4)Equations (26) are in essence the integral form of the renormalization-group equations. We have recently received a preprint by Buras,[13] in which distributions analogous to (36) were obtained by diagonalization of the differential equations of the renormalization group.[3]

5)In papers on reggeization no account was taken of the effects due to the distance dependence of the charge $\bar{g}^2$.

6)In the abelian case one of the poles of (69) is a standing one, $\lambda_G = 0$. This agrees with the fact that the photon is not reggeized in electrodynamics.

7)In the general case we can construct the following "non-interpretable" relation

$$\int_0^{g_\mu^2} \frac{d\bar{g}^2}{\bar{g}^2} (\bar{g}^2)^{2(N-n_f)(N^2-1)/3(N^2+1)} \left\{ \frac{n_f}{N} D_F^{\langle F \rangle}(x) + D_F{}^G(x) - D_G{}^F(x) - D_G{}^G(x) \right\} = 0.$$

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Critical charge for anomalous nuclei and the effect of screening

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(Submitted April 20, 1977)
Zh. Eksp. Teor. Fiz. 73, 1241–1257 (October 1977)

The dependence of the critical charge of nuclei on photon density is found. Values of Z_{cr} for anomalous nuclei are obtained. The calculations of Z_{cr} have been carried out taking into account the diffuse nature of the nuclear boundary and also the effect of screening of the Coulomb nuclear potential by the electron shell. The Thomas–Fermi statistical method is employed for describing the electron density in the shell. Two cases are considered: 1) screening by the usual electron shell formed by electrons occupying levels of the discrete spectrum ($-m < \epsilon < m$); 2) screening by a vacuum shell which is formed by electrons situated in levels of energy $\epsilon < -m$. In the first case the dependence of Z_{cr} on the degree of ionization of the atom $q = (Z-N)/Z$ is also obtained. The properties of electron states at the critical point are considered in detail. Asymptotic formulas for the solutions of the Thomas–Fermi equation in two limiting cases have been obtained in the Appendix: $q \to 0$ (weakly ionized atom) and $q \to 1$.

PACS numbers: 31.20.Lr

1. INTRODUCTION

The critical charge of a nucleus[1] and the spontaneous production of positrons for $Z > Z_{cr}$ have been investigated in many papers (a discussion of the different aspects of this problem, and of its significance for the verification of quantum electrodynamics in strong external fields and references to the literature of the subject can be found in Refs. 2–5). The usually quoted[6–9] values of Z_{cr} refer to the normal density of nuclear matter $n_0 \approx 0.17$ nucleon $\cdot$ F^{-3}. At the same time there are theoretical indications[10–15] of the possibility of existence of anomalous nuclei with a density which differs significantly from n_0. Such a possibility was investigated for the first time by Migdal[10] who showed that nuclear matter beginning with a certain density $n = n_c$ becomes unstable with respect to the production of π mesons, and this leads to a phase transition of the nucleus into a superdense state with the formation of a pion condensate. Subsequently this problem was considered in greater detail[11–13]; also the possibility of the existence of neutron ($N \gg Z$) and supercharged ($Z \sim 137^{3/2}$) nuclei was discussed.[12] Lee and Wick also gave arguments in favor of the existence of stable superdense nuclei.[14,15] At present a large number of papers is devoted to the problem of the π condensate, to its effect on different properties of nuclei and

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to the search for nuclei with anomalous values of Z, A and of the nucleon density n. Apparently superdense states of nuclear matter can be obtained in a collision of two heavy ions.

In connection with these papers we have carried out a calculation of Z_{cr} for anomalous nuclei, i.e., we have obtained the dependence of Z_{cr} on the nucleon density n and on the parameter $\eta = Z/A$. The calculation has been carried out both for a nuclear model with a sharp boundary and also taking into account the diffuse nature of the nuclear boundary (Sec. 2). In Secs. 3 and 4 an estimate is given of the change in the critical charge ΔZ_{cr} as a result of the screening of the Coulomb field of the nucleus by electrons. An investigation is made of the screening by the vacuum shell[2,16] of an above critical atom and also by the usual electron shell (electrons in levels of energy ε_n with $-m < \varepsilon_n < m$). We have obtained the dependence of ΔZ_{cr} on the degree of ionization of the atom $q = 1 - N/Z$. The calculation is carried out in the Thomas–Fermi approximation. The results obtained are of interest in connection with design of an experiment on observing the spontaneous production of positrons in the collision of two heavy ions, since when the nuclei approach to a distance $R < R_{cr} \approx 0.1\hbar/m_e c$ they are surrounded by the electron shell of the united atom. In Sec. 5 we have examined in detail the physical properties of atomic levels in the case of Z close to Z_{cr}. In the Appendix we have given the asymptotic expansions in the Thomas–Fermi equation utilized in calculating the effect of screening.

In this paper we use the system of units $\hbar = c = m_e = 1$ and we have introduced the notation $\zeta = Z\alpha$, $\alpha = e^2 = 1/137$, Z is the nuclear charge, A is the number of nucleons in the nucleus, $R = r_0 A^{1/3}$ is the nuclear radius, n is the density of nuclear matter, $n_p = \eta n$ is the proton density, $\delta = n_p/n_p^{(0)}$. For normal heavy nuclei $r_0 = 1.1$ F, $A = 2.6Z$, $n = n_0 = 3/4\pi r_0^3 \approx 0.18$ nucleon/F^3, $n_p \equiv n_p^{(0)} = 0.385 n_0$.

2. CRITICAL CHARGE FOR ANOMALOUS NUCLEI.[1)]

We write the potential energy of the electron in the form

$$V(r) = -\zeta v(r),$$

where

$$v(r) = \begin{cases} R^{-1} f(r/R), & 0 < r \leq R \\ r^{-1}, & r \geq R \end{cases}. \tag{2.1}$$

The cut-off function $f(r/R)$ takes into account the finite dimensions of the nucleus. Its introduction eliminates the "collapse to the center" in the Dirac equation[1] and is required for the mathematically correct formulation of the problem for $\zeta > 1$.

The value of Z for which the discrete level drops to the boundary of the lower continuum $\varepsilon = -1$ is called the critical nuclear charge. For its determination we have the transcendental equation[2)]

$$zK_{i\nu}'(z)/K_{i\nu}(z) = 2(\xi + s) - 1, \tag{2.2}$$

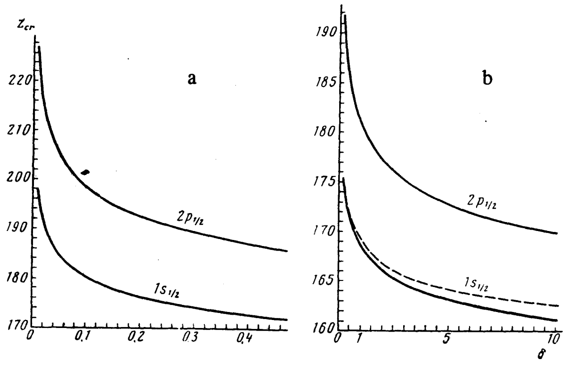

FIG. 1. The critical nuclear charge for the $1s_{1/2}$ levels. Along the horizontal axis is plotted the ratio of the proton densities $\delta = n_p/n_p^{(0)}$. The dotted curve takes into account the diffuse nature of the nuclear boundary; the other curves refer to a nucleus with a sharp boundary (the cut-off model II of Ref. 7 is used, i.e., $f(\rho) = (3 - \rho^2)/2$ for $\theta < \rho = r/R < 1$).

