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Functional for the hydrodynamic action and the Bose spectrum of superfluid Fermi systems of the He^3 type

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A hydrodynamic-action functional is constructed by a continual-integration method for Fermi systems with pairing in the p state. A simplified model is considered in which it is possible to introduce local fields that describe tensor Bose condensates. It is shown that the most stable is the B phase, which undergoes a second-order phase transition into a planar $2D$ phase in a sufficiently strong magnetic field. The A phase is metastable in the considered model and is destroyed by an arbitrarily weak magnetic field. The Bose spectrum of the system is investigated. In the model in question it contains four phonon branches in the B phase, six in the $2D$ phase, and nine in the A phase. In the more general case of a type-He^3 Fermi system there are four branches each in the B and $2D$ phases and five in the A phase. Qualitative conclusions are deduced for real superfluid He^3. In particular, arguments are advanced favoring a second-order phase transition from the B phase to a planar $2D$ phase in a sufficiently strong magnetic field.

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1. INTRODUCTION

At temperatures on the order of 10^{-3} K, several superfluid phases can exist in He^3 and go over into one another when the external parameters of the system are changed.[1–5] The difficulty of constructing a complete microscopic theory of He^3 makes it expedient to study simplified models similar to the Bose-gas model for He^4.

We investigate here a simplified He^3 model using a continual-integral formalism that is convenient, in particular, for the description of collective excitations. We change for this purpose from an integral over Fermi fields to an integral over an auxiliary Bose field that in fact corresponds to collective excitations. This device was used in[6,7] for a model of a superfluid Fermi gas with pairing in the s state. In this approach, the transition of a Fermi system into the superfluid state constitutes Bose condensation of the model Bose system.

We have considered a technically more complicated case, that of pairing in the p state. In this case we deal of necessity not with a scalar wave function, but with a tensor one that describes the superfluid state, i.e., 9 complex or 18 real independent functions (18 degrees of freedom). This makes possible the coexistence of several superfluid phases (including the phases A and B typical of He^3) and to a rich spectrum of collective excitations. In the simplified model we can get along with a local Bose field that describes the collective excitations. In the general case, however, a bilocal formalism is needed (see, e.g.,[8]).

In Sec. 2 we describe a simplified model that admits of a transition from an integral over Fermi fields to an integral over an auxiliary local Bose field, and construct a "hydrodynamic action" functional S_h that describes the collective excitations.

In Sec. 3 the functional S_h is investigated in the Ginzburg–Landau region $|\Delta T| \ll T_c$. We obtain the Bose-condensation point and the density of the Bose condensate for various superfluid phases. The investigation shows that in the absence of a magnetic field the energywise most favored and stable (with respect to small perturbations) is the B phase. The A phase is energywise less profitable, but is also stable to small perturbations. However, application of an arbitrarily weak magnetic field destroys the A phase. With increasing magnetic field, the B phase becomes deformed and goes over con-

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tinuously, at a certain critical field $H=H_c$, into a planar 2*D* phase (in the terminology of the review[1]) which is stable at $H>H_c$.

An investigation of the Bose spectrum leads to the conclusion that in the *B* phase there exist four branches that start at zero as $k\to 0$, the 2*D* phase has six such branches, and the *A* phase has nine. The phonon branches are analogs of the Bogolyubov-sound branch[9] for the superfluid Fermi systems with pairing in the *s* state. In neutral Fermi systems, these branches really exist, in contrast to superconductors, in which the Coulomb interaction converts the phonon branch into a plasma-oscillation branch.

The case of low temperatures is investigated in Sec. 4. The conclusion that the *B* phase is stable at $H=0$ and the 2*D* phase is stable at $H>H_c$ remains valid in the entire region $T<T_c$. At $T\ll T_c$ the spectral branches that start out from zero are phonon branches, and one can speak of speeds of various sounds. These speeds have been calculated in the limit as $T\to 0$ for the *A*, *B*, and 2*D* phases.

The comparison of the properties of the model and of real He^3 is the subject of Sec. 5.

A study of the model leads to the natural conclusion that He^3 goes over into the 2*D* phase in sufficiently strong magnetic fields. As to the Bose spectrum, the number of phonon branches in the general case of a Fermi system of type He^3 is smaller in the *A* and 2*D* phases than the corresponding number for the model, and is equal to four each for the *B* and 2*D* phases and to five for the *A* phase.

2. CONTINUAL INTEGRAL AND FUNCTIONAL OF THE HYDRODYNAMIC ACTION

The continual-integration method offers extensive possibilities for the construction of nontrivial perturbation-theory schemes and is convenient for the description of the collective excitations of statistical-physics systems.

In the case of a nonrelativistic Fermi system at a finite temperature *T*, we must integrate over the space of the anticommuting functions $\chi_s(\mathbf{x},\tau)$ and $\bar\chi_s(\mathbf{x},t)$ with the Fourier expansion

$$\chi_s(\mathbf{x},\tau)=\frac{1}{(\beta V)^{1/2}}\sum_{\mathbf{k},\omega}a_s(\mathbf{k},\omega)\exp[i(\omega\tau-\mathbf{kx})]. \quad (2.1)$$

