta^2+\sinh^2\theta d\varphi^2$) and zero curvature ($d\Omega^2=d\theta^2+d\varphi^2$). After the substitution $T=e^\tau$ the metric (1.2) takes the form

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$$l^{-2}\,ds^2=\sigma e^{\nu(r)}dT^2-T^2(\sigma e^{\lambda(r)}\,dr^2+r^2\,d\Omega^2). \tag{1.3}$$

The energy density ε and the matter four-velocity u^i are given by

$$\varepsilon=\varepsilon_0\bar{\varepsilon}(r)\,e^{-2\tau},$$
$$(u^0,u^1,u^2,u^3)=e^{-\tau}\left[\left(\frac{\sigma e^{-\nu}}{1-u^2}\right)^{1/2},u\left(\frac{\sigma e^{-\lambda}}{1-u^2}\right)^{1/2},0,0\right], \tag{1.4}$$

where ε_0 is a constant with the dimensions of energy density. If $\sigma=+1$, then $u(r)=v/c$, $|u|<1$, but if $\sigma=-1$ then $u(r)=c/v$, $|u|>1$, where $v(r)$ is the three-dimensional radial velocity of the gas.

Under a certain coordinate transformation (see Sec. 3), the metric (1.2) goes over into the metric (1) of self-similar solutions of Einstein's equations.

2. To investigate solutions of the form (1.2), it is convenient to use the system consisting of two of Einstein's equations and one equation of hydrodynamics:

$$\alpha(T_0^0-T_1^1)=R_0^0-R_1^1,\quad \alpha T_2^2=R_2^2-{}^1/{}_2R,\quad T_{i;k}^k=0. \tag{1.5}$$

From the equation $R_{01}=\alpha T_{01}$, we obtain an expression for the energy density:

$$\mu=\frac{\bar{\varepsilon}}{|1-u^2|}=-\frac{\nu' e^{-(\nu+\lambda)/2}}{(1+k)\beta u},\quad \beta=\frac{8\pi k l^2\varepsilon_0}{c^4}. \tag{1.6}$$

It follows from the condition of positivity of $\bar{\varepsilon}$ that $\operatorname{sign}u=-\operatorname{sign}\nu'$.

The system (1.5) after substitution of the expression (1.6) and transition to the new variables

$$\zeta=\ln r,\quad Q=re^{(\lambda-\nu)/2}>0,\quad w=d\nu/d\zeta=\nu' r \tag{1.7}$$

is transformed into the system

$$dQ/d\zeta=\dot{Q}=Q(1-w-Q^2-wQ(1+u^2)/2u),$$
$$\dot{w}=w\left[-1-w-Q^2-Q\left\{\frac{w(1+u^2)}{2u}+\frac{2k(1-u^2)}{(1+k)u}+\frac{1+u^2}{u}\right\}\right], \tag{1.8}$$
$$\dot{u}=-\frac{1-u^2}{u^2-k}\left[\frac{1}{2}wu(1-k)-2ku+\frac{Q\{u^2(1-k)-k(1+3k)\}}{1+k}\right].$$

Here, $\operatorname{sign}u=-\operatorname{sign}w$.

The Einstein equation $R_1^1-\frac{1}{2}R=\alpha T_1^1$ after the substitution (1.6) determines a constraint that is preserved by virtue of the system (1.8):

$$L=w+1-Q^2+wQ(k+u^2)u^{-1}(1+k)^{-1}=\sigma\varkappa e^{\lambda}. \tag{1.9}$$

Here, the constant $\varkappa=\pm1$ or 0 is equal to the constant curvature of the two-dimensional metric $d\Omega^2$ (see (1.2)). In accordance with (1.9), the system (1.8) describes spherically symmetric solutions ($\varkappa=\pm1$) in the region $\sigma L>0$; solutions with negative curvature, i.e., $\varkappa=-1$, corresponding to the symmetry group $SL(2,R)$ in the region $\sigma L<0$; and solutions with $\varkappa=0$, corresponding to flat symmetry on the manifold $L=0$ (it is easy to verify directly that the manifold $L=0$ is invariant for integral curves of the system (1.8)). Two forms of the metric (1.2) ($\sigma=\pm1$) are described by the system (1.8) in different regions: if $\sigma=+1$, then $|u|<1$ but if $\sigma=-1$ then $|u|>1$.

On the invariant manifolds $u=\pm1$, the system (1.8) describes directed fluxes of neutrinos (for classical specification of the energy–momentum tensor, see[17]) and can be integrated explicitly.

The presence of the denominator u^2-k in the third equation in (1.8) means that the solutions for certain $r=r_0$ cannot be continued through the surfaces $V_\pm$: $u=\pm k^{1/2}$ since the vector field of the system (1.8) on the two sides of this surface ($u^2-k<0$ and $u^2-k>0$) is directed in opposite directions and in the limit $|u|\to k^{1/2}$ is perpendicular to the surfaces $V_\pm$ since $|\dot{u}|\to\infty$, and and $|\dot{w}|$ and $|\dot{Q}|$ are bounded.

Because of this impossibility of continuing the solutions of the system (1.8), a shock wave is formed in the actual gas flow, just as in classical gas dynamics. The real solution with shock wave can exist for all values of r: $0<r<\infty$.