where

$$z = (8\zeta_{cr} R)^{1/2}, \quad \nu = 2[\zeta_{cr}^2 - (j + 1/2)^2]^{1/2},$$

ξ is the logarithmic derivative of the internal ($r < R$) wave function at the edge of the nucleus, s is the spin of the particle ($s = 0, \frac{1}{2}$). For scalar particles one should replace in the definition of the parameter ν the total angular momentum j by the orbital angular momentum l. For the ground 1s level we have in both cases

$$\nu = 2[\zeta_{cr}^2 - (s + 1/2)^2]^{1/2}.$$

In subsequent discussion we set $\rho = r/R, f(\rho) = (3 - \rho^2)/2$ (the cut-off model II according to Ref. 7), which corresponds to a uniform charge distribution:

$$n_p(r) = n_p \theta(R - r).$$

The numerical solution of (2.2) yields the curve $\zeta_{cr}(R)$ for the corresponding level. Its intersection with the curve $R = r_0 A^{1/3}$ (where $r_0 = 1.1$ F and $Z = 0.385A$) determines the critical charge $\zeta_{cr} \equiv \zeta_{cr}^{(0)}$ and the radius R_0 for nuclei with a normal proton density. The values of $\zeta_{cr}^{(0)}$ and R_0 are given in Table II (the subscript zero is omitted for them)—see Sec. 5.

Further, the relation

$$\delta = \frac{n_p}{n_p^{(0)}} = \frac{\zeta_{cr}}{\zeta_{cr}^{(0)}} \left(\frac{R}{R_0}\right)^{-3} \tag{2.3}$$

enables us to obtain from the curve $\zeta_{cr} = \zeta_{cr}(R)$ the desired dependence of ζ_{cr} on the proton density. The result of the calculation is shown in Fig. 1.

The critical charge Z_{cr} decreases with increasing δ and in the limit $\delta \to \infty$ approaches the value $Z = 137$, corresponding to a point charge. However, this approach is quite slow. This follows from Fig. 1, and also from the asymptotic relation:

$$\zeta_{cr} = 1 + a_1(\ln\delta + a_2)^{-2} + \ldots, \quad \delta \gg 1 \tag{2.4}$$

(for the ground $1s_{1/2}$ level). In order to obtain this formula we note that $\zeta_{cr}(R)$ for $R\to 0$ can be expanded in powers of the small parameter $(\ln R)^{-1}$. In particular, for the ground state we have[7]

$$\zeta_{cr}(R)=1+\pi^2/2\ln^2 R+\dots .$$

Taking into account the fact that

$$\zeta=c_0R^3\delta,\quad c_0=4\pi e^2 n_p^{(0)}/3,$$

we obtain formula (2.4) in which $a_1=9\pi^2/2$, $a_2=\ln c_0=11.7$. For excited levels the dependence of Z_{cr} on δ is more pronounced (cf., the curves for the $1s_{1/2}$ and $2p_{1/2}$ levels in Fig. 1, and also Fig. 2 in Ref. 17.

The opposite limiting case $\delta\to 0$ corresponds to $R\gg 1$. Here the dependence of ζ_{cr} on R becomes linear: $\zeta_{cr}(R)=\beta_1R+\beta_2+\cdots$, with the coefficients β_1 and β_2 depending on the distribution of the electric charge in the nucleus.[18] From this we obtain

$$\zeta_{cr}=b_1\delta^{-1/3}+b_2+O(\delta^{1/3}),\quad \delta\to 0, \tag{2.5}$$

where b_1 and b_2 are certain constants. This formula qualitatively describes the increase in ζ_{cr} as the proton density diminishes, but in the region $\delta\gtrsim 10^{-2}$ of interest to us the accuracy of the asymptotic formula (2.5) is yet insufficiently good, and therefore a numerical calculation is required.

Until now we have been discussing the model of a nucleus with a sharp edge. We make an estimate of the amount by which Z_{cr} changes when the diffuse nature of the nuclear boundary is taken into account. Taking for $n_p(r)$ the Woods–Saxon distribution

$$n_p(r)=C[1+\exp((r-R)/b)]^{-1}, \tag{2.6}$$

we assume that the width of the surface layer b is determined by the range of nuclear forces and therefore has the same value as in ordinary nuclei. The condition $R\gg b$ enables us to calculate the correction $\Delta\zeta_{cr}^{(1)}$ by means of perturbation theory.

Let the value $\zeta=\zeta_{cr}$ correspond to the potential $V_{cr}(r)=-\zeta_{cr}v(r)$. We find the change in ζ_{cr} as the perturbing potential $\delta V(r)$ is switched on from the condition $\delta\varepsilon=\langle\delta V-\delta\zeta_{cr}v\rangle=0$:

$$\delta\zeta_{cr}=\beta^{-1}\langle\delta V\rangle. \tag{2.7}$$

Here we have

$$\langle\delta V\rangle=\int_0^\infty \delta V(r)\chi_{cr}^2\,dr,\quad \beta=\int_0^\infty v(r)\chi_{cr}^2(r)\,dr, \tag{2.8}$$

$$\chi_{cr}^2(r)=r^2(g^2+f^2),\quad \int_0^\infty \chi_{cr}^2(r)\,dr=1, \tag{2.9}$$

$g(r)$ and $f(r)$ are the radial functions for the upper and lower components of the Dirac bispinor defined in accordance with Ref. 19. The wave functions $g(r)$, $f(r)$ and $\chi_{cr}(r)$ refer to the critical point $\zeta=\zeta_{cr}$, $\varepsilon=-1$. The normalization condition (2.9) usual for the states of the discrete spectrum $(-1<\varepsilon<1)$ is preserved also at the edge of the lower continuum, due to the presence of the Coulomb barrier in the effective potential.[2,7]

We note that β is the slope of the level at the point where it intersects the boundary of the negative continuum:

$$\beta=-[d\varepsilon/d\zeta|_{\zeta=\zeta_{cr}}. \tag{2.10}$$

The value of β determines the threshold behavior of the probability for the spontaneous production of positrons.[20] The values of ζ_{cr} and β for the first four levels of the discrete spectrum are given below (cf., Table II).

Since $R\gg b$, one can utilize in the integration the formula

$$\int_0^\infty \frac{f(r)\,dr}{1+\exp[(r-R)/b]}=\int_0^R f(r)\,dr+\frac{\pi^2}{6}b^2f'(R)+O\left[\left(\frac{b}{R}\right)^4\right]$$

and determine the constant C:

$$C=n_p\{1-(\pi b/R)^2+O[(b/R)^4]\}. \tag{2.11}$$

The smearing out of the edge of the nucleus is equivalent to perturbing the proton density

$$\delta n_p(r)=C\{1+\exp[(r-R)/b]\}^{-1}-n_p\theta(R-r)$$
$$=-(\pi b/R)^2n_p[\theta(R-r)+{}^1/_6R^2\delta'(R-r)+\dots], \tag{2.12}$$

which corresponds to the perturbation of the potential

$$\delta V(r)=e^2\int_r^\infty 4\pi r'^2\,dr'\left(\frac{1}{r}-\frac{1}{r'}\right)\delta n_p(r').$$

Substituting these expressions in (2.8) we obtain:

$$\delta\zeta_{cr}=\frac{\zeta_{cr}}{\beta R}\left(\frac{\pi b}{R}\right)^2\int_0^R\left(1-\frac{r^2}{2R^2}\right)\chi_{cr}^2(r)\,dr. \tag{2.13}$$

Calculation using this formula with $b=0.5$ F (the same value as for ordinary nuclei[21]) yields the dotted curve in Fig. 1. The numerical values of the correction $\Delta Z_{cr}^{(0)}$ for $n_p=n_p^{(0)}$ are given in Table I. The correction $\Delta Z_{cr}^{(1)}$ increases sharply for superdense nuclei which is explained by the factor R^{-3} in front of the integral in (2.13). In the case of low density of nuclear matter $(\delta<0.5)$ the correction for the diffuse nature of the nuclear boundary is negligibly small.