Here $s=\pm$ is the spin index $\mathbf{x}\in V=L^3, \tau\in[0,\beta], \beta=T^{-1}$ (in units $\hbar=c_B=1$), $k_i=2\pi n_i/L$, $\omega=(2n+1)\pi/\beta$; n and n_i are integers. The temperature Green's functions are obtained by averaging the products of several fields with different arguments with weight e^S, where the functional

$$S=\int_0^\beta d\tau\int d\mathbf{x}\sum_s\bar\chi_s(\mathbf{x},\tau)\partial_\tau\chi_s(\mathbf{x},\tau)-\int_0^\beta H'(\tau)d\tau \quad (2.2)$$

has the meaning of the action corresponding to the Hamiltonian

$$H'(\tau)=\int d\mathbf{x}\sum_s\left(\frac{1}{2m}\nabla\bar\chi_s(\mathbf{x},\tau)\nabla\chi_s(\mathbf{x},\tau)-(s\mu_0H+\lambda)\bar\chi_s(\mathbf{x},\tau)\chi_s(\mathbf{x},\tau)\right)$$
$$+\frac{1}{2}\int d\mathbf{x}\,d\mathbf{y}\,u(\mathbf{x}-\mathbf{y})\sum_{s,s'}\bar\chi_s(\mathbf{x},\tau)\bar\chi_{s'}(\mathbf{y},\tau)\chi_{s'}(\mathbf{y},\tau)\chi_s(\mathbf{x},\tau), \quad (2.3)$$

in which λ is the chemical potential, μ_0 is the magnetic moment of the Fermi particle, and *H* is the magnetic field.

We integrate first over the "fast" Fermi fields and then over the "slow" ones, using during these two stages different perturbation-theory schemes.[7] The integral over the fast fields χ_1 and $\bar\chi_1$, for which $|k-k_F|>k_0$ or $|\omega|>\omega_0$, will be written in the form

$$\int\exp S\,d\bar\chi_1\,d\chi_1=\exp\tilde S[\bar\chi_{0s},\chi_{0s}]. \quad (2.4)$$

The functional $\tilde S$ has the meaning of the action of the "slow" fields χ_{0s} and $\bar\chi_{0s}$, for which $|k-k_F|<k_0$ and $|\omega|<\omega_0$. The auxiliary parameters k_0 and ω_0 are defined only accurate to their order of magnitude, and the physical results should not depend on their concrete choice.

The general form of the functional $\tilde S$ is a sum of functionals of the even powers in the fields χ_{0s} and $\bar\chi_{0s}$:

$$\tilde S=\sum_{n=0}^{\infty}\tilde S_{2n}. \quad (2.5)$$

Neglecting the higher functionals $\tilde S_6, \tilde S_8, \ldots$ and omitting the constant $\tilde S_0$, which is no longer significant, we examine the forms of $\tilde S_2$ and $\tilde S_4$. The form of $\tilde S_2$ corresponds to non-interacting quasiparticles near the Fermi surface, and is given by

$$\sum_{\mathbf{k},\omega,s}\varepsilon_s(\mathbf{k},\omega)a_s^+(\mathbf{k},\omega)a_s(\mathbf{k},\omega),\quad |k-k_F|<k_0,\quad |\omega|<\omega_0, \quad (2.6)$$

with

$$\varepsilon_s(\mathbf{k},\omega)\approx Z^{-1}(i\omega-c_F(k-k_F)+s\mu H). \quad (2.7)$$

Here, assuming that $\varepsilon_s(\omega=0, k=k_F, H=0)=0$, we have expanded ε_s in powers of ω, $k-k_F$, and *H* and retained only the linear terms. The coefficient c_F has the meaning of the velocity on the Fermi surface, μ is the magnetic moment of the quasiparticle, and *Z* is a normalization constant.

The form $\tilde S_4$ describes the interaction of the quasiparticles and is given by

$$-\frac{1}{\beta V}\sum_{p_1+p_2=p_3+p_4}t_0(p_1,p_2,p_3,p_4)a_+^+(p_1)a_-^+(p_2)a_-(p_4)a_+(p_3)$$
$$-\frac{1}{2\beta V}\sum_{p_1+p_2=p_3+p_4}t_1(p_1,p_2,p_3,p_4)[2a_+^+(p_1)a_-^+(p_2)a_-(p_4)a_+(p_3)$$
$$+a_+^+(p_1)a_+^+(p_2)a_+(p_4)a_+(p_3)+a_-^+(p_1)a_-^+(p_2)a_-(p_4)a_-(p_3)]. \quad (2.8)$$

Here $p=(\mathbf{k},\omega)$ is the 4-momentum; $t_0(p_i)$ and $t_1(p_i)$ are respectively the symmetrical and antisymmetrical scattering amplitudes under the permutations $p_1\rightleftarrows p_2$ and $p_3\rightleftarrows p_4$. In the vicinity of the Fermi sphere we can put $\omega_i=0$, $\mathbf{k}_i=k\mathbf{n}_i$ $(i=1,2,3,4)$, where $\mathbf{n}_i$ are unit vectors such that $\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}_3+\mathbf{n}_4$. The amplitudes t_0 and t_1 should depend only on two invariants, for example on $(\mathbf{n}_1, \mathbf{n}_2)$ and $(\mathbf{n}_1 - \mathbf{n}_2, \mathbf{n}_3 - \mathbf{n}_4)$, with t_0 even and t_1 odd in the second invariant. We therefore have the expressions

$$t_0 = f((\mathbf{n}_1, \mathbf{n}_2); (\mathbf{n}_1 - \mathbf{n}_2, \mathbf{n}_3 - \mathbf{n}_4)),$$
$$t_1 = (\mathbf{n}_1 - \mathbf{n}_2, \mathbf{n}_3 - \mathbf{n}_4) g((\mathbf{n}_1, \mathbf{n}_2), (\mathbf{n}_1 - \mathbf{n}_2, \mathbf{n}_3 - \mathbf{n}_4)). \tag{2.9}$$

Here t_0 and t_1 are expressed in terms of the functions f and g, which are even in the second argument.