3. The position of the shock front is determined by a certain constant value of the coordinate r. Therefore, the coordinate system (1.2) is comoving with the shock wave. On the shock front, one fits the solutions of the system (1.8) on the different sides of the surfaces $V_\pm$, i.e., the subsonic solution ($|u|^2<k=dp/d\varepsilon$) is fitted to the supersonic ($|u|>k^{1/2}$). One can consider a more general situation for which the matter in the region behind the shock wave has the equation of state $p=k_1\varepsilon$ but in the region in front of it the equation of state $p=k_2\varepsilon$ ($0\leqslant k_2\leqslant k_1$). The most interesting cases are $k_2=k_1$ and $k_2=0$. The limiting values of the parameters on the two sides of the shock are related by the following natural conditions. 1) The metric coefficients ν, λ, and r are everywhere continuous. 2) At the shock, the conservation laws (see[13]) are satisfied: $[T_i^k n_k]=0$ (here, $n_k=(0,1,0,0)$ is a vector orthogonal to the shock front). These conditions lead to the following equations (the indices 1 and 2 determine the parameters on the two sides of the shock):

$$\bar{\varepsilon}_1(u_1^2+k_1)/(1-u_1^2)=\bar{\varepsilon}_2(u_2^2+k_2)/(1-u_2^2)\quad (i=1),$$
$$(1+k_1)\bar{\varepsilon}_1u_1/(1-u_1^2)=(1+k_2)\bar{\varepsilon}_2u_2/(1-u_2^2)\quad (i=0). \tag{1.10}$$

From this we obtain the relations

$$\frac{(1+k_1)u_1}{u_1^2+k_1}=\frac{(1+k_2)u_2}{u_2^2+k_2}\qquad \frac{\bar{\varepsilon}_1}{\bar{\varepsilon}_2}=\frac{1+k_2}{1+k_1}\frac{u_2}{u_1}\frac{1-u_1^2}{1-u_2^2}, \tag{1.11}$$

which determine u_2 and $\bar{\varepsilon}_2$ from the u_1 and $\bar{\varepsilon}_1$ obtained from the solution, the supersonic value ($u_2^2>k_2$) being chosen from the two values of u_2 (for $k_2\leqslant k_1$). It follows from the condition $|u_2|<1$ and (1.11) that $|u_1|>k_1$. If $k_2=k_1$, then $u_1u_2=k_1$, and if $k_2=0$ then $u_2=(u_1^2+k_1)/u_1(1+k_1)$. It follows from the expression (1.9), the continuity of the functions ν, λ, r, and the first relation in (1.11) that the function w (and $\nu'=wr$) is continuous on the shock front. Thus, the functions ν, λ, r, ν', w, Q are continuous across the shock, and the discontinuities of u and $\bar{\varepsilon}$ are determined by (1.11). It is natural to say that the shock is strong if $|u_2|\approx1$. Then $|u_1|\approx k_1$ and

$$\varepsilon_1/\varepsilon_2 \approx (1-k_1)(1+k_2)/k_1(1-u_2^2) \gg 1.$$

In the actual solution of the problem, the shock must be fairly strong. Indeed, if it is weak, i.e., $\varepsilon_1 \approx \varepsilon_2$, $u_1 \approx u_2 \approx \pm k^{1/2}$, the impossibility of continuing such a solution cannot be avoided because $|\dot{u}| = \infty$ on the surfaces $V_\pm$.

§2. INVESTIGATION OF THE DYNAMICAL SYSTEM

We consider the system (1.9) for spacelike variable r (i.e. $\sigma = +1$ and $|u| < 1$). The system (1.9) is defined in the region s_1: $Q > 0$, $|u| < 1$, $L \geq 0$, and $\operatorname{sign} u = -\operatorname{sign} w$. To investigate the system (1.9) in the region S_1 by the methods of the qualitative theory of differential equations, we transform this system into one defined on a certain three-dimensional manifold S with boundary Γ (one can imagine S as a parallelepiped, and then Γ consists of the six faces).

1. We show first of all that in each solution for $u > 0$ it becomes impossible to continue the solutions, and this cannot be avoided by introducing a discontinuity (it is helpful to remember that the region $0 \leq u \leq 1$, $L \leq 0$ is bounded: here, $-1 \leq w \leq 0$, $0 \leq Q \leq 1$). This assertion follows readily from the behavior of the integral curves of the system (1.9) in the region $1 > u > 0$, $L \geq 0$; for $0 < u < k^{1/2}$, by virtue of $\dot{u} < 0$, $\dot{w} > 0$, all integral curves with increasing r leave the surface $V_+(u = k^{1/2})$ at a certain finite $r = r_1$ and at a certain finite $r = r_0 > r_1$ enter the line $l_1(u = w = 0, 0 \leq Q \leq 1)$. All integral curves in the region $k^{1/2} < u < 1$ leave the surface V_+ at $r = r_1$ and in the limit $r \to \infty$ enter, by virtue of $w > 0$, one of the singular points: $Y_1(w = 0, Q = u = 1)$ or Y_2:

$$w=0, \quad Q=1, \quad u=[k(1+k)+\{k^2(1+k^2)+k(1-k)(1+3k)\}^{1/2}]/(1-k).$$

Thus each integral curve in the region $0 < u < 1$ leaves the surface V_+ at some $r = r_1$. Therefore, in this region, even if a discontinuity is introduced, one cannot obtain any solution defined in the limit $r \to 0$. Because the solutions cannot be continued to $u > 0$, we shall seek solutions that remain in the region $u < 0$ for all $r > 0$.

2. For $-1 \leq u \leq 0 (w \geq 0)$ the region S is not bounded with respect to w, and if $-k < u < 0$ then $Q < 1$ and if $-1 < u < -k$ then $Q < (k + u^2)/(1+k)|u|$. On the plane $u = 0$ the condition $L \geq 0$ cuts out two straight lines: $l_1(u = w = 0$, $0 \leq Q \leq 1)$ and $l_2(u = Q = 0, w \geq 0)$, on which the system (1.8) becomes singular. To investigate the system (1.8) in the neighborhood of the straight lines l_1 and l_2, and also for $w \gg 1$, we make the following substitution of the coordinates Q, u, w and the variable $\zeta = \ln r$:

a) in the neighborhood of the line l_1 we use the coordinates Q, u, w and the new variable ρ:

$$d\rho/d\zeta = -1/u > 0; \qquad (2.1)$$

b) in the neighborhood of the line l_2, we use the variable ζ and the coordinates

$$q = Q/u, \quad u, \quad w; \qquad (2.2)$$

c) in the region $w \gg 1$, we use the coordinates q, u, v and the variable ρ_1:

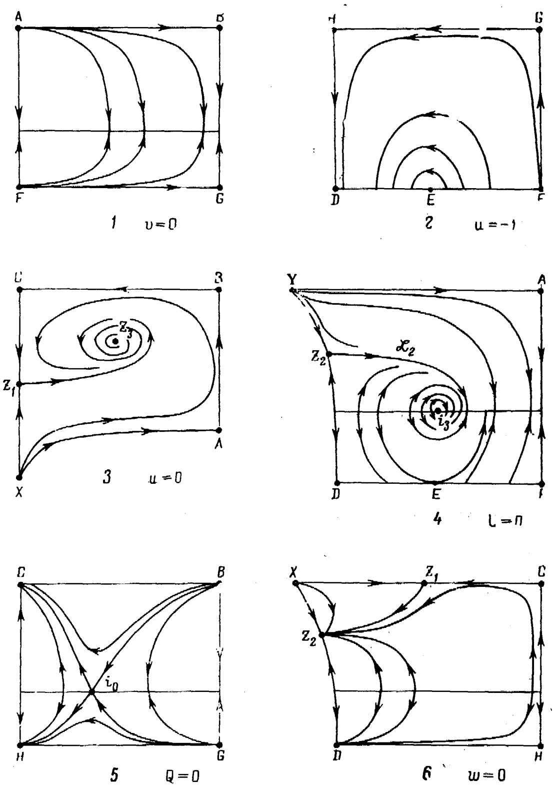

FIG. 1. Integral curves of the dynamical system on the boundary component Γ_k (the numbers under the figures correspond to the values of $k = 1, \ldots, 6$).

$$d\rho_1/d\zeta = w, \quad q = Q/u, \quad v = 1/w. \qquad (2.3)$$

The system (1.8), transformed to these coordinates, has only nondegenerate singular points. Below, we shall also denote it by (2.1), (2.2), (2.3).

We describe the three-dimensional manifold S (with boundary Γ) on which the system (1.8) has nondegenerate singular points. In the neighborhood of the lines l_1 and l_2 and for $w \gg 1$ we shall use the coordinates we have introduced and the equivalent systems (2.1), (2.2), and (2.3), while within S we shall use the system (1.8). The manifold S is distinguished by the conditions $-1 \leq u \leq 0$, $Q \geq 0$, $w \geq 0$, $L \geq 0$ and has boundary Γ consisting of six components: $\Gamma_1(v = 0)$, $\Gamma_2(u = -1)$, $\Gamma_3(u = 0)$ (here system (2.2) is used), $\Gamma_4(L = 0)$, $\Gamma_5(Q = 0)$, $\Gamma_6(w = 0)$. The integral curves of the dynamical system on the boundary component $\Gamma_k (k = 1, \ldots, 6)$ are shown in Fig. 1 (the symbols on the figures is explained below).

When the coordinates (1.7), (2.1)–(2.3) are used, S is represented as in Fig. 2. Here, there is a single point of noncompactness X: $q = \infty$, $u = 0$, $w = 0$, corresponding in the coordinate system (2.1) to the line l_1: $u = w = 0$, $0 \leq Q \leq 1$. The line l_1 consists of repulsive singular points of the system (2.1): the eigenvalues at these singular points are $\lambda_u = \lambda_w = (1+3k)/(1+k) > 0$, $\lambda_Q = 0$. Therefore, near l_1 (for $Q \neq 0$) every integral curve for $\rho = -\infty$ comes out of a certain singular point $u = w = 0$, Q (one can also show directly that the point of noncompactness X is repulsive or that for all integral curves in the neighborhood of X we have $q(r) \to \infty$, $u(r) \to 0$, $w(r) \to 0$ as

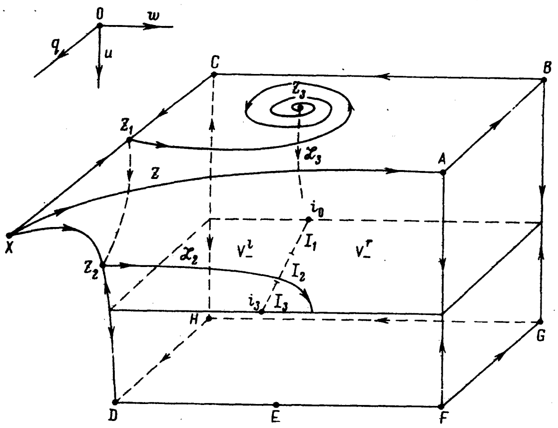

FIG. 2. Singular points and separatrices of the dynamical system on the manifold S. The surface $V_-(u=-k^{1/2})$ is the surface through which the solutions cannot be continued.

r decreases). It is important that at the same time $(\rho \to -\infty) r \to r_0 > 0$. In the corresponding solution $u(r_0)=0$ and for $r < r_0$ the solution is defined in the region $u > 0$ $(w < 0)$. As was shown above, none of these solutions can be continued to $r = 0$ even if a discontinuity is introduced. Therefore, none of the integral curves that come out of the point of noncompactness X have physical significance, and we shall therefore ignore them.

3. All singular points of the dynamical system on the manifold S are nondegenerate and lie on the boundary Γ. Among the singular points, there are eight isolated singular points:

$$A\ (v=0,\ q=-1,\ u=0),\ B\ (v=q=u=0),\ C\ (w=q=u=0),$$
$$G\ (v=q=0,\ u=-1),\quad H\ (w=q=0,\ u=-1),$$
$$Z_2\ (Q=1,\ w=0,\ u=u_2=[k(1-k)-\{k^2(1+k)+k(1-k)(1+3k)\}^{1/2}]/(1-k)),$$
$$Z_1\ (w=u=0,\ q=-3(1+k)/(1+3k)),\ Z_3\ (u=0,\ q=-1,\ w=4k/(1+k))$$

and the straight line $DF(Q=1, u=-1, 0 \leqslant u \leqslant \infty)$ of singular points (see Fig. 2). On the surface V_- there lies the line of singular points I, and these will be considered separately below.

The singular points A, B, C, G, H are unstable saddles and all their separatrices lie on the unphysical components of Γ, i.e., no exact physical solutions correspond to their separatrices.

