Quantum singularities of a ponderomotive meter of electromagnetic energy

V. B. Braginskiĭ, Yu. I. Vorontsov, and F. Ya. Khalili

Moscow State University
(Submitted May 24, 1977)
Zh. Eksp. Teor. Fiz. 73, 1340–1343 (October 1977)

It is shown that when energy is measured in an electromagnetic resonator by determining the ponderomotive force, high sensitivity can be attained without greatly disturbing the value of the energy. An energy-measurement procedure and conditions under which the measurement error is less than one quantum are described.

PACS numbers: 03.65.Bz, 06.30.Lz

In connection with the problem of raising the sensitivity in a number of physical experiments, a need arose for developing methods of measuring the energy of the electromagnetic field in microwave resonators with a minimum possible energy-measurement error and with a minimum perturbation introduced by the measurement.[1,2] It is well known that an oscillator can be in a state with a given energy, and quantum theory presupposes the existence of a measurement method such that the energy is not perturbed in the course of the measurement. So far, however, there are no known realizations of such a method.

The general principle of constructing a non-perturbing meter is known[3]—the Hamiltonian of the interaction of a quantum system with the measuring instrument must be diagonal in the same representation as the measured quantity. If the Hamiltonian of the interaction of an electromagnetic resonator with the measuring device is not purely diagonal in the energy representation, then the energy perturbation will be less the smaller the relative value of the off-diagonal terms and the farther their frequency spectrum (if they oscillate) from the resonator frequency. The purpose of the present article is to show that a method wherein energy is measured in an electromagnetic resonator by using the ponderomotive interaction with a mechanical oscillator can be a weakly perturbing one.

Let us examine the scheme of such an experiment (see Fig. 1). One of the capacitor plates (with mass m) together with a spring of stiffness $k=m\omega_m^2$ make up a mechanical oscillator. The ponderomotive force F of the attraction of the capacitor plates changes the equilibrium position of the plate and causes it to oscillate. The displacement of the equilibrium position is proportional to the energy in the resonator $\mathscr{E}=q^2/2C$, where q is the electric charge on the plate. The Hamiltonian of the interaction of the electric field with the mechanical system is of the form

$$H_I=-q^2x/2C_0d, \tag{1}$$

where q is the instantaneous value of the charge, x is the displacement of the plate, and C_0 and d are the initial capacity and the distance between the plate. If $x/d\ll 1$, then $q\approx q_0\cos\omega_e t$ and

$$H_I\approx-\mathscr{E}_0\frac{x}{2d}-\frac{1}{2}\mathscr{E}_0\frac{x}{d}\cos 2\omega_e t, \tag{1'}$$

where $\mathscr{E}_0=q_0/2C_0$. The first term in (1′) is diagonal in the energy representation, and the second is off-diagonal. At $\omega_m\ll\omega$, however, and at an interaction time $t\gg 1/\omega_e$, the spectrum of the second term has no resonant terms. The motion of the analyzed system at small x/d can be approximately described as follows: the operator of the plate coordinate is equal to

$$\hat{x}(t)\approx\frac{1}{2}\frac{\hat{\mathscr{E}}_0}{kd}+\left(\hat{x}(0)-\frac{1}{2}\frac{\hat{\mathscr{E}}_0}{kd}\right)\cos\omega_m't+\frac{\hat{\dot{x}}(0)}{\omega_m'}\sin\omega_m't, \tag{2}$$

and the operator of the resonator energy is

$$\hat{\mathscr{E}}(t)\approx\hat{\mathscr{E}}_0(1-\hat{x}(t)/2d). \tag{3}$$

We have left out of (2) the term

$$\frac{\hat{\mathscr{E}}_0}{2kd}\left(\frac{\omega_m'}{2\omega_e}\right)^2\cos 2\omega_e t,$$

since its effect on the interaction between the electric and mechanical systems is negligibly small at $\omega_m'/\omega_e\ll 1$. The frequency ω_m' of the mechanical oscillations depends on the measured value of $\mathscr{E}_0$, and if the resonator is in a state with given energy ($\mathscr{E}_0=(n+\frac{1}{2})\hbar\omega_e$), then

$$\omega_m'=\omega_m(1-\mathscr{E}_0/4kd^2)^{1/2}. \tag{4}$$

The uncertainty in the quantity $\mathscr{E}$ after the measurement is determined according to (2) by the error Δx_0 in the measurement of the value of $x(t)=x_0$ averaged over the period, and according to (3) also by the perturbation of the energy on account of the uncertainty of the position