The functional $S_2 + S_4$, defined by formulas (2.6)–(2.9), is the most general expression describing Fermi quasiparticles and their pair interaction near the Fermi sphere. The method of obtaining this functional in the continual-integral formalism, and its investigation that follows below, constitute an alternative approach to that developed in the Landau theory of the Fermi liquid.[10]

The functions f and g can be easily calculated for the gas model. For high-density systems they must be determined from experiment.

We consider hereafter a model with

$$f = 0, \quad g = \text{const} < 0 \tag{2.10}$$

as the simplified model of He^3 with pairing in the p state. We use a previously developed[7] method (see also[6]) for the case $g = 0$ and $f = \text{const} < 0$. The main idea is to introduce a new field that describes the Cooper pair. The condition $g = \text{const}$ (just as the condition $f = \text{const}$ in[7]) allows us to make do with a local field and not to resort to the bilocal formalism. (In fact, this can be done also in a more general case, namely when the functions f and g depend only on the first argument.)

The indicated idea can be realized by introducing under the sign of the integral over the Fermi field a Gaussian integral of $\exp(\bar{c}\hat{A}c)$ with respect to the Bose field c, where $\bar{c}\hat{A}c$ is a quadratic form with a certain operator $\hat{A}$. We then shift the Bose field by a quadratic form of the Fermi fields, so as to annihilate the form $\tilde{S}_4$ of fourth-degree in the Fermi fields. The integral over the Fermi fields is then transformed into a Gaussian integral and is equal to the determinant of the operator $\hat{M}(c, \bar{c})$ that depends on the Bose fields c and $\bar{c}$. We arrive at the functional

$$S_h = \bar{c}\hat{A}c + \ln\det[\hat{M}(c, \bar{c}) / \hat{M}(0, 0)], \tag{2.11}$$

in which the ln det has been regularized by dividing $\hat{M}(c, \bar{c})$ by the operator $\hat{M}(0, 0) = \hat{M}(c, \bar{c})|_{c=\bar{c}=0}$.

The functional S_h was dubbed in[7] the "hydrodynamic action functional." It defines the point of the phase transition of the initial Fermi system as Bose condensation of the fields c and $\bar{c}$, and determines the density of the condensate at $T < T_c$ and the spectrum of the collective excitations.

For the case of pairing in the s state ($g = 0, f = \text{const}$) it suffices to introduce a Gaussian integral over the "scalar" complex functions $c(\mathbf{x}, \tau)$ and $\bar{c}(\mathbf{x}, \tau)$. In our case ($f = 0, g = \text{const}$) it is necessary to integrate over the space of the complex functions $c_{ia}(\mathbf{x}, \tau)$ and $\bar{c}_{ia}(\mathbf{x}, \tau)$ with the vector index i and the isostopic index a ($i, a = 1, 2, 3$). The Gaussian integral inserted under the sign of the integral over the Fermi fields, is of the form

$$\int d\bar{c}_{ia}\, dc_{ia} \exp\left(\frac{1}{g}\sum_{p,i,a} c_{ia}^+(p) c_{ia}(p)\right), \tag{2.12}$$

where g is the constant (2.10). It is easily verified that the shift

$$c_{i1}(p) \to c_{i1}(p) + \frac{g}{2(\beta V)^{1/2}} \sum_{p_1+p_2=p} (n_{1i} - n_{2i})[a_+(p_2)a_+(p_1) - a_-(p_2)a_-(p_1)],$$
$$c_{i2}(p) \to c_{i2}(p) + \frac{gi}{2(\beta V)^{1/2}} \sum_{p_1+p_2=p} (n_{1i} - n_{2i})[a_+(p_2)a_+(p_1) + a_-(p_2)a_-(p_1)], \tag{2.13}$$
$$c_{i3}(p) \to c_{i3}(p) + \frac{g}{(\beta V)^{1/2}} \sum_{p_1+p_2=p} (n_{1i} - n_{2i}) a_-(p_2) a_+(p_1)$$

does indeed eliminate the form $\tilde{S}_4$.

To calculate the Gaussian integral over the Fermi fields, we introduce a column $\psi_a(p)$ with elements

$$\psi_1(p) = a_+(p),\ \psi_2(p) = -a_-(p),\ \psi_3(p) = a_-^+(p),\ \psi_4(p) = a_+^+(p) \tag{2.14}$$

and write down a quadratic form in the Fermi fields in the form

$$K = {}^1/_2 \sum_{p_1, p_2, a, b} \psi_a^+(p_1) M_{a,b}(p_1, p_2) \psi_b(p_2). \tag{2.15}$$

The fourth-order matrix $M(p_1, p_2)$ with elements $M_{ab}(p_1, p_2)$ is given by

$$M = \begin{pmatrix} Z^{-1}(i\omega - \xi + \mu H\sigma_3)\delta_{p_1,p_2}, & \frac{1}{(\beta V)^{1/2}}(n_{1i} - n_{2i}) c_{ia}(p_1 + p_2)\sigma_a \\ -\frac{1}{(\beta V)^{1/2}}(n_{1i} - n_{2i}) c_{ia}^+(p_1 + p_2)\sigma_a, & Z^{-1}(-i\omega + \xi + \mu H\sigma_3) \end{pmatrix}, \tag{2.16}$$

where $\xi = c_F(k - k_F)$ and σ_a ($a = 1, 2, 3$) are 2×2 Pauli matrices.