The singular points on the straight line DF have the following eigenvalues: $\lambda_1 = \lambda_2 = w - 2$, $\lambda_3 = 0$. Thus, the segment $DE(0 \leqslant w \leqslant 2)$ consists of attractive singular points, and the segment $EF(2 < w \leqslant \infty)$ of repulsive ones. All integral curves of the system (1.8) on the boundary component $\Gamma_2(u=-1)$ begin at a certain $w = w_0$ on the segment EF and end for $w = 4/w_0$ on the segment DE; this follows from the existence of the first integral $K = wQ(1+w+Q)^{-2}$. The integral curves of the system (1.8) near the boundary component $\gamma_2(u \approx -1)$ behave similarly. The metric corresponding to integral curves that enter the singular points of DE (as $r \to \infty$) or EF (as $r \to 0$) is incomplete and can be smoothly continued in the synchronous frame of reference (see Sec. 4 below).

The singular points Z_1, Z_2, Z_3 are the most important ones in the problem we are considering. At them, the system (1.8)–(2.2) has the following eigenvalues (the subscript indicates the corresponding proper direction):

$$Z_1:\ \lambda_q=-3,\ \lambda_w=2,\ \lambda_u=1;$$
$$Z_2:\ \begin{array}{l} \lambda_Q=-2,\ \lambda_w=-(1-k)u_2/k>0, \\ \lambda_u=2(1-u_2^2)(u_2(1-k)-k(1+k))/(k-u_2^2)(1+k)<0; \end{array} \tag{2.4}$$
$$Z_3:\ \lambda_{q,w}=(-(1+3k)\pm i(7+42k-k^2)^{1/2})/2(1+k),\ \lambda_u=(1-k)/(1+k).$$

Thus, the singular points Z_1 and Z_2 are unstable saddles; the singular point Z_3 (also unstable) on the boundary component Γ_3 is an attractive focus and has a one-dimensional separatrix $\mathscr{L}_3$ emerging from it within the manifold S. All integral curves on the boundary components Γ_6 (for $0 > u > -k^{1/2}$) and Γ_3 go into, respectively, the attractive (on these boundary components) singular points Z_2 and Z_3. On the boundary component Γ_6 this follows from the fact that $\mathring{Q} > 0$ on Γ_6, while on Γ_3 it follows from the absence of limit cycles on Γ_3 (this last can be shown by going over to the coordinates q, $L_1 = w + 1 + kwq/(1+k)$ and using the Dulac–Bendixson criterion[18]). In accordance with (2.4), a two-dimensional separatrix Z leaves the singular point Z_1. The one-dimensional separatrices obtained by the intersection of Z and the boundary components Γ_6 and Γ_3 join the singular point Z_1 to Z_2 and Z_3. Therefore, the one-dimensional separatrix $\mathscr{L}_2$ that comes out of the singular point Z_2 (see Fig. 1, $k = 4$) is the intersection of the separatrix Z with the boundary component $\Gamma_4(L=0)$, and the one-dimensional separatrix $\mathscr{L}_3$ that comes out of the singular point Z_3 is a limiting line onto which the two-dimensional separatrix Z is wound (see Fig. 2).

We give the asymptotic behavior of the metric (1.3) in the limit $r \to 0$ corresponding to the separatrices Z, $\mathscr{L}_2(\varkappa = 0)$, and $\mathscr{L}_3$:

Z:

$$l^{-2}ds^2 \cong (1+(c/2)r^2)dT^2 - T^2[(1+(c/(1+3k)-1)r^2)dr^2 + r^2 d\Omega^2],$$
$$u \cong -\frac{1+3k}{3(1+k)}r,\quad \varepsilon \cong \frac{3\varepsilon_0 c}{\beta(1+3k)}T^{-2}; \tag{2.5}$$

$\mathscr{L}_2$:

$$l^{-2}ds^2 \cong dT^2 - T^2\left(\frac{dr^2}{r^2} + r^2(d\theta^2 + d\varphi^2)\right),$$
$$u \cong u_2 < 0,\quad \varepsilon \cong \frac{\varepsilon_0(1-u_2^2)}{\beta(1+k)|u_2|}\frac{r^d}{T^2},\quad d=(1-k)|u_2|/k; \tag{2.6}$$

$\mathscr{L}_3$:

$$l^{-2}ds^2 \cong \frac{m}{u_0^2} r^{4k/(1+k)} dT^2 - T^2(m\,dr^2 + r^2 d\Omega)^2,$$
$$u \cong u_0 r^{(1-k)/(1+k)},\quad \varepsilon \cong \frac{4k\varepsilon_0}{\beta(1+k)^2} m^{-1}(rT)^{-2},\quad m=1+4k/(1+k)^2. \tag{2.7}$$

4. The surface $V_-(u=-k^{1/2})$ through which the solutions cannot be continued is cut by the line I:

$$w(1-k) + 8Qk^{1/2}(1+k)^{-1} = 4k$$

into left- and right-hand parts: V_-^l and V_-^r (see Fig. 2). The left-hand part V_-^l repels the integral curves into the manifold S (in the neighborhood of V_-^l we have $\dot{u} > 0$ for $u > -k^{1/2}$ and $\dot{u} < 0$ for $u < -k^{1/2}$), while the right-hand part V_-^r attracts the integral curves out of the manifold S. In both cases, the integral curves intersect the surface V_- at a finite value of r.

To study the behavior of the integral curves in a small neighborhood of the line I, we make the following change of coordinates and (nonmonotonic) change of variable ζ:

$$\alpha=1/Q, \quad l=((1-k)w-4k)/2Q, \quad d\tau_1/d\zeta=-Q/(u^2-k). \tag{2.8}$$

The line $I(l=-4k^{3/2}(1+k)^{-1}, u=-k^{1/2}, \alpha>0)$ consists of nondegenerate singular points of the system (1.8) in the coordinates (2.8). The eigenvalues of the resulting system at these singular points have the form

$$\lambda_\pm=-k^{1/2}(1-k)\pm k^{1/2}(1-k)(1+2Z_0(1-k)^{-1})^{1/2}, \quad \lambda_3=0,$$
$$Z_0=\frac{2k(1+3k)}{1-k}\left[\alpha^2-\frac{2k^{1/2}(3+k)}{1+3k}\alpha+1-\frac{(1-k)^2(1+4k-k^2)}{4k^{3/2}(1+3k)(1+k)}\right]. \tag{2.9}$$

The line of singular points I within the manifold S is split into three segments: $I_1(\infty>\alpha>\alpha_1)$, $I_2(\alpha_1>\alpha>\alpha_2)$, $I_3(\alpha_2>\alpha>\alpha_3)$, where α_1 and α_2 are the largest roots of the quadratic equations $Z_0(\alpha_1)=0$ and $Z_0(\alpha_2)=-(1-k)/2$; α_3 is the coordinate of the point I_3 of intersection of the line I with the boundary component Γ_4 ($L=0$, see Fig. 1, $k=4$); we shall not give expressions for α_1, α_2, α_3 because they are cumbersome.