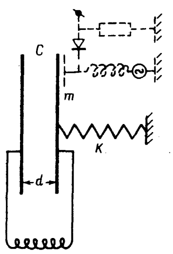

FIG. 1.

0038-5646/77/4604-0705$02.40

of the plate (Δx) after the measurement

$$\langle\Delta\mathscr{E}^2_{meas}\rangle=(2kd)^2\langle\delta x_0^2\rangle, \qquad \langle\Delta\mathscr{E}^2_{pert}\rangle=\langle\mathscr{E}_0^2\rangle\langle(\Delta x/2d)^2\rangle. \tag{5}$$

We assume next that the time consumed in the measurement is much shorter than the time τ^*_m of the relaxation of the mechanical oscillations. In this case a decrease of δx_0 is accompanied by an increase of Δx.

The limit of the measurement accuracy is determined by the Heisenberg uncertainty relation for the coordinate and momentum of the mechanical system and does not depend on the measurement procedure. This limit can be easily obtained by considering one of the measurement methods. If the period of the mechanical oscillations were exactly known, then the value of x_0 could be determined with zero error by measuring x twice at times separated by half the period. However, ω'_m depends on the unknown quantity $\mathscr{E}_0$ (see (4)). Therefore the error in the determination of the period leads to an error in the measurement of $\mathscr{E}_0$. If the first measurement of x is made with accuracy δx_1, then the momentum perturbation is $\Delta p=\hbar(2\delta x_1)^{-1}$. At the optimal $\delta x_1 =(\pi\omega_m\hbar\mathscr{E}_0^{1/2})/4kd$, the value of $\Delta\mathscr{E}_{meas}$ is

$$\Delta\mathscr{E}_{meas}=(n+{}^1/_2)^{1/2}\hbar\omega_e(\pi\omega_m/\omega_e)^{1/2}. \tag{6}$$

The coordinate uncertainty corresponding to Δp is $\Delta x =\Delta p/m\omega_m$ and leads according to (5) to a resonator-energy uncertainty

$$\Delta\mathscr{E}_{pert}=(n+{}^1/_2)^{1/2}\hbar\omega_e(\omega_m/4\pi\omega_e)^{1/2}. \tag{7}$$

An analysis of other measurement methods for an infinitely large value of τ^*_m leads to values of $\Delta\mathscr{E}_{meas}$ and $\Delta\mathscr{E}_{pert}$ that differ from those obtained only by numerical factors close to unity.

The oscillations of aplate with random amplitude and phase introduce an uncertainty in the value of the phase of the field oscillations in the resonator. It is easy to show that the standard deviation of the phase during the measurement time is $\Delta\varphi=\frac{1}{2}(\omega_e/\omega_m\pi n)^{1/2}$ for large n, and accordingly

$$\frac{\Delta\mathscr{E}}{\hbar\omega_e}\Delta\varphi=\frac{1}{2}. \tag{8}$$

The sensitivity of a realistic measurement system will depend also on the thermal fluctuations in the device that registers x. If we use a capacitive pickup for this purpose (shown dashed in the figure), then we can turn on a pump signal for short intervals, twice during each half-period of the mechanical oscillations. At optimal pumping (see[4]) the error $\Delta\mathscr{E}_{meas}$ turns out to be

$$\Delta\mathscr{E}_{meas}=(n+{}^1/_2)^{1/2}\hbar\omega_e(\pi\omega_m/\omega_e)^{1/2}(2\varkappa T_p/\hbar\omega_p)^{1/2}, \tag{9}$$

where T_p is the temperature of the pickup circuit, ω_p is the natural frequency of the pickup circuit, and $\varkappa$ is Boltzmann's constant.