Integrating over the Fermi fields

$$\int e^K d\bar{\chi}_{0s}\, d\chi_{0s} = (\det \hat{M})^{1/2}, \tag{2.17}$$

we arrive at the "hydrodynamic action" functional in the form

$$S_h = \frac{1}{g}\sum_{p,i,a} c_{ia}^+(p) c_{ia}(p) + \frac{1}{2}\ln\det\frac{\hat{M}(c, \bar{c})}{\hat{M}(0,0)}. \tag{2.18}$$

3. THE REGION $|\Delta T| \ll T_c$

The functional (2.18) contains all the information on the physical properties of the model system. It can be most easily investigated in the Ginzburg–Landau region $|\Delta T| \ll T_c$, where ln det can be expanded in powers of the fields c and $\bar{c}$. Putting

$$\hat{M}(0, 0) = \hat{G}^{-1}, \quad \hat{M}(c, \bar{c}) = \hat{G}^{-1} + \hat{u}, \tag{3.1}$$

we retain in the expansion

$$\frac{1}{2}\ln\det\frac{\hat{M}(c, \bar{c})}{\hat{M}(0,0)} = \frac{1}{2}\operatorname{Sp}\ln(1 + \hat{G}\hat{u}) = -\sum_{n=1}^{\infty}\frac{1}{4n}\operatorname{Sp}(\hat{G}\hat{u})^{2n} \tag{3.2}$$

the first two terms ($n=1, 2$).

Consider the second-order form. If $H=0$, the form is diagonal in the isotopic index a and is given by

$$-\sum_{p,i,a} A_{ij}(p)c_{ia}^+(p)c_{ja}(p)$$
$$=-\sum_{p,i,j}(A_{ij}(p)-A_{ij}(0))c_{ia}^+(p)c_{ja}(p)-\sum_{p,i,j}A_{ij}(0)c_{ia}^+(p)c_{ja}(p), \quad (3.3)$$

where

$$A_{ij}(0)=-\frac{4Z^2}{\beta V}\sum_{p_1}\frac{n_{1i}n_{1j}}{\omega_1^2+\xi_1^2}$$
$$=-\delta_{ij}\frac{2Z^2k_F^2}{3\pi^2c_F}\int_0^{c_Fk_0}\frac{d\xi}{\xi}\,\mathrm{th}\,\frac{\beta\xi}{2}=-\delta_{ij}A(0), \quad (3.4)$$

$$A_{ij}(p)-A_{ij}(0)=\frac{Z^2}{2V}\sum_{|k_1-k_F|<k_0}(n_{1i}-n_{2i})(n_{1j}-n_{2j})$$
$$\times\left[\frac{1}{i\omega-\xi_1-\xi_2}\left(\mathrm{th}\,\frac{\beta\xi_1}{2}+\mathrm{th}\,\frac{\beta\xi_2}{2}\right)+\frac{1}{\xi_1}\,\mathrm{th}\,\frac{\beta\xi_1}{2}\right]. \quad (3.5)$$

At small $H(\mu H\ll T)$ it suffices to take into account the increment to $A_{ij}(0)$ and neglect the H-dependence of the function $A_{ij}(p)-A_{ij}(0)$ at small p. After summing over the frequencies we get the integrals

$$\int_{-c_Fk_0}^{c_Fk_0}\frac{1}{\xi\pm\mu H}\,\mathrm{th}\,\frac{\beta}{2}(\xi\pm\mu H)\,d\xi, \quad a=1,2;$$
$$\int_{-c_Fk_0}^{c_Fk_0}\frac{d\xi}{2\xi}\left(\mathrm{th}\,\frac{\beta}{2}(\xi+\mu H)+\mathrm{th}\,\frac{\beta}{2}(\xi-\mu H)\right), \quad a=3. \quad (3.6)$$

Only the last of them, which receives an increment $\sim H^2$, depends on H. The H-dependent increment to S_h is of the form

$$-\frac{7\zeta(3)Z^2k_F^2\mu^2H^2}{6\pi^4T^2c_F}\sum_{n,i}c_{i3}^+(p)c_{i3}(p). \quad (3.7)$$

In the fourth-order form we shall hereafter be interested in terms with small 2-momenta of all the fields $c_{ia}(p)$ and $c_{ia}^+(p)$ $(c_F|\mathbf{k}_i|\ll T)$, which can be written in the form

$$-\frac{7\zeta(3)k_F^2Z^4}{30\pi^4T^2c_F\beta V}\sum_{p_1+p_2=p_3+p_4}[2c_{ia}^+(p_1)c_{jb}^+(p_2)c_{ia}(p_3)c_{jb}(p_4)$$
$$+2c_{ia}^+(p_1)c_{ib}^+(p_2)c_{ja}(p_3)c_{jb}(p_4)+2c_{ia}^+(p_1)c_{jb}^+(p_2)c_{ja}(p_3)c_{ib}(p_4)$$
$$-2c_{ia}^+(p_1)c_{ja}^+(p_2)c_{ib}(p_3)c_{jb}(p_4)-c_{ia}^+(p_1)c_{ia}^+(p_2)c_{jb}(p_3)c_{jb}(p_4)]. \quad (3.8)$$

We find the Bose-condensation temperature T_c from the condition that the coefficient in the quadratic form at $c_{ia}^+(p)c_{ia}(p)$ vanish for $p=0$. At $H=0$ we have the equality

$$0=\frac{1}{g}+A(0)=\frac{1}{g}+\frac{2Z^2k_F^2}{3\pi^2c_F}\int_0^{c_Fk_0}\frac{d\xi}{\xi}\,\mathrm{th}\,\frac{\beta\xi}{2}. \quad (3.9)$$

The integral with respect to ξ depends logarithmically on k_0. In order for T_c not to depend on k_0 it is necessary that g^{-1} depend on k_0 in accord with the formula

$$\frac{1}{g}=\frac{1}{g_0}-\frac{2Z^2k_F^2}{3\pi^2c_F}\ln\frac{k_0}{k_F}, \quad (3.10)$$

where g_0 no longer depends on k_0 (we must have here $g_0<0$). Substitution of (3.10) in (3.9) yields

$$T_c=\frac{2\gamma}{\pi}c_Fk_F\exp\left(-\frac{3\pi^2c_F}{2|g_0|Z^2k_F^2}\right), \quad (3.11)$$

where $\ln\gamma=C$ is the Euler constant.