It follows from (2.9) that the singular points of the segment I_1 are unstable saddles; the singular points of I_2 are attractive nodes; and the singular points of I_3 are attractive focuses. The segment I_1 intersects the boundary component $\Gamma_5(Q=0)$ at the point $i_0(\alpha=\infty$, see Fig. 1, $k=5)$.

Returning to the original coordinates w, Q, u, we find that from the subsonic region of the manifold $S(|u|<k^{1/2})$ the two-dimensional separatrix X_1 enters the segment I_1 and the two-dimensional separatrix Y_1 leaves it (i.e., I_1) (see Fig. 3). The three-dimensional separatrix Y enters the segment I_2. At the same time, there is the two-dimensional separatrix X_2, which is the smooth continuation of X_1 corresponding to the eigenvalues $\lambda_-(\alpha)$; all the other integral curves that go into the segment touch the two-dimensional separatrix Y_2, which corresponds to the eigenvalues $\lambda_+(\alpha)$. In all cases, when the integral curve enters the segment I the parameter $\zeta=\ln r$ remains finite.

§3. TRANSFORMATION OF THE CONFORMALLY STATIC METRIC TO SELF-SIMILAR FORM

1. The metric of self-similar spherically symmetric solutions of Einstein's equations has the form

$$ds^2=e^{\nu_0(ct/R)}c^2dt^2-e^{\lambda_0(ct/R)}dR^2-R^2e^{\mu_0(ct/R)}d\Omega^2, \tag{3.1}$$

where the functions ν_0, λ_0, μ_0 satisfy the arbitrary (nondegenerate) constraint $F(\nu_0, \lambda_0, \mu_0, ct/R)=0$. The energy density ε and the matter four-velocity u_i are determined by

$$\varepsilon=\varepsilon(ct/R)/R^2, \quad (u^0,u^1,u^2,u^3)=\left(\frac{e^{-\nu_0/2}}{(1-v^2)^{1/2}}, v\frac{e^{-\lambda_0/2}}{(1-v^2)^{1/2}}, 0, 0\right), \tag{3.2}$$

where $cv(ct/R)$ is the three-dimensional radial velocity of the matter.

In this paper, we use the expression (3.1) of the metric in coordinates of the form (1) ($\mu_0=0$):

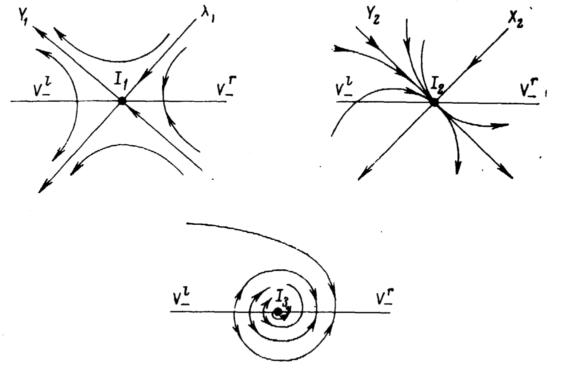

FIG. 3. Qualitative picture of the behavior of the trajectories in the vicinity of the singular points on the segments I_1, I_2, and I_3.

$$ds^2=X(ct/R)c^2dt^2-Y(ct/R)dR^2-R^2d\Omega^2 \tag{3.3}$$

and in the synchronous frame of reference ($\nu_0=0$):

$$ds^2=c^2dt^2-U(ct/R)dR^2-R^2V(ct/R)d\Omega^2. \tag{3.4}$$

2. The metric (1.2)–(1.3) is mapped as follows into the synchronous metric of the form (3.4):

$$ct/l=f_1(\zeta)e^\tau, \quad R=lf_2(\zeta)e^\tau, \quad \zeta=\ln r, \tag{3.5}$$

where the functions f_1 and f_2 satisfy the equations

$$df_1/d\zeta=Q(f_1^2-\sigma e^\nu)^{1/2},$$
$$df_2/d\zeta=f_1f_2Q(f_1^2-\sigma e^\nu)^{-1/2}. \tag{3.6}$$

Under this mapping, the coefficients of the metric (3.4), the radial velocity v, and the energy density ε have the form

$$U=(f_1^2-\sigma e^\nu)f_2^{-2}, \quad V=e^{2\zeta}f_2^{-2},$$
$$v=\frac{(f_1^2-\sigma e^\nu)^{1/2}+uf_1}{f_1+u(f_1^2-\sigma e^\nu)^{1/2}}, \quad \varepsilon=-\frac{\varepsilon_0\sigma(1-u^2)wf_2^2e^{-\nu}}{(1+k)\beta uQR^2}. \tag{3.7}$$

Using Eqs. (3.5)–(3.7) we convert to the synchronous system (3.4) the asymptotic behaviors of the solutions that in the limit $\zeta\to+\infty$ enter attractive singular points on the segment DE (see §2). We obtain the asymptotic behaviors

$$f_1\approx e^\zeta, \quad f_2\approx C_2e^\zeta, \quad R/ct=f_2/f_1\approx C_2,$$
$$U\approx V\approx C_2^{-2}, \quad \varepsilon\to\text{const}, \quad v\to\text{const}. \tag{3.8}$$

Thus, under the mapping (3.5) the metric (1.2) in the limit $\zeta\to\infty$ goes over into the metric (3.4) defined in the region $R/ct<C_2$, and in the limit $R/ct\to C_2$ the metric (3.4) is nondegenerate. Therefore, the solution can be continued smoothly to the region $ct/R<C_2^{-1}$. At the same time, the self-similar variable ct/R becomes timelike, and the straight line $ct/R=C_2^{-1}$ is isotropic. Such continuation is possible, for example, to $ct/R=0$; under it, all parameters of the solution remain regular and no new qualitative features of the solution arise. In the region $ct/R<C_2^{-1}$, the metric (3.4) corresponds to the metric (1.2) with $\sigma=-1$, and the integral curve of the system

(1.8) corresponding to it continues smoothly into the region $u<-1$ of the original integral curve.