Figure 2 shows the dependence of the slope of the ground $1s_{1/2}$ level at the boundary of the lower continuum on $\delta=n_p/n_p^{(0)}$.

We obtain estimates of possible values of the parameter δ. According to the latest results in the theory of π condensation[13] two regions of stability of anomalous nuclei are possible:

1) superdense nuclei for which we have

$$A<A_1=200f(n,{}^1/_2)\sim 10^2-10^3,\quad \eta=Z/A=0{,}5,\quad n\sim(5-10)n_0;$$

2) neutron nuclei with

TABLE I. The critical nuclear charge for the first four levels of the discrete spectrum.

	$1s_{1/2}$	$2p_{1/2}$	$2s_{1/2}$	$3p_{1/2}$
Z_{cr}	168.8	181.3	231.7	254.5
$\Delta Z_{cr}^{(1)}$	0.45	0.63	0.85	0.98
$\Delta Z_{cr}^{(2)}$	1.23	1.03	3.50	3.30
$\Delta Z_{cr}^{(3)}$	$0 (N_e=0)$ $0.78 (N_e=1)$	$1.46 (N_e=2)$ $2.18 (N_e=3)$	$3.11 (N_e=4)$ $3.89 (N_e=5)$	$4.55 (N_e=6)$ $5.31 (N_e=7)$
μ (cm. (4. 7))	0,224	0,272	0.222	0,241

Note. 1) Z_{cr} is the critical charge for a bare nucleus with a sharp edge (cut-off model II).
2) The origin of the corrections $\Delta Z_{cr}^{(i)}$: $\Delta Z_{cr}^{(1)}$—diffuse nature of the nuclear edge, $\Delta Z_{cr}^{(2)}$—screening by the shell of the neutral atom ($q=0$), $\Delta Z_{cr}^{(3)}$—screening by the vacuum shell.
3) After the values of $\Delta Z_{cr}^{(3)}$ there is indicated in brackets the number of electrons which were already present in the vacuum shell at the moment that the given sublevel enters the lower continuum.
4) All numbers refer to nuclei with normal proton density ($\delta = 1$).

$$A > A_2 \sim 2\cdot 10^5 (n/n_0)^4 [f(n,0)]^{-3}, \quad \eta \sim 50 (n/n_0)^{2/3} A^{-2/3}.$$

Here $f(n,\eta) = \varepsilon(n,\eta)/\varepsilon(n_0, \frac{1}{2})$; $\varepsilon(n,\eta)$ is the volume energy per particle in the nucleus. For these two regions of stability we have

$$\delta \approx \begin{cases} 1.3 n/n_0 \sim 3-10, & A < A_1 \\ 100 (n/n_0)^{2/3} A^{-2/3} \ll 1, & A > A_2 \end{cases}. \tag{2.14}$$

For neutron nuclei with $A \sim 10^6$ we obtain: $\delta \sim 0.01$ for $n = n_0$ and $\delta \sim 0.1$ for $n = 5n_0$. The values of Z_{cr} shown in Fig. 1 include both stability regions.

Superdense nuclei ($A < A_1$) can be both below critical and above critical. The boundary between these two regions is quite sharp; for critical nuclei $A = 320-330$ for $3 < \delta < 10$. On the other hand, neutron nuclei ($A > A_2$) have $Z/Z_{cr} \sim 20-100$ and lie in the beyond critical region. Such nuclei, if they exist in nature, must be surrounded by a vacuum shell including many electrons.[16]

3. EFFECT OF SCREENING ON Z_{cr}

The attraction of an electron by the nucleus is weakened due to the screening action of the electron shell, and this leads to an increase in Z_{cr}. Two types of screening are possible: a) screening by atomic electrons situated in the usual ($-1 < \varepsilon_n < 1$) levels of the discrete spectrum; b) screening by the vacuum shell ("levels" with $\varepsilon_n < -1$), forming around a nucleus with a

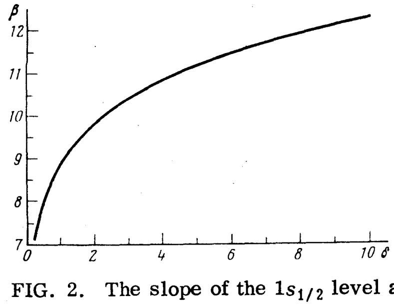

FIG. 2. The slope of the $1s_{1/2}$ level at the critical point (cut-off model II).

charge $Z > Z_{cr}$. An estimate of these effects is important in connection with designing an experiment on the spontaneous production of positrons in collisions of heavy nuclei. Such an experiment can be carried out not only with bare nuclei, but also with beams of bare nuclei Z_1 incident on a usual heavy target Z_2 if the following conditions are satisfied[22]

$$Z_1 + Z_2 > Z_{cr} \approx 170, \quad Z_1 \geq Z_2.$$

The possibility of working with an ordinary target consisting of neutral atoms (Z_2) makes the carrying out of the experiment much easier. But in this case the quasimolecule formed during the time of close approach of the nuclei is surrounded by electrons of the external shells and it is necessary to calculate Z_{cr} taking screening into account. In the region of uranium

$$R_{cr} \ll r_K \ll \bar{r}_a, \tag{3.1}$$

where $R_{cr} \sim 0.1$ is the critical distance between nuclei[3)], r_K is the radius of the K-shell with $R = R_{cr}$, $\bar{r}_a$ is the mean radius of the atom. In virtue of the conditions (3.1) it is sufficient for carrying out a calculation of screening in a system of two nuclei separated by distance $R < R_{cr}$ to consider the problem of a spherical superheavy nucleus with the total charge $Z = Z_1 + Z_2$. For $Z = Z_{cr}$ the velocity of K-electrons is of the order of the velocity of light, but for the majority of the electrons of the atomic shell the distance from the nucleus $r \sim \bar{r}_a \gg 1$, and the energy $\varepsilon_n \sim (Ze^3)^{4/3} = \zeta^{4/3}\alpha^{2/3} \ll 1$, and therefore the nonrelativistic Thomas–Fermi model is applicable to describe them.