In the region $|\Delta T|\ll T_c$ we have

$$\frac{1}{g}+A(0)=\frac{2Z^2k_F^2}{3\pi^2c_F}\ln\frac{T_c}{T}\approx\frac{2Z^2k_F^2}{3\pi^2c_F}\frac{T_c-T}{T_c}. \quad (3.12)$$

To find the density of the condensate at $T<T_c$ we make the substitution

$$c_{ia}(p)\to(\beta V)^{1/2}\delta_{p0}a_{ia}\left(\frac{10T_c|\Delta T|}{7\zeta(3)}\right)^{1/2}\frac{\pi}{Z},$$
$$c_{ia}^+(p)\to(\beta V)^{1/2}\delta_{p0}\bar{a}_{ia}\left(\frac{10T_c|\Delta T|}{7\zeta(3)}\right)^{1/2}\frac{\pi}{Z}, \quad (3.13)$$

which transforms S_h into

$$-\frac{20k_F^2(\Delta T)^2\beta V}{21\zeta(3)c_F}\Pi, \quad (3.14)$$

an expression that depends on the matrix A with elements a_{ia} as well as on its hermitian conjugate A^+, its transpose A^T, and its complex conjugate A^*:

$$\Pi=-\mathrm{tr}\,AA^++\nu\,\mathrm{tr}\,A^+AP+(\mathrm{tr}\,A^+A)^2+\mathrm{tr}\,AA^+AA^+$$
$$+\mathrm{tr}\,AA^+A^*A^T-\mathrm{tr}\,AA^TA^*A^+-{}^1/_2\,\mathrm{tr}\,AA^T\,\mathrm{tr}\,A^+A^*, \quad (3.15)$$

where

$$\nu=7\zeta(3)\mu^2H^2/4\pi^2T_c\Delta T, \quad (3.16)$$

P is the projector on the third axis which is directed along the magnetic field.

We note that Π is invariant to the transformations

$$A\to e^{i\alpha}UA, \quad (3.17)$$

where α is a real parameter and U is a real orthogonal matrix.

Minimizing Π, we obtain the matrix A that determines the density of the Bose condensate. The equation $\delta\Pi=0$, i.e.,

$$-A+\nu AP+2(\mathrm{tr}\,A^+A)A+2AA^+A$$
$$+2A^*A^TA-2AA^TA^*-A^*\,\mathrm{tr}\,AA^T=0, \quad (3.18)$$

has several nontrivial solutions corresponding to different superfluid phases. We consider the following possibilities:

$$A_1=c_1P,\quad A_2=c_2P_2,\quad A_3=c_3'P+c_3''P_2,$$
$$A_4=c_4\hat{C}_4,\ A_5=c_5\hat{C}_5,\ A_6=c_6'\hat{C}_4+c_6''\hat{C}_5,\ A_7=c_7\hat{C}_7. \quad (3.19)$$

Here P_2 is the projector on the two-dimensional subspace and is orthogonal to P; $\hat{C}_4$, $\hat{C}_5$, and $\hat{C}_7$ are third-order matrices:

$$\hat{C}_4=\begin{pmatrix}0&0&1\\0&0&i\\0&0&0\end{pmatrix},\quad \hat{C}_5=\begin{pmatrix}0&0&0\\0&0&0\\1&i&0\end{pmatrix},\quad \hat{C}_7=\begin{pmatrix}1&i&0\\i&-1&0\\0&0&0\end{pmatrix}. \tag{3.20}$$

We write down the squares of the moduli of the coefficients $|c_i|^2$ as well as the corresponding values of Π_i:

$$\begin{gathered}|c_1|^2={}^1/_3(1-\nu),\quad |c_2|^2={}^1/_4,\quad |c_3'|^2={}^1/_5(1-2\nu),\\ |c_3''|^2={}^1/_{10}(2+\nu),\ |c_4|^2={}^1/_4(1-\nu),\ |c_5|^2={}^1/_{12},\end{gathered} \tag{3.21}$$

$$\begin{gathered}|c_6'|^2={}^1/_8(2-3\nu),\quad |c_6''|^2={}^1/_8\nu,\quad |c_7|^2={}^1/_{16};\\ \Pi_1=-{}^1/_6(1-\nu)^2,\quad \Pi_2=-{}^1/_4,\quad \Pi_3=-{}^1/_{10}(3-2\nu+\nu^2),\\ \Pi_4=-{}^1/_4(1-\nu)^2,\quad \Pi_5=-{}^1/_{12},\quad \Pi_6=-{}^1/_8(2-4\nu+3\nu^2),\quad \Pi_7=-{}^1/_8.\end{gathered} \tag{3.22}$$

At $H=0$ ($\nu=0$) the minimal of all the variants, $\Pi=-0.3$, is obtained for $A_3=c_3I$. This is the symmetrical Balian–Werthamer phase (the B phase). The next values of equal magnitude $\Pi=-\frac{1}{4}$ yield c_2P_2—the planar $2D$ phase (according to the terminology of the review[11]) and $c_4\hat{C}_4$—the Anderson–Morel–Brinkman A phase. The remaining four phases, A_1, A_5, A_6, and A_7 are energywise unprofitable compared with the B, A, and $2D$ phases.