Similarly, in the synchronous system (3.4) one can continue the metrics (1.2) corresponding to integral curves that in the limit $\zeta\to-\infty$ leave the segment EF of repulsive singular points.

3. The metric (1.2) can be transformed to the form (3.3) by means of the transformation

$$R=le^{\tau+\zeta},\quad ct/l=e^{\tau+\varphi(\zeta)}, \tag{3.9}$$

where $d\varphi/d\zeta=Q^2$. The components of the metric, the matter velocity v, and the energy density ε have the form

$$X(ct/R)=\sigma e^{\nu-2\varphi}/(1-Q^2),\quad Y(ct/R)=\sigma e^{\nu-2\zeta}Q^2/(1-Q^2),$$
$$v=(Q+u)/(1+uQ),\quad \varepsilon=-\varepsilon_0\sigma(1-u^2)we^{2\zeta-\nu}/\beta Q(1+k)uR^2. \tag{3.10}$$

The asymptotic behavior of the solutions corresponding to the separatrices Z in the limit $\zeta\to-\infty$ have, after the transformation (3.9), the form (as $R/ct\to 0$)

$$ds^2\cong\left(1+\frac{d}{2}\left(\frac{R}{ct}\right)^2\right)c^2dt^2-\left(1+\frac{d}{1+3k}\left(\frac{R}{ct}\right)^2\right)dR^2-R^2d\Omega^2,$$
$$cv\cong\frac{2}{3(1+k)}\frac{R}{t},\quad \varepsilon\cong\frac{3\varepsilon_0 d}{\beta(1+3k)}\frac{1}{t^2},\quad d=\text{const}. \tag{3.11}$$

As $\zeta\to+\infty$, we have $R/ct=C_1$, as under the mapping (3.5). The metric (3.3) can be smoothly continued to the region $ct/R<C_1^{-1}$, but in the limit $ct/R\to 0$ this metric has in general an unphysical singularity with asymptotic behavior $X\approx(ct/R)^\gamma$, $Y\to$ const. This singularity can be eliminated by means of the transition (3.5) to the synchronous frame of reference.

§4. SELF-SIMILAR SOLUTIONS WITH EXPANDING AND COLLAPSING SHOCK WAVES

1. Our investigation in §2 of the system (1.8) shows that in the subsonic region of the manifold $S(0>u>-k^{1/2})$ the system (1.8) has five types of integral curve: A) integral curves that come out of the repulsive singular point X and go into the surface V_-^r; B) integral curves that begin on V_-^l and end on V_-^r; C) integral curves that fill the two-dimensional separatrix Z of the singular point Z_1 (in particular, the separatrices L_2 and L_3 of the singular points Z_2 and Z_3); D) integral curves that fill the two-dimensional separatrix Y_1 that comes out of the segment of singular points I_1; E) integral curves which fill the separatrices X_1 and Y and enter the segments I_1 and I_2 of singular points. (The hypothesis is made that there are no integral curves in S wound onto any invariant subsets.)

All five types of integral curves determine the solution (metric) only on a finite interval of r values. Integral curves of the types A and B are defined for $0<r_1<r<r_2$, and, therefore, to continue the corresponding solution for all $r>0$ it is necessary to introduce at least two discontinuities in the region $u<0$ (for integral curves of the type B this follows from their definition, while for integral curves of the type A continued smoothly into the region $u>0$ this follows from the fact that in the region $u>0$ the impossibility of continuing the integral curves cannot be avoided, even by introducing a discontinuity; see §2). Then to an external discontinuity there will correspond a compression shock wave and to an internal discontinuity, a rarefaction shock wave (see (1.11)). Since rarefaction shock waves in matter with normal properties are impossible (see[15]), all integral curves of the types A and B are unphysical.

2. Integral curves of type C filling the two-dimensional separatrix Z are defined for $0<r<r_1$. These integral curves can be continued for all $r>0$ by introducing a single discontinuity, to which there corresponds an expanding shock wave; for suppose the discontinuity is introduced at $r=r_0$. In the subsonic region we have $r<r_0$, and therefore the gas velocity $u<0$ is directed into the subsonic region, i.e., the shock wave is a compression wave. The radius R_0 of the shock wave is $le^\tau r_0=R_0$; therefore, $R_0\to\infty$ as $\tau\to\infty$, i.e., the shock wave is an expanding one.

In the case of a strong discontinuity, i.e., for $u_1\approx-k$, the supersonic solution fitted to the given solution has $u_2\approx-1$ at the discontinuity, and therefore, as was noted in §2, the corresponding integral curve moves along the boundary component $\Gamma_2(u=-1)$ until it reaches in the limit $r\to\infty$ a certain attractive singular point on the segment DE. At the same time, the self-similar variable $\zeta=\ln r$ becomes isotropic and, as is shown in §3, for timelike variable ζ the solution can be continued in the synchronous system (3.4) in the region $ct/R\geq 0$. A discontinuity in the solution arises for constant $\zeta=\ln r_0$, i.e., $ct/R=$ const corresponds to a shock wave. Therefore, in the synchronous system (3.4) the shock wave moves with a certain constant velocity v_0.

The shock wave is formed (the explosion occurs) in the symmetry center $R=0$ at $t=0$. At this instant, the three-dimensional metric (3.4) $u(0\ dR^2+R^2V(0)d\Omega^2$ has a conical singularity at the origin for $U(0)\neq V(0)$; at infinity, the metric is obviously flat; the gas velocity $v(0)$ is constant throughout space and directed toward the center; the energy density is $\varepsilon=c_0/R^2$ (such a distribution of the energy density in a finite region can be realized, for example, by the continuous escape of matter from a star).