The selfconsistent potential for the electron taking screening into account takes on the form $V(r) = -\zeta v_q(r)$, where

$$v_q(r) = \begin{cases} \varphi(x) v(r) + q r_0^{-1}, & r \leq r_0 \\ q r^{-1}, & r \geq r_0 \end{cases}. \tag{3.2}$$

The function $v(r)$ is defined in (2.1), r_0 is the radius of the positive ion, $q = (Z-N)/Z$ is the degree of ionization,[4)] $\varphi(x)$ is the solution of the Thomas–Fermi equation

$$x^{1/2}\varphi'' = \varphi^{3/2} \tag{3.3}$$

with the boundary conditions

$$\varphi(0) = 1, \quad \varphi(x_0) = 0, \quad x_0\varphi'(x_0) = -q. \tag{3.4}$$

Here x is the dimensionless variable:

$$x = \left(\frac{128Z}{9\pi^2}\right)^{1/3} \frac{r}{a_B} = C_1 \zeta^{1/3} r, \quad C_1 = \frac{1}{0.885\alpha^{-2/3}} = 0.0425.$$

Substituting the perturbation $\delta V(r) = \zeta[v(r) - v_q(r)]$ into (2.7) we take into account that the principal contribution to the matrix element $\langle \delta V \rangle$ comes from the region $r \sim r_K \ll \bar{r}_a$ in which one can utilize the expansion (A.1). We denote by $\Delta Z_{cr}^2(q)$ the increase in the critical charge due to screening. In the given approximation the dependence on the degree of ionization q can be factored:

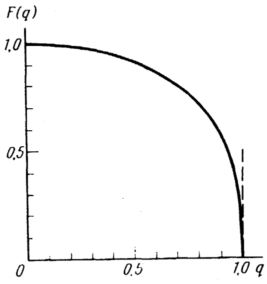

FIG. 3. Dependence of the correction due to screening on the degree of ionization of the atom q.

$$\Delta Z_{cr}^{(2)}(q)=\Delta Z_{cr}^{(2)}(0)F(q), \tag{3.5}$$

$$F(q)=\frac{1}{\gamma(0)}\left[\gamma(q)-\frac{q}{x_0(q)}\right]; \tag{3.6}$$

here $\gamma=\gamma(q)$ is the slope of the curve $\varphi(x)$ at the origin, cf., formula (A.2). Numerical calculation yields for the function $F(q)$ the curve shown in Fig. 3. We also give its asymptotic expansion:

$$F(q)=\begin{cases}1-a_3q^{4/3}+\ldots, & q\to 0\\ a_4(1-q)^{1/2}+\ldots, & q\to 1\end{cases} \tag{3.7}$$

(the derivation of these formulas, and also certain details of the numerical calculation are discussed in the Appendix: the constants a_3 and a_4 are also given there).

The correction $\Delta Z_{cr}^{(2)}$ for neutral ($q=0$) atoms was obtained by a numerical solution of the Dirac equation with the potential $V(r)=-\zeta r^{-1}\varphi_0(x)$ and the energy $\varepsilon=-1$; to carry this out we used the phase method described in Ref. 23. The values of $\Delta Z_{cr}^{(2)}$ obtained in this manner are shown in Table I.

For q not too close to unity the function $F(q)$ varies slowly (see Fig. 3). Therefore the correction for screening in an ion with $q\approx 0.5$ is almost the same as in the case of complete screening: $F(0.5)=0.907$. This is explained by the fact that as the degree of ionization increases the electron shell draws in towards the nucleus ($x_0(q)$ decreases with increasing q), and this partially compensates for the decrease in the screening charge of the shell equal to $Z(1-q)$.

In deriving formulas (3.5) and (3.6) the potential $\delta V(r)$ was replaced by its value $\delta V(0)$ at the centre of the nucleus. Such an approximation is justified for $r_K \ll r_0$. This inequality is satisfied in virtue of the fact that the radius of the K-shell is $r_K\sim 1$, while the radius of the ion is $r_0\sim\alpha^{-1}$.

Utilizing the asymptotic expansion (A.11) for $x_0(q)$ we write the condition for the applicability of formula (3.5) in the form

$$1-q\gg {}^1/_3\alpha({}^1/_2\zeta_{cr})^{1/3}r_K^{4/3}\sim 3\cdot 10^{-4}. \tag{3.8}$$

This condition is violated only in the case of very high degree of ionization ($q\to 1$) when the Thomas–Fermi approximation itself ceases to be applicable. Since in this case the correction for screening also disappears formulas (3.5), (3.6) can be utilized practically always.

4. VACUUM SCREENING

After the spontaneous production of positrons and their departure to infinity a vacuum shell remains surrounding the nucleus. We take the distribution of charge in it to be given by the relativistic Thomas–Fermi model.[16] For the first levels of the discrete spectrum the parameter $Z_{cr}e^3=0.085\zeta_{cr}\ll 1$. In this case the selfconsistent potential $V(r)$ can be expanded in series in powers of Ze^3:

$$V(r)=V_0(r)+\frac{4}{3\pi}(Ze^3)^2V_1(r)+\ldots \tag{4.1}$$

In this expression $V_0(r)$ given by formula (2.1), while

$$\begin{aligned}V_1(r)&=\frac{\zeta}{R}\left[1+\frac{1}{x}\int_0^x f^3(x')x'^2dx'+\int_x^1 f^3(x')x'dx'\right], \quad 0<r<R,\\ V_1(r)&=(\ln(\zeta/R)+c_1)\zeta/r-{}^1/_4\psi_1(x), \quad R<r<r_a;\\ V_1(r)&=(\ln(\zeta/R)+c_1)\zeta/r, \quad r>r_a.\end{aligned} \tag{4.2}$$

Here $x=r/r_a=2r/\zeta$, $r_a=\zeta/2$ is the radius of the vacuum shell; the values of c_1 and $\psi_1(x)$ are given in Ref. 16. The correction $\sim V_1$ in (4.1) describes the distortion of the electrostatic potential of the nucleus by the vacuum electrons situated in "levels" of energy $\varepsilon_n<-1$.

We represent (4.2) in the following form:

$$V_1(r)=\left(\ln\frac{\zeta}{R}+c_1\right)\zeta v(r)-w(r), \tag{4.3}$$

where

$$\begin{aligned}w(r)&=R^{-1}\zeta[(\ln(2\zeta/R)-{}^{11}/_3)f(\rho)-g(\rho)] \quad \text{for } 0<r<R,\\ w(r)&=\frac{1}{x}\left[(2+3x)\ln\frac{1+(1-x)^{1/2}}{1-(1-x)^{1/2}}-\frac{5}{2}(1-x)^{1/2}+\frac{2}{3}(1-x)^{3/2}\right], \quad \text{for } R<r<r_a,\\ w(r)&=0 \quad \text{for } r>r_a.\end{aligned} \tag{4.4}$$

Here $\rho=r/R$, $f(\rho)$ is the cut-off function for the Coulomb potential inside the nucleus (cf., (2.1)),

$$g(\rho)=1-f(\rho)\int_0^1 f^3(x)x^2dx-f(\rho)+\rho^{-1}\int_0^\rho f^3(x)x^2dx+\int_\rho^1 f^3(x)x\,dx.$$

We note that $g(1)=0$ for any kind of cut-off. If $f(\rho)=1$, which corresponds to the cut-off model I (cf., Ref. 7), then $g(\rho)=(1-\rho^2)/6$. For model II we have

$$\begin{aligned}f(\rho)&={}^1/_2(3-\rho^2),\\ g(\rho)&={}^1/_{192}(1-\rho^2)(279.1-27.59\rho^2+4.810\rho^4-0.3333\rho^6).\end{aligned} \tag{4.5}$$

Substituting (4.3) into (2.7) and taking into account the fact that the total number of electrons in the vacuum shell is equal to

$$N_e=\frac{4}{3\pi}\zeta^3\left(\ln\frac{\zeta}{R}+c_1\right),$$

$c_1=-1.38$ for model II, we calculate the shift in Z_{cr} due to the screening effect of the vacuum shell to be given by:

$$\Delta Z_{cr}^{(3)}=(1-\mu)N_e, \tag{4.6}$$

$$\mu=[\beta\zeta_{cr}(\ln(\zeta_{cr}/R)+c_1)]^{-1}\int\limits_0^{\zeta_{cr}/2} w(r)\chi_{cr}^2(r)\,dr, \qquad (4.7)$$

where β is the slope of the level at the boundary of the lower continuum. If the radius of the vacuum shell were small compared to the radius of the K shell

$$r_K=\int\limits_0^\infty \chi_{cr}^2(r)\,r\,dr,$$

then $\Delta Z_{cr}^{(3)}$ would have been equal to N_e. In actual fact r_a and r_K are quantities of the same order of magnitude; this leads to the appearance of the factor $(1-\mu)$ which decreases the correction $\Delta Z_{cr}^{(3)}$.