When the magnetic field is increased the value of Π for the $2D$ phase remains unchanged, while Π for the A and B phases increases, the B phase becoming "deformed." At $\nu=\frac{1}{2}$ we have $A_2=A_3$ and $\Pi_2=\Pi_3$. Moreover, at $\nu>\frac{1}{2}$ the solution A_2 is meaningless, since $|c_3'|^2=(1-2\nu)/5$ becomes negative. Thus, at

$$\nu={}^1/_2,\quad H_c^2=2\pi^2T_c\Delta T/7\zeta(3)\mu^2 \tag{3.23}$$

a transition takes place from the B phase into the planar $2D$ phase whose energy, in first-order approximation, does not depend on H at all. The continuity of the transition of A_3 into A_2 favors the assumption that this is a second-order phase transition.

The conclusion concerning the phase transition is confirmed by a calculation of the second variation of the function Π. In order for the phase to be stable to small perturbations, $\delta^2\Pi$ must be non-negative. We present the expressions for $\delta^2\Pi_i$ for the most interesting cases $i=2, 3, 4$:

$$\begin{gathered}\delta^2\Pi_2=(\nu-{}^1/_2)u_{33}^2+(\nu+{}^1/_2)v_{33}^2+\nu(u_{13}^2+u_{23}^2)\\ +(\nu+2)(v_{13}^2+v_{23}^2)+{}^1/_2[3u_{11}^2+3u_{22}^2+2u_{11}u_{22}\\ +(u_{12}+u_{21})^2]+{}^1/_2[3v_{12}^2+3v_{21}^2-2v_{12}v_{21}+(v_{11}-v_{22})^2];\\ \delta^2\Pi_3=\frac{\nu+2}{5}[3u_{11}^2+3u_{22}^2+2u_{11}u_{22}+(u_{12}+u_{21})^2+u_{13}^2+u_{23}^2]\\ +\frac{2(1-2\nu)}{5}(u_{31}^2+u_{32}^2+3u_{33}^2)\\ +\frac{4}{5}\left(\frac{(1-2\nu)(2+\nu)}{2}\right)^{1/2}(u_{11}u_{33}+u_{22}u_{33}+u_{13}u_{31}+u_{23}u_{32})\\ +\frac{4-3\nu}{5}(v_{11}^2+v_{22}^2)+\frac{8-\nu}{5}(v_{12}^2+v_{21}^2)+\frac{2(2+\nu)}{5}(v_{33}^2-v_{11}v_{22}-v_{12}v_{21})\\ +\frac{8+9\nu}{5}(v_{13}^2+v_{23}^2)+\frac{8(1-2\nu)}{5}(v_{31}^2+v_{32}^2)\\ -\frac{4}{5}\left(\frac{(1-2\nu)(2+\nu)}{2}\right)^{1/2}(v_{11}v_{33}+v_{22}v_{33}+v_{13}v_{31}+v_{23}v_{32}),\\ \delta^2\Pi_4={}^1/_2(1-\nu)[2(u_{13}+v_{23})^2+(u_{13}-v_{23})^2+(u_{23}+v_{13})^2\\ +2(u_{21}-v_{11})^2+(u_{11}-v_{21})^2+(u_{21}+v_{11})^2\\ +2(u_{22}-v_{12})^2+(u_{12}-v_{22})^2+(u_{22}+v_{12})^2]\\ -\nu(u_{11}^2+u_{21}^2+u_{31}^2+u_{12}^2+u_{22}^2+u_{32}^2+v_{11}^2+v_{21}^2+v_{31}^2+v_{12}^2+v_{22}^2+v_{32}^2).\end{gathered} \tag{3.24}$$

Here $u_{ia}=\mathrm{Re}\,\delta a_{ia}$, $v_{ia}=\mathrm{Im}\,\delta a_{ia}$.

At $\nu<\frac{1}{2}$ the variation $\delta^2\Pi_3$ is non-negative and $\delta^2\Pi_2$ is of alternating sign, while at $\nu>\frac{1}{2}$ the variation $\delta^2\Pi_2$ is non-negative and the solution A_3 has no meaning. The variation $\delta^2\Pi_4$ is non-negative at $\nu=0$ and is of alternating sign for any $\nu>0$. This means that the A phase, which is metastable at $H=0$, is destroyed in this model by an arbitrarily weak magnetic field.

The second variations $\delta^2\Pi_1$, $\delta^2\Pi_5$, $\delta^2\Pi_7$ turn out to be of alternating sign at all ν. The corresponding phases (the one-dimensional c_1P, the "conjugate with the A phase" $c_5\hat{C}_5$, and the A_1 phase $c_7\hat{C}_7$) are destroyed in this model by small perturbations, and are therefore not realized. The expression for $\delta^2\Pi_6$ (as well as for $\delta^2\Pi_4$) turns out to be of alternating sign for all $\nu>0$, while at $\nu=0$ the corresponding phase coincides with the A phase.

Thus, in the considered model system, the condition for stability in the small is satisfied only for the A phase (at $H=0$), for the B phase (at $H\leqslant H_c$), and for the $2D$ phase (at $H\geqslant H_c$).

The quadratic form $\delta^2\Pi_3$ with 18 variables u_{ia} and v_{ia} has at $\nu<\frac{1}{2}$ four null eigenvectors, the form $\delta^2\Pi_2$ has at $\nu>\frac{1}{2}$ six null vectors, and the form $\delta^2\Pi_4$ has at $\nu=0$ nine null vectors. This is due to the existence in the B phase of four branches of the Bose spectrum, which start out from zero ($E(\mathbf{k}=0)=0$), and to the existence of six such branches in the $2D$ phase and nine in the A phase. We note that the presence of at least four $\delta^2\Pi$ null vectors in the B phase is the consequence of the symmetry of Π relative to the transformations with one parameter α and with three parameters that define the orthogonal matrix U.