Solutions of type C depend on two essential parameters: the choice of the one-dimensional separatrix of the singular point Z_1 and the choice of the point of discontinuity on this separatrix. One can consider an additional condition: The gas velocity at spatial infinity in the static frame is $v(0)=0$ (and then at the initial instant $v(t=0)=0$). This condition will be satisfied if the discontinuity is introduced in the solutions on a certain line P on the separatrix Z (on the line P we have $-k^{1/2}<u<-k$). The points of the line P are determined by the requirement that for the solution corresponding to them the functions f_1 and v in (3.6) and (3.7) vanish simultaneously.

As is shown in §3, the solutions corresponding to the separatrix Z as $\zeta\to-\infty$ in the frame of reference (3.3) have the asymptotic behavior (3.11), where $R/ct\to 0$. From the form of the asymptotic behavior (3.11) we obtain an important conclusion: 1) Solutions of type C with expanding shock wave can be continued to the symmetry center $R=0$, and at it the metric after the departure of the shock wave ($t>0$) does not have a singularity; 2) the gas in the neighborhood of the symmetry center moves away from the center, and for $R/ct \ll 1$ the velocity cv has the same asymptotic behavior as in the well-known solution of the explosion problem in classical gas dynamics[2]; 3) the energy density near the center is approximately constant with respect to R and decreases as c_0/t^2.

Let us consider the solutions corresponding in the subsonic region to the separatrices of the singular point Z_1 that pass close to the boundary component Γ_3. These separatrices for small $u<0$ make an arbitrarily large number of turns around the singular point Z_3; see Fig. 1, $k=3$, and Fig. 2. We write the expression (3.10) for the velocity v in the coordinates (2.2):

$$v=u(1+q)/(1+qu).$$

It is obvious that along these separatrices the velocity v has an arbitrarily large number of zeros (since $1+q=0$ at the singular point Z_3). Therefore, for such solutions, the coordinate r along the trajectories of the motion of the gas (the streamlines) in the region behind the shock wave varies nonmonotonically and executes an arbitrarily large number of oscillations.

3. Integral curves of the type D can be smoothly continued for decreasing parameter ξ through the segment I_1 into the supersonic region (see Fig. 3). After this unique continuation, the integral curves of type D (of the separatrix Y_1) passing through the segment I_1 in the neighborhood of the point i_0 can be approximated by the sequence of separatrices $\overrightarrow{FG}, \overrightarrow{Gi_0}, \overrightarrow{i_0C}, \overrightarrow{CZ_1}, Z$ (see Fig. 1, $k=5$, and Fig. 2). Thus, integral curves of type D with increasing ξ move in the neighborhood of the separatrix Z and therefore, like the integral curves of type C, correspond to solutions with expanding shock wave. After the introduction of a discontinuity, these integral curves can also be continued in the static frame of reference (3.4) in the region V_1: $C_1 \geq ct_1/R_1 \geq 0$ (where the self-similar variable ξ is timelike).

However, the behavior of solutions of the type D in the region behind the shock wave is very different from that of the type C solutions. It follows from the separatrix approximation given above that solutions of type D after the smooth continuation through the points of the segment I_1 into the supersonic region in the limit $\xi \to -\infty$ go into attractive (for this direction of ξ) singular points of the segment EF. At the same time, the self-similar variable ξ again becomes isotropic and, as is shown in §3, the solution can be continued smoothly for timelike variable ξ in the static frame of reference (3.4) into the new region V_2: $C_2 \geq ct_2/R_2 \geq 0$.

The complete solution of type D is fitted together from the regions V_2, V_{2c}, V_g, V_{1c}, V_1, which are contiguous in the given order. The subsonic region V_g is described by the motion of the integral curve along the separatrics $\overrightarrow{i_0C}, \overrightarrow{CZ_1}, Z$ to the point of discontinuity (see Fig. 1, $k=5$, and Fig. 2). The region V_{2c} is described by the motion of the integral curve along the separatrices $\overrightarrow{FG}, \overrightarrow{Gi_0}$. The region V_{1c} is fitted to the region V_g through the shock wave and is described by the motion of the integral curve in the supersonic region of the manifold S from the discontinuity point to the attractive segment DE.

The regions V_1 and V_2 are bounded by the isotropic surfaces $ct_1/R_1=C_1$ and $ct_2/R_2=C_2$, and therefore cannot be connected by physical signals. The shock wave moves along the common boundary of the regions V_g and V_{1c}. The formation of the shock wave (the explosion) takes place at the symmetry center, as in the solutions of type C, but for $t \neq 0$ the spacelike section has a "throat" and is topologically the product of a two-dimensional sphere and a straight line, as in the Kruskal solution.[1]

4. With increasing parameter ξ, the integral curves of type E can be continued smoothly through the segment I into the supersonic region of the manifold S (see Fig. 3). The separatrices X_1 passing through the segment I_1 in the neighborhood of the point i_0 can be approximated as the parameter ξ decreases by the stable sequence of separatrices $\overrightarrow{i_0B}, \overrightarrow{BA}, \overrightarrow{AX}$ and they arrive at the attractive (when ξ decreases) point X. With increasing ξ, the separatrices X_1 move along the separatrices $\overrightarrow{i_0H}, \overrightarrow{HD}$ and, therefore, arrive at the segment of attractive singular points DE. The separatrices X_1 determine solutions with a collapsing shock wave, the discontinuity in them being introduced in the region $u<0$ in the neighborhood of the separatrix $\overrightarrow{Bi_0}$ (see Fig. 2).

It follows from the investigation of the singular points on the segment I_2 that there is an entire region of integral curves that leave V_-^1 and enter the segment I_2 touching the separatrix Y_2 (see Fig. 3). To these integral curves, like the separatrices X_1 and X_2, there correspond solutions with collapsing shock wave.[1)]

Indeed, the solutions of Einstein's equations of the form (1.2) again go over into solutions under the transformation $\tau_1=-\tau$, $u_1(r)=-u(r)$, all the other functions of r remaining unchanged. Under this transformation, the metric (1.2) takes the form

$$ds^2=l^2e^{-2\tau_1}(e^{\nu(r)}d\tau_1^2-e^{\lambda(r)}dr^2-r^2d\Omega^2).$$

Suppose the discontinuity on these integral curves is introduced at $r=r_0$ (a discontinuity is needed because these integral curves come out of X and V_-^1). In the region behind the shock wave, where the gas flow is subsonic, $r>r_0$. The gas velocity $u_1(r)=-u(r)>0$ is directed into this region, and therefore the shock wave is a compression wave. The radius of the shock wave is $R_0=le^{-\tau_1}r_0$; therefore, $R_0 \to 0$ as $\tau_1 \to +\infty$, i.e., the shock wave is a converging wave.