The values of μ and $\Delta Z_{cr}^{(3)}$ obtained from formula (4.7) are shown in Table I. It should be emphasized that the correction $\Delta Z_{cr}^{(3)}$ refers to the case when at the moment that a particular level drops into the lower continuum all the preceding states ($\varepsilon_n<-1$) are filled by vacuum electrons. If the initial nucleus is bare (i.e., completely stripped), then for this it is necessary to wait a time $\tau\sim\gamma^{-1}$ in order for the positrons from the corresponding quasistationary levels to be removed to infinity.[5)]

Of the effects considered by us which increase Z_{cr} the greatest is the increase in Z_{cr} due to vacuum screening. This can be easily understood since the corresponding charge density $n_e(r)$ is situated at the same distances from the nucleus as the electron at the critical point $\zeta=\zeta_{cr}$. But the effect of vacuum screening is not great for the lowest levels ($1s_{1/2}$ and $2p_{1/2}$), which are of the greatest interest from the experimental point of view.

We make two additional remarks.

1. Let a level of angular momentum j be lowered into the lower continuum. From this level $2j+1$ positrons will be emitted with the number of electrons in the vacuum shell N_e increasing by unity with the emission of each positron. A consequence of this is the splitting of Z_{cr} for a given level into $2j+1$ equidistant values separated by an interval $1-\mu$ (cf., formula (4.6)). The energy spectrum of the positrons emitted by the nucleus consists of $2j+1$ closely spaced lines. For the lowest levels of the discrete spectrum Z_{cr} is split into two values, since $j=\frac{1}{2}$. The corresponding values of $\Delta Z^{(3)}$ are shown in Table I with the number of electrons N_e in the vacuum shell being shown for each case.

The final value of Z_{cr} is obtained by summing $Z_{cr}^{(0)}$ and the different corrections $\Delta Z_{cr}^{(i)}$. For example, for the K shell in the case of the bare nucleus $Z_{cr}=169.2$ when the first positron is emitted and $Z_{cr}=170.0$ when the second positron is emitted.

2. For the lowest levels the number of electrons in the vacuum shell N_e is not great and therefore the question arises of the accuracy of the Thomas–Fermi statistical method. With this aim in mind the calculation of $\Delta Z_{cr}^{(3)}$ was carried out by a different method utilizing the density of the vacuum K-electrons $n_e(r)$ obtained by Gyulassy[24] by means of a numerical solution of the Dirac equation. The correction for screening has the following form

$$\Delta Z_{cr}^{(3)}=\frac{1}{\beta}\int \chi_{cr}^2(r)\frac{e^2}{|\mathbf{r}-\mathbf{r}'|}n_e(r')\,d^3r\,d^3r'=N_e(1-\mu). \qquad (4.8)$$

After certain transformations we obtain:

$$\mu=\frac{I_1+I_2}{\beta},\quad I_1=\int\limits_R^\infty dr\chi_{cr}^2(r)\int\limits_r^\infty\left(\frac{1}{r}-\frac{1}{r'}\right)\rho_e(r')\,dr',$$

$$I_2=\int\limits_0^R dr\chi_{cr}^2(r)\left[1-\frac{1}{r}\int\limits_0^r\rho_e(r')\,dr'-\int\limits_r^\infty\rho_e(r')\frac{dr'}{r'}\right], \qquad (4.9)$$

where

$$\rho_e(r)=\frac{4\pi n_e(r)r^2}{N_e},\quad \int\limits_0^\infty\rho_e(r)\,dr=1.$$

The normalized density $\rho_e(r)$ for the beyond critical K shell ($1s_{1/2}$, $N_e=2$) has been calculated by Gyulassy[24] for the following values of the parameters: $\zeta_{cr}=1.383$, $R=0.0259$, which corresponds to the critical charge of the nucleus for the $2p_{1/2}$ level (with $f(r/R)=\theta(R-r)$, i.e., for the cut-off model I). A calculation of the slope of the $2p_{1/2}$ level according to formula (2.8) yields $\beta=9.66$, as a result of which we obtain from (4.9) $\mu=0.310$. On the other hand the use of the Thomas–Fermi model for $n_e(r)$ (cf., formula (4.7)) yields[6)] $\mu_{TF}=0.247$. This leads to a difference of ~10% in the values of the correction $\Delta Z_{cr}^{(3)}$. It is natural to expect that in going over to the next levels the accuracy of the Thomas–Fermi method improves.

5. PROPERTIES OF ELECTRON STATES FOR $Z\geqslant 137$

Calculations were made of different quantities characterizing the state of the electron for $Z\geqslant 137$; in doing so particular attention was paid to the critical point $Z=Z_{cr}$, $\varepsilon=-1$. The results of the calculations for the first four levels of the discrete spectrum are collected in Table II. We explain the notation.

The meaning of the quantities ε_1, ζ_0, ζ_{cr} and $\beta=-[d\varepsilon/d\zeta]_{\zeta=\zeta_{cr}}$ is clear from Fig. 4. Further, $\varkappa=\mp(j+\frac{1}{2})$ is an integral of the motion for a Dirac electron in a central field,[19] R is the nuclear radius at the critical point. In obtaining ζ_{cr} and R the dependence $R=r_0A^{1/3}$ was utilized with the parameters $r_0=1.1$ F and $A/Z=2.6$. The quantities $\bar{r}$, r_{max}, r_0, w_R, w_0 and ν characterize the probability distribution for an electron for $Z=Z_{cr}$: r_{max} is the point at which the electron density $\chi_{cr}^2(r)$ has

TABLE II.

	$1s_{1/2}$	$2p_{1/2}$	$2s_{1/2}$	$3p_{1/2}$		$1s_{1/2}$	$2p_{1/2}$	$2s_{1/2}$	$3p_{1/2}$
$\varkappa$	−1	1	−1	1	$\bar{r}$	0.303	0.229	0.534	0.439
ε_1	0.218	0.779	0.802	0.916	r_0	0.211	0.284	0.549	0.660
ζ_0	1.068	1.220	1.415	1.634	σ	1.123	1.283	0.803	0.881
ζ_{cr}	1.232	1.323	1.691	1.858	$w_R\cdot 100$	2.00	3.73	1.92	2.65
β	8.87	11.82	4.62	5.61	w_0	0.466	0.238	0.361	0.187
ξ	0,290	1.004	−0,723	0.396	ν	1.645	0.333	1.028	0.333
$R\cdot 100$	2,17	2.22	2,41	2.48	$-\left.\frac{d\zeta_{cr}}{d\delta}\right\|_{\delta=1}$	0.030	0,044	0,074	0.093
$r_{max}\cdot 100$	5,97	4.50	2,33	2.12					

Note. All the quantities given in the table have been calculated for $\delta=1$, i.e., for the same value of the proton density as in ordinary heavy nuclei. The lengths R, r_{max} and $\bar{r}$ are expressed in units of $\hbar/m_ec=386.1$ F.