Consider now the Bose spectrum of the system at $|\Delta T|\ll T_c$. At $T>T_c$ all the branches of the spectrum are pure imaginary, with $|E(\mathbf{k})|\ll T_c$. Continuing analytically the function (3.5) into the region $|\omega|\ll T$, we obtain at $\omega>0$

$$A_{ij}(p)-A_{ij}(0)=\frac{Z^2k_F^2}{12\pi Tc_F}\left[\omega\delta_{ij}+\frac{7\zeta(3)c_F^2}{10\pi^3T}(\delta_{ij}k^2+2k_ik_j)\right]. \tag{3.25}$$

Taking into account the terms $g^{-1}+A(0)$ [Eq. (3.12)] and the magnetic increment (3.7) we obtain at $T>T_c$ the following branches of the spectrum:

$$\begin{gathered}E_{a,\parallel}(\mathbf{k})=-i\left(\frac{21\zeta(3)c_F^2k^2}{10\pi^3T_c}+\frac{8}{\pi}(T-T_c)+\delta_{a3}\frac{14\zeta(3)\mu^2H^2}{\pi^3T_c}\right)\\ E_{a,\perp}(\mathbf{k})=-i\left(\frac{7\zeta(3)c_F^2k^2}{10\pi^3T_c}+\frac{8}{\pi}(T-T_c)+\delta_{a3}\frac{14\zeta(3)\mu^2H^2}{\pi^3T_c}\right)\end{gathered} \tag{3.26}$$

Here a is the isotopic index of the corresponding branch, and the symbol $\parallel$ or $\perp$ indicates that the vector index is "parallel" or "perpendicular" to the propagation direction.

At $T<T_c$ it is necessary to take into account the fourth-degree forms. Separating from S_h the quadratic form after making the shift $c_{ia}(p)\to c_{ia}(p)+c_{ia}^{(0)}$, it is easy to obtain the Bose spectrum. All its branches are of the form $E=-iak^2-i\Gamma$ with $\alpha>0$ and $\Gamma\geqslant 0$, with $\Gamma=0$ for four branches in the B phase, six branches in the $2D$

phase, and nine branches in the A phase. These are precisely the branches that go over into the phonon-spectrum branches with decreasing temperature.

4. THE REGION $T_c - T \approx T_c$

The "condensate density" can be sought here as before in one of the forms (3.19). Let initially $H=0$. The solution corresponding to the B phase is of the form

$$c_{ia}^{(0)} = (\beta V)^{1/2}\delta_{p0}\delta_{ia}c, \tag{4.1}$$

and the equation obtained for the constant c is

$$\frac{3}{g}+\frac{4Z^2}{\beta V}\sum_p(\omega^2+\xi^2+4c^2Z^2)^{-1}$$

$$=\frac{3}{g}+\frac{2Z^2k_F^2}{\pi^2 c_F}\int_0^{c_F k_0}\frac{d\xi}{(\xi^2+4c^2Z^2)^{1/2}}\,\mathrm{th}\,\frac{\beta}{2}(\xi^2+4c^2Z^2)^{1/2}=0. \tag{4.2}$$

Using (3.9), we rewrite this equation in the form

$$\int_0^\infty d\xi\left(\frac{1}{(\xi^2+4c^2Z^2)^{1/2}}\,\mathrm{th}\,\frac{\beta}{2}(\xi^2+4c^2Z^2)^{1/2}-\frac{1}{\xi}\,\mathrm{th}\,\frac{\beta_c\xi}{2}\right)=0. \tag{4.3}$$

For $T=0$ we have an explicit formula for cZ:

$$cZ=\frac{\pi}{2\gamma}T_c=c_F k_F\exp\left(-\frac{3\pi^2 c_F}{2|g_0|Z^2k_F^2}\right), \tag{4.4}$$

as well as for the functional S_h:

$$\Delta p_3=\lim_{T\to 0, V\to\infty}(S_h/\beta V)=T_c^2k_F^2/4\gamma^2c_F. \tag{4.5}$$

The symbol Δp_3 reminds us that this quantity has the physical meaning of a correction to the pressure in the corresponding superfluid phase.

For the remaining superfluid phases at $H=0$ and $T=0$ we can also obtain explicit formulas

$$|c_1|/|c_3|=e^{1/6},\quad |c_2|/|c_3|={}^1/_2e^{5/6},$$
$$|c_4|=|c_6|=|c_2|,\quad |c_5|=|c_1|,\quad |c_7|={}^1/_2|c_2|; \tag{4.6}$$

$$\Delta p_1/\Delta p_3={}^1/_3e^{1/3},\quad \Delta p_2/\Delta p_3={}^1/_6e^{5/3},$$
$$\Delta p_4=\Delta p_6=\Delta p_2,\quad \Delta p_5={}^1/_2\Delta p_1,\quad \Delta p_7={}^1/_2\Delta p_2. \tag{4.7}$$

It is seen that at $T=0$, just as in the region $|\Delta T| \ll C_c$, the symmetrical B phase is energywise most favored, while the two-dimensional $2D$ phase and the A phase have "equal opportunities." All the remaining phases are energywise less profitable and can be shown to be unstable to small perturbations at all values of H.

Just as in the Ginzburg-Landau region, the A phase is metastable but is destroyed when an arbitrarily weak magnetic field is turned on.