We have assumed so far that τ^*_m is long enough, so that thermal fluctuations could be neglected in the mechanical oscillator. At a finite value $\tau^*_m=2Q_m/\omega_m$ (where Q_m is the quality factor of the oscillator), the additional error $(\Delta\mathscr{E}_{meas})_{T_m}$ at $\varkappa T_m\gg\hbar\omega_m$ will be

$$(\Delta\mathscr{E}_{meas})_{T_m}=4d(\varkappa T_m m\omega_m/\hat{\tau}Q_m)^{1/2}, \tag{10}$$

where T_m is the temperature of the mechanical oscillator and $\hat{\tau}$ is the time of measurement of the value of x_0. In a practical realization of such a measurement scheme, the principal restriction on the sensitivity will be due to thermal fluctuations in the mechanical oscillator. Indeed, if $\pi\omega_m/\omega_e=1\times10^{-10}$, then even at $\varkappa T_p=10^4\hbar\omega_p$ the thermal fluctuations in the coordinate-recording device limit the sensitivity to a level $\Delta\mathscr{E}_{meas}\approx1\times10^{-2}(n+\frac{1}{2})^{1/2}\times\hbar\omega_e$ (see the condition (9)). At the same time, even at $T_m=2$ K, $m=1\times10^{-2}$ g, $\omega_m=1$ rad/sec, $\hat{\tau}=3$ sec, $d=1\times10^{-3}$ cm and at $Q_m=5\times10^9$ (such values of Q have already been attained[5]), we have $(\Delta\mathscr{E}_{meas})_{T_m}\approx6\times10^{-17}$ erg. At these data we have $(\Delta\mathscr{E}_{meas})_{T_m}<\hbar\omega_e$ only at $\omega_e>6\times10^{10}$ rad/sec.

Summarizing the foregoing, we emphasize that the considered method makes it possible in principal to register with practically no perturbation a one-quantum change in the resonator energy.

In conclusion, let us consider another method of measuring the resonator energy. A scheme for energy measurement with the aid of an electron probe was considered in[6]. In this method, the interaction energy is proportional to the instantaneous value of the charge q on the resonator plates. An analysis of the simplest measurement schemes for this type of interaction has shown that the smallest error is $\Delta\mathscr{E}_{meas}=n^{1/2}\hbar\omega_e$. In a scheme with a two passes of the electrons, the calculations in[6] yielded a much smaller error. Possible causes were considered in the estimate of the perturbation of the resonator energy, including diffraction by the receiving electrodes. However, in the analysis of this last effect account was taken of only the change of the law governing the electron distribution in the second pass. A more detailed analysis has shown that this is not enough, since a correlation exists between the initial coordinate of the electron and the perturbation of the coordinate due to the diffraction by the receiving electrodes. The resultant compensation of the resonator-energy perturbation with allowance for this correlation turned out to be such as to make the measurement error also equal to $n^{1/2}\hbar\omega_e$.

[1]K. S. Thorne, General Relativistic Astrophysics, Preprint, Chicago University, May, 1975.

[2]V. B. Braginskiĭ and Yu. I. Vorontsov, Usp. Fiz. Nauk **114**, 41 (1974) [Sov. Phys. Usp. **17**, 644 (1975)].

[3]D. Bohm, Quantum Theory, Prentice-Hall, 1951.

[4]V. B. Braginskiĭ and A. B. Manukin, Izmerenie malykh sil v fizicheskikh ėksperimentakh (Measurement of Small Forces in Physical Experiments), Nauka, 1974.

[5]Kh. S. Bagdasarov, V. B. Braginskiĭ, V. P. Mitrofanov, and V. S. Shiyan, Vestn. Mosk. Univ. Fiz. Astron. No. 1, 98 (1977).

[6]V. B. Braginskiĭ, Yu. I. Vorontsov, and V. D. Krivchenko, Zh. Eksp. Teor. Fiz. **68**, 55 (1975) [Sov. Phys. JETP **41**, 28 (1975)].

Translated by J. G. Adashko