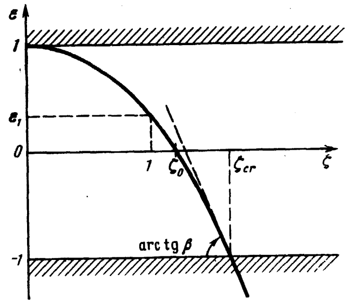

FIG. 4. Approximate variation of the energy of the level ε with increasing $\zeta = Z\alpha$.

its maximum value, r_0 is the quasiclassical turning point,[7] w_R is the probability of finding the electron within the nucleus. They are equal to

$$\bar{r}=\int_0^\infty \chi_{cr}^2(r)\,r\,dr,\quad r_0=(\zeta^2-\varkappa^2)/2\zeta, \tag{5.1}$$

$$w_R=\int_0^R \chi_{cr}^2(r)\,dr,\quad w_0=\int_{r_0}^\infty \chi_{cr}^2(r)\,dr,\quad \nu=\frac{w_2}{w_1}, \tag{5.2}$$

where

$$w_1=\int_0^\infty g^2(r)r^2dr=\frac{1}{1+\nu},\quad w_2=\int_0^\infty f^2(r)r^2dr=\frac{\nu}{1+\nu}. \tag{5.3}$$

The function $\chi_{cr}(r)=r(g^2+f^2)^{1/2}$ refers to the energy $\varepsilon=-1$ and is normalized in accordance with the condition (2.9).

We discuss the results obtained above. From Table II it can be seen that $r_{max}\sim R$, while the average radius $\bar{r}$ is by a factor of 10–20 greater than R. Thus, for $Z=Z_{cr}$ the electron is basically situated outside the nucleus. In this respect the situation reminds one of the deuteron, but with the significant difference that the electron is completely relativistic and is held at distances $r\sim\hbar/mc=1$ by the Coulomb barrier in the effective potential $U(r)$. Therefore it is not accidental that $\bar{r}$ and r_0 are quantities of the same order of magnitude. In Table II $\sigma=[(\overline{r^2}-\bar{r}^2)/\bar{r}^2]^{1/2}$ is the dispersion of the electron cloud. For $Z<137$ one can neglect the nuclear radius and utilize the wave functions for a point charge. For example, for the ground $1s_{1/2}$ level we have

$$\bar{r}=(2\zeta)^{-1}[1+2(1-\zeta^2)^{1/2}],\quad \sigma=[1+2(1-\zeta^2)^{1/2}]^{-1/2} \tag{5.4}$$

(for $\zeta\ll 1$ these formulas go over into the well-known expressions for the nonrelativistic hydrogen atom). With increasing Z the average radius of the K shell decreases monotonically, while the dispersion σ increases. This tendency is preserved also in going over into the region $\zeta>1$.

If one departs from the relation between R and Z specified by the formula $R=r_0A^{1/3}$, one can obtain the curve $\zeta_{cr}\equiv\zeta_{cr}(R)$ (cf., Fig. 2 in Ref. 7), and also one can study the dependence of different quantities[8] on the nuclear radius R. For the probability w_R we obtain with good accuracy the linear dependence $w_R=\text{const}\cdot R$ (cf., Fig. 5). We show that this fact is associated with the "collapse towards the center" in the Dirac equation with the Coulomb potential $V(r)=-\zeta/r$.

Retaining in the system of equations[19] for the radial function $g(r)$ and $f(r)$ the terms most singular at the origin, we obtain the behavior of g and f as $r\to 0$:

$$f(r),g(r)\propto r^{\sqrt{\varkappa^2-\zeta^2}-1}. \tag{5.5}$$

For the second (singular) solution we have

$$f,g\propto r^{-\sqrt{\varkappa^2-\zeta^2}-1}.$$

As long as $\zeta<|\varkappa|=j+\frac{1}{2}$ there exists a criterion for the choice of the solution of the Dirac equation regular at the origin. Collapse towards the center arises when the coupling constant has the value $\zeta=|\varkappa|=j+\frac{1}{2}$. In this case f and g have at the origin a singularity $\propto r^{-1}$, which is not permissible from the point of quantum mechanics.

The cutting off of the Coulomb potential at $r<R$ makes the wave function finite at the origin even for $Z=Z_{cr}$, with

$$\psi_0(0)\propto 1/R,\quad w_R\propto R,\quad R\ll 1$$

($\psi_0(r)=\chi_{cr}(r)/r$, $w_r\sim R^3\psi_0^2(0)$). Although the probability w_R is small for $R\ll\hbar/mc$, nevertheless the limiting transition to $R=0$ is impossible. This can also be seen from the asymptotic formulas for ε_1 (the position of the levels for $Z=137$), ζ_0 and ζ_{cr}. These quantities are nonanalytic with respect to R at the point $R=0$. Thus in the case of the ground $1s_{1/2}$ state we have

$$\varepsilon_1=\Lambda^{-1},\ \zeta_0=1+{}^1/_8\pi^2\Lambda^{-2},\quad \zeta_{cr}=1+{}^1/_2\pi^2\Lambda^{-2} \tag{5.6}$$

(here $\Lambda=-\ln R\gg 1$).

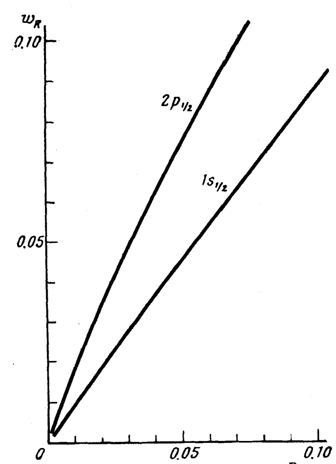

FIG. 5. Probability of finding the electron inside the nucleus for $Z=Z_{cr}$ (the radius R is measured in units of $\hbar/m_ec$ = 386 F).

The quantity w_0 is equal to the probability of penetration of the electron into the classically forbidden region $r>r_0$ and is quite large, particularly for the lowest levels. For highly excited states with $\zeta\gg|\varkappa|$ we have

$$w_0=c_2\zeta^{-1/3},\quad \bar{r}/r_0={}^3/_5+c_3\zeta^{-2}+\ldots,$$
$$\sigma=(17/63)^{1/2}(1+c_4\zeta^{-2}+\ldots), \tag{5.7}$$

where

$$c_2=0.158,\ c_3={}^1/_{20}(2\varkappa^2-11\varkappa+9),\ c_4={}^5/_{51}(3\varkappa^2-\varkappa+12).$$

The ratio of the probabilities $\nu = w_2/w_1$ characterizes the degree of "relativism" of the electron: at the critical point ν is of order of magnitude unity. In terms of this parameter one can express the magnetic moment of the electron for $\zeta = \zeta_{cr}$:

$$\mu_{cr} = \frac{2\varkappa(1-\varkappa)(\nu^2+1)}{(1-4\varkappa^2)(\nu^2+6\varkappa^2-3\varkappa+1)}. \tag{5.8}$$

All the quantities collected in Table II refer to definite values of the parameters r_0 and A/Z. In order to make a recalculation for other values of these parameters it is not difficult to obtain the following formulas:

$$d\zeta_{cr}/d\delta = (1-3\lambda^{-1})^{-1}\zeta_{cr}/\delta, \quad \Delta\delta = \Delta\eta/\eta - 3\Delta r_0/r_0,$$

where $\lambda = d\ln\zeta_{cr}/d\ln R$. The values of $-d\zeta_{cr}/d\delta$ for $\delta = 1$ are given in Table II; they enable us to recalculate ζ_{cr} for any desired values of the proton density, which are not too different from the value of $n_p^{(0)}$ adopted by us.