The complete solution of type E is fitted from the following regions: V_2, V_{2c}, V_g, V_{1c}, V_1. The region V_g is described by the motion of the integral curve with increasing variable ξ in the subsonic region of the manifold S from the discontinuity until it reaches the segment of singular points I_2. The region V_{1c} is described by the motion of the integral curve in the supersonic region of the manifold S from the segment I_2 until the segment DE of attractive singular points. As is shown in §3, the solution can then be continued in the region V_1, in which ζ is timelike. In the static system (3.4), the region V_1 is defined by the condition $-C_1 \leq ct_1/R_1 \leq 0$; the region $V_\varepsilon + V_{1c}$ is defined by the condition $-C_0 \leq ct_1/R_1 \leq -C_1$, and the equation of the shock wave is $ct_1/R_1 = -C_0$. The shock wave collapses with a certain constant velocity into the symmetry center at $t_1 = 0$.

The region V_{2c} is fitted to the region V_ε through the shock wave and is described, as the variable ζ decreases, by the motion of the integral curve in the supersonic region of the manifold S from the discontinuity to the segment EF. The solution then goes over into the region V_2, where the variable ζ is again timelike. The region V_2 in the static frame of reference is determined by the condition $-C_2 \leq ct_2/R_2 \leq 0$. Since the surfaces $ct_1/R_1 = C_1$ and $ct_2/R_2 = -C_2$ are timelike, the regions V_1 and V_2 cannot communicate by physical signals. The spacelike sections in these solutions are topologically the product of a two-dimensional sphere and a straight line, as in the Kruskal solution, and have a throat, which contracts into a point as the shock wave collapses into the center.

The listed properties of the type E solutions indicate that these solutions are a certain asymptotic regime for the collapse of matter (in the presence of pressure) leading to the formation of a shock wave.

1) When the integral curve is continued through the singular point on the segment $I_1 + I_2$, the solution as a function of ζ ceases, in general, to be smooth (infinitely differentiable). Smoothness of the solution is preserved only for the separatrices Y_2, X_1, X_2. This property can evidently be used to restrict the set of solutions of type E that have physical meaning to the solutions corresponding to the separatrices Y_2, X_1, X_2.

[1] L. D. Landau and E. M. Lifshitz, Teoriya Polya (The Classical Theory of Fields, Pergamon Press, Oxford (1976)), Fizmatgiz (1973).

[2] L. I. Sedov, Metody Podobiya i Razmernosti v Mekhanike, Nauka (1973). (Translation of earlier edition: Similarity and Dimensional Methods, London (1959)).

[3] K. P. Stanyukovich, Dokl. Akad. Nauk SSSR **140**, 77 (1961) [Sov. Phys. Dokl. **6**, 765 (1962)].

[4] V. Ts. Gurovich, K. P. Stanyukovich, and O. Sh. Sharshekeev, Dokl. Akad. Nauk SSSR **165**, 510 (1965) [Sov. Phys. Dokl. **10**, 1030 (1965)].

[5] K. P. Stanyukovich, Zh. Eksp. Teor. Fiz. **66**, 826 (1974) [Sov. Phys. JETP **39**, 399 (1974)].

[6] M. E. Cahill and A. H. Taub, Commun. Math. Phys. **21**, 1 (1971).

[7] A. H. Taub, Commun. Math. Phys. **29**, 79 (1973).

[8] A. N. Golubyatnikov, Dokl. Akad. Nauk SSSR **233**, 318 (1977) [Sov. Phys. Dokl. **22**, 122 (1977)].

[9] O. I. Bogoyavlenskiĭ and S. P. Novikov, Zh. Eksp. Teor. Fiz. **64**, 1475 (1973) [Sov. Phys. JETP **37**, 747 (1973)].

[10] O. I. Bogoyavlenskiĭ, Zh. Eksp. Teor. Fiz. **70**, 361 (1976) [Sov. Phys. JETP **43**, 187 (1976)].

[11] I. S. Shikin, Zh. Eksp. Teor. Fiz. **68**, 1583 (1975) [Sov. Phys. JETP **41**, 794 (1976)].

[12] R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves, New York (1948) (Russian translation, I. I. L. (1950)).

[13] L. D. Landau and E. M. Lifshitz, Mekhanika Sploshnykh Sred (Mechanics of Continuous Media), Gostekhizdat (1954) [Pergamon, 1958].

[14] K. P. Stanyukovich, Neustanovivshiesya Dvizheniya Sploshnoĭ Sredy (Nonstationary Motions of a Continuous Medium), Gostekhizdat (1955).

[15] Ya. B. Zel'dovich and Yu. P. Raĭzer, Fizika Udarnykh Voln i Vysokotemperaturnykh Gidrodinamicheskikh Yavleniĭ (Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena, Vols. 1 and 2, Scipta Technica, New York (1966, 1967)), Gostekhizdat (1963).

[16] B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, Gravitational Theory and Gravitational Collapse, Univ. of Chicago Press (1965) (Russian translation, Mir (1967)).

[17] V. N. Lukash and A. A. Starobinskiĭ, Zh. Eksp. Teor. Fiz. **66**, 1515 (1974) [Sov. Phys. JETP **39**, 742 (1974)].

[18] A. A. Andronov, E. A. Leontovich, I. M. Gordon, and A. G. Maĭer, Kachestvennaya Teoriya Dinamicheskikh Sistem (Qualitative Theory of Dynamical Systems), Nauka, 1966, Ch. IV, §12.

[19] G. Guderley, Luftfahrtforschung **19**, 302 (1942).

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