6. CONCLUSION

1. In connection with the problem of anomalous nuclei the dependence of the critical charge Z_{cr} on the proton density has been obtained. It is shown that superdense nuclei, the possibility of which is indicated by the theory of Migdal,[10–13] will be above critical for $A > 320$–330.

2. For an atom with an arbitrary degree of ionization we have calculated Z_{cr}, and also we have obtained the change in Z_{cr} due to the screening of the field of the nucleus by the vacuum shell. An estimate of these effects is particularly important in connection with designing an experiment on the spontaneous production of positrons in collision of heavy nuclei (of the type $U+U$). Indeed, the total charge of the two nuclei from the neighborhood of uranium exceeds by only 15–20 units the value of $Z_{cr} \approx 170$ calculated without taking screening into account.[6,7] Therefore an increase in Z_{cr} by 10–15 units would have made the carrying out of such an experiment considerably more difficult.

However it turns out that screening decreases the critical charge only by an amount $\Delta Z_{cr} \sim 1.5$, which does not close off the possibility of carrying out an experiment with heavy elements known at present.

3. Numerical calculations have been carried out of different physical quantities characterizing the state of an electron at the boundary of the negative continuum ($Z = Z_{cr}$). From these results, which were discussed in detail in Sec. 5, we note the following one.

It is shown that the probability of an electron existing within the nucleus is $w_R = cR \sim 10^{-2}$, while the mean radius of the electron state $\bar{r}$ exceeds the nuclear radius R by an order of magnitude. Therefore the quantities ζ_{cr}, β, ν, ... have only a weak dependence on the specific form of cutting off the Coulomb potential inside the nucleus; basically they are determined by the region $r > R$ in which the potential $V(r)$ is known. This guarantees good accuracy for theoretical calculations of the critical parameters (ζ_{cr}, β, $\chi^2_{cr}(r)$, etc.), in spite of the fact that these quantities are nonanalytic with respect to R at the point $R=0$ and the limiting transition to the point charge $R \to 0$ in the region $\zeta > 1$ is in principle impossible.

APPENDIX. ALLOWANCE FOR SCREENING IN THE THOMAS-FERMI MODEL

As $x \to 0$ the solution of (3.3) has the expansion

$$\varphi(x) = 1 - \gamma x + \sum_{n=3}^{\infty} c_n x^{n/2} \tag{A.1}$$

(the Baker series). Here

$$\gamma = -\varphi'(0), \tag{A.2}$$

$c_3 = \frac{4}{3}$, $c_4 = 0$; the remaining coefficients c_n are expressed as polynomials in terms of γ (explicit expressions for them up to $n = 1$ are given in the paper by Feynman *et al.*[25] and up to $n = 17$ in the paper by Kobayashi *et al.*[26]) It is more convenient to utilize the series (A.1) for small x, however in this case it is necessary to know the dependence of the slope γ on the degree of ionization q and for this it is necessary to solve equation (3.3) over the whole region $0 \leqslant x \leqslant x_0(q)$ taking into account the boundary conditions (3.4). We obtain $\gamma = \gamma(q)$ in two limiting cases: $q \to 0$ and $q \to 1$.

In the former case $x_0(q) \to \infty$. Setting

$$\varphi(x) = \varphi_0(x) y(x), \tag{A.3}$$

where $\varphi_0(x)$ is the well-known[25,27] solution for a neutral atom we obtain

$$y'' + 2y'\varphi_0'/\varphi_0 + (\varphi_0/x)^{1/2}(y - y^{3/2}) = 0. \tag{A.4}$$

Since $\varphi_0(x) \approx 144x^{-3}$ as $x \to \infty$, then in the region $1 \ll x \ll x_0$ the exact equation (A.4) is simplified:

$$x^2 y'' - 6xy' + 12(y - y^{3/2}) = 0, \tag{A.5}$$

and its solution has the scaling property: $y(x) = w(t)$, $t = x/x_0$, $0 < t < 1$. The solution which we require is fixed by the boundary conditions $w(0) = 1$, $w(1) = 0$. Numerical integration of equation (A.5) shows that the function $w(t)$ is a monotonically decreasing one with

$$w(t) = \begin{cases} 1 - p_1 t^\rho + \ldots, & t \to 0 \\ p_2(1-t) + \ldots, & t \to 1 \end{cases}; \tag{A.6}$$
$$\rho = (\sqrt{73}+7)/2 = 7.772, \quad p_1 = 1.040, \quad p_2 = -w'(1) = 7.439.$$

For $x \ll x_0$ the function $\varphi(x)$ can also be obtained by another method, specifically by means of the perturbation theory developed by Fermi[27]:

$$\varphi(x) = \varphi_0(x) - k\eta(x) + O(k^2). \tag{A.7}$$

Here k is a small parameter:

$$k = \gamma(q) - \gamma(0); \tag{A.8}$$

$$\eta(x) = \psi(x)\int_0^x \frac{dz}{\psi^2(z)}, \quad \psi(x) = \frac{1}{3x^2}[x^3\varphi_0(x)]'. \tag{A.9}$$

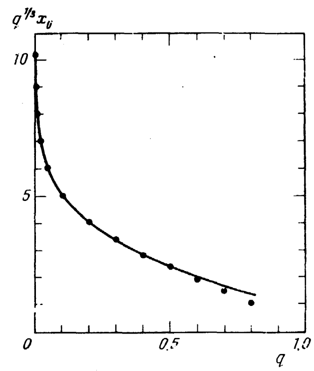

FIG. 6. Dependence of the quantity $q^{1/3}x_0$ on the degree of ionization q for positive ions; the radius of the ion in atomic units is $r_0 = 0.885Z^{-1/3}x_0(q)$. The solid curve is constructed using equation (A. 11); in doing so the first two terms of the expansion are taken into account. Points denote results of a numerical solution of the Thomas-Fermi equation.

As $q \to 0$ there is a connecting region $1 \ll x \ll x_0(q)$ in which the solutions of (A. 3) and (A. 7) coincide; from here one determines the dependence of γ on q. In carrying out this calculation we utilize the expansion

$$\varphi_0(x) = 144x^{-3}\left[1 + \sum_{n=1}^{\infty}(-1)^n q_n x^{-n\sigma}\right], \qquad (A. 10)$$

where[27,28] $\delta = (\sqrt{73}-7)/2 = 0.7720\ldots$, $q_1 = 13.271$. From (A. 9) as $x \to \infty$ we obtain:

$$\psi(x) = 48\sigma q_1 x^{-(3+\sigma)}, \quad \eta(x) = [48q_1\sigma(2\sigma+7)]^{-1}x^{\sigma+4}.$$

We state the final formulas for $q \to 0$:

$$x_0 = C_0 q^{-1/3}\left[1 + \sum_{n=1}^{\infty}\alpha_n q^{ns}\right], \qquad (A. 11)$$

$$k = C_1 q^r[(1+\alpha_1 q^s)^{-\rho} + \ldots). \qquad (A. 12)$$

Here

$$r = \rho/3 = 2.591, \quad s = \sigma/3 = 0.2573. \quad C_0 = (144p_2)^{1/3} = 10.23,$$

$$C_1 = 2^8 \cdot 3^3\sqrt{73}\sigma p_1 q_1 C_0^{-\rho} = 8.90 \cdot 10^{-3}, \quad \alpha_1 = -0.913.$$

The expansion parameter in (A. 11) is q^s. We can verify this by substituting $y(x)$ into (A. 4) in the form

$$y(x) = \sum_{n=0}^{\infty} w_n(t) x^{-n\sigma}.$$

For $w_n(t)$ we obtain a chain of equations from which these functions are determined sequentially. The first of these coincides with (A. 5); we note that $w_0(t) \equiv w(t)$. The boundary conditions have the form $w_n(0) = \delta_{n,0}$, $w_n(1) = 0$. The smallness of the exponent s leads to the fact that in the practically important region $0.01 < q < 0.1$ one cannot restrict oneself to only the first term of the series (A. 11). Taking into account two terms of this series already guarantees very good accuracy (cf., Fig. 6).

The dependence of the slope $\varphi'(0) = -\gamma(0) - C_1 q^r + \ldots$ on q is exceedingly weak for small q, and this creates definite difficulties in numerical calculations. The only attempt known to us of determining the asymptotic expressions for x_0 and δ as $q \to 0$ is due to Fermi and Amaldi[29] who have proposed on the basis of numerical calculations the interpolation formula $k = 0.083q^3$. As can be seen from (A. 12) the power exponent r in the exact asymptotic expression differs somewhat from 3.

The case $q \to 1$ is simpler. The substitution $\varphi = q\psi$, $t = x/x_0$ brings the Thomas-Fermi equation into the form ($\dot\psi = d\psi/dt$)

$$t^{1/2}\ddot\psi = \beta\psi^{3/2}, \quad \beta = (qx_0^3)^{1/2}, \qquad (A. 13)$$

with the boundary conditions

$$\psi(0) = q^{-1}, \quad \psi(1) = 0, \quad \dot\psi(1) = -1. \qquad (A. 14)$$

Since x_0 and $\beta \to 0$, it is convenient to represent $\psi(t)$ in the form of a series in powers of β (cf., the work of Plindov and Dmitrieva[30]). As a result we have as $q \to 1$

$$\beta \approx \frac{16}{\pi}(1-q), \quad x_0 = q^{-1/3}\beta^{2/3} \approx \left[\frac{16}{\pi}(1-q)\right]^{2/3}. \qquad (A. 15)$$

The expansions (3. 7) follow directly from (A. 11) and (A. 15); in these expansions $a_3 = [C_0\gamma(0)]^{-1} \approx 0.062$, $a_4 = \gamma(0)^{-1}(\pi/16)^{2/3} \approx 0.21$.

The evaluation of the quantities $\gamma(q)$, $x_0(q)$ and $F(q)$ for $q \geq 0.1$ was carried out utilizing equation (A. 13). The values of the corrections for screening $\Delta Z_{cr}^{(2)}$ given in Table I were obtained by a numerical solution of the Dirac equation with the aid of the phase method. For the ground state this correction had been calculated previously.[23]

1) A brief explanation of the results of this section has been published previously.[17] Taking advantage of the present opportunity we wish to correct a misprint on p. 256 of Ref. 17 [p. 228 of the translation]: the estimate of the change ΔZ_{cr} due to the diffuse nature of the nuclear boundary amounts to 0.7% (instead of the 7% given there).

2) See Ref. 7. The possibility of obtaining for Z_{cr} an equation of the form (2. 2) is associated with the fact that the Dirac and the Klein-Gordon equations in a Coulomb field $V(r) = -\zeta/r$ at an energy $\varepsilon = -1$ have exact solutions expressed in terms of the MacDonald function. We note that the function $K_{i\nu}(z)$ is real for $-\infty < \nu < \infty$ and $z > 0$.

3) All lengths are measured in units of $\hbar/m_e c = 386$ F; in these units the Bohr radius is given by $a_B = \alpha^{-1} = 137$. According to the Thomas-Fermi model the average radius of the atom is $\bar{r}_a \sim Z^{-1/3}a_B = 27\zeta^{-1/3} \gg 1$.

4) Here N is the total number of electrons in the atomic shell. For a neutral atom $q = 0$, for a bare nucleus $q = 1$. The quasi-molecule formed in the collision of a bare nucleus with a neutral atom corresponds to the values $q = Z_2/(Z_1 + Z_2) \sim 0.5$.

5) As long as the positron is inside the Coulomb barrier its contribution to the charge density completely compensates the charge of the electron of the vacuum shell.[2] Therefore the correction to Z_{cr} due to screening by the vacuum shell is absent for $\tau \ll \gamma^{-1}$.

6) The value of the parameter $\mu = 0.247$ differs from the one given in Table I for the $2p_{1/2}$ level ($\mu = 0.272$) due to the fact that here we have used the cut-off model I.

7) In investigating the states near the boundary of the lower continuum it is convenient to reduce the Dirac equation[2,7] to the Schrödinger equation with the effective potential $U(r)$. For a Coulomb field $U(r) = \zeta/r - (\zeta^2 - \kappa^2)/2r^2$ for $\varepsilon = -1$. The formula given above for r_0 follows from the condition $U(r_0) = 0$. We note that for the 1s-level $\bar{r}$ coincides with the radius r_K introduced above.

8) In the course of this, one in fact determines the dependence

of the critical parameters (ζ_{cr}, $\bar{r}$, r_{max} etc.) on the proton density n_p.

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Translated by G. Volkoff

Single-nucleon absorption of slow pions by atomic nuclei and π condensation

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(Submitted May 4, 1977)
Zh. Eksp. Teor. Fiz. 73, 1258–1270 (October 1977)

The problem of the single-nucleon absorption of slow pions by atomic nuclei is solved. The presence of a pion condensate significantly increases the single-nucleon absorption probability. The measurement of the single-nucleon absorption probability may be a critical experiment for the elucidation of the question of the existence of a condensate in nuclear systems.

PACS numbers: 21.65.+f, 25.80.+f

1. INTRODUCTION

In 1971 Migdal pointed out the possibility of a reconstruction of the pion field in a sufficiently dense nucleon system, i.e., the formation of a "pion condensate." The main physical consequence of such a phase transition is the possibility in principle of the existence of abnormally dense nuclei.[1] In these nuclei the energy loss due to the change in the nucleon density is compensated by the energy gain from the phase transition. The quantitative theory developed by Migdal[2] led to a critical-density value (the density at which the phase transition occurs) of $n_c \gtrsim 0.6 n_0$ for nuclear matter with $N=Z$ and $n_c \gtrsim 0.8 n_0$ for a neutron material ($N \gg Z$), where n_0 is the normal nuclear density. This allowed the existence of a condensate in real atomic nuclei to be postulated. In subsequent investigations[3,4] the estimate for the critical density did not change in comparison with the estimate obtained in Migdal's first papers. A more exact computation of the critical density is not possible,

 0038-5646/77/4604-0662$02.40