In the presence of a magnetic field, B and $2D$ are competitively possible. For the $2D$ phase the functional S_h is not dependent on H at all. To prove this, we substitute $c_{ia}=(\beta V)^{1/2}\delta_{p0}c_2(P_2)_{ia}$ in the functional

$$\ln\det\frac{\hat M(c,\bar c)}{\hat M(0,0)}=\sum_p\ln\left[\left(1+\frac{4c^2Z^2\sin^2\theta}{\omega^2+(\xi-\mu H)^2}\right)\left(1+\frac{4c^2Z^2\sin^2\theta}{\omega^2+(\xi+\mu H)^2}\right)\right]. \tag{4.8}$$

We write down the logarithm of the product as the sum of two terms, in which we make the respective substitutions $\xi\to\xi-\mu H$ and $\xi\to\xi+\mu H$ as we integrate with respect to ξ, thus eliminating the dependence on H.

To find the condensate density in the B phase, we substitute

$$c_{ia}=(aP+bP_2)_{ia}\delta_{p0}(\beta V)^{1/2}$$

in S_h. We obtain

$$S_h=\frac{\beta V}{g}[a^2+2b^2]$$
$$+\frac{1}{2}\sum_p\ln\left\{1+\frac{1}{[\omega^2+(\xi+\mu H)^2][\omega^2+(\xi-\mu H)^2]}\right.$$
$$\times[16(a^2\cos^2\theta+b^2\sin^2\theta)^2+8(\omega^2+\xi^2)(a^2\cos^2\theta+b^2\sin^2\theta)$$
$$\left.+8\mu^2H^2(b^2\sin^2\theta-a^2\cos^2\theta)]\right\}. \tag{4.9}$$

The parameters a and b are determined from the equations

$$\partial S_h/\partial a=0,\quad \partial S_h/\partial b=0. \tag{4.10}$$

We shall use these equations to find the critical magnetic field H_c at which the B phase goes over into the two-dimensional $2D$ phase. At the transition point we have $a=0$. The second equation in (4.10) coincides with the corresponding equation for the $2D$ phase where, as already shown S_h is independent of H. We can therefore obtain from the second equation the value of b (which does not depend on H) and then substitute it in the first, which now takes the form

$$\frac{8\mu^2H^2}{\beta V}\sum_p\frac{\cos^2\theta}{M}=\frac{k_F^2}{3\pi^2c_F}, \tag{4.11}$$

where

$$M=(\omega^2+\xi^2)^2+2\mu^2H^2(\omega^2-\xi^2)+\mu^4H^4$$
$$+16b^4\sin^4\theta+8(\omega^2+\xi^2+\mu^2H^2)b^2\sin^2\theta,$$

with b independent of H. It can be shown that the left-hand side of (4.11) is monotonic in H, varying from zero at $H=0$ to infinity as $H\to\infty$. Therefore the equation has a single root $H=H_c$.

Investigation of the second variations of the phases B and $2D$ shows the B phase to be stable at $H<H_c$ and the $2D$ phase at $H>H_c$. Just as in the Ginzburg-Landau region, the form of the second variation in the B phase has four null eigenvectors and six in the $2D$ phase. The A phase (at $H=0$) has nine null eigenvectors and as many phonon branches.

Let us indicate the calculation of the phonon branches and of the sound velocities for the B, $2D$, and A phases at $T=0$ and $H=0$.

We begin with the B phase. We make in S_h the substitutions

$$c_{ia}(p)\to c_{ia}(p)+c_{ia}^{(0)}(p),\quad c_{ia}^+(p)\to c_{ia}^+(p)+c_{ia}^{(0)}(p) \tag{4.12}$$

and separate in S_h a quadratic form in the new variables:

$$
\begin{aligned}
&-\sum_p c_{ia}^+(p)c_{ja}(p)\left[\frac{\delta_{ij}}{g}-\frac{4}{\beta V}\sum_{p_1+p_2=p} n_{1i}n_{1j}\varepsilon(-p_1)\varepsilon(-p_2)G(p_1)G(p_2)\right]\\
&\quad-\frac{1}{2}\sum_p (c_{ia}(p)c_{jb}(-p)+c_{ia}^+(p)c_{jb}^+(-p))\\
&\quad\times\frac{16c^2Z^2}{\beta V}\sum_{p_1+p_2=p} G(p_1)G(p_2)(2n_{1a}n_{1b}n_{1i}n_{1j}-\delta_{ab}n_{1i}n_{1j}),\\
&\quad\varepsilon(p)=i\omega-\xi,\quad G(p)=Z(\omega^2+\xi^2+4c^2Z^2)^{-1}.
\end{aligned}
\tag{4.13}
$$

We express the coefficient tensors of c^+c and $cc+c^+c^+$ in the form $A(0)+(A(p)-A(0))$ and $B(0)+(B(p)-B(0))$. We have

$$
\begin{aligned}
A_{ij}(0)&=\delta_{ij}\frac{1}{g}-\frac{4}{\beta V}\sum_p n_in_j\varepsilon_+\varepsilon_-G^2\\
&=\delta_{ij}\left(\frac{1}{g}-\frac{4Z}{3\beta V}\sum_p G\right)+\frac{16c^2Z^2}{\beta V}\sum_p n_in_jG^2=\frac{Z^2k_F^2}{3\pi^2c_F}\delta_{ij},\\
B_{ijab}(0)&=\frac{16c^2Z^2}{\beta V}\sum_p(2n_an_bn_in_j-\delta_{ab}n_in_j)G^2\\
&=\frac{Z^2k_F^2}{15\pi^2c_F}[2\delta_{ai}\delta_{bj}+2\delta_{aj}\delta_{bi}-3\delta_{ab}\delta_{ij}].
\end{aligned}
\tag{4.14}
$$

The contribution of the terms with $A(0)$ and $B(0)$ is

$$
\begin{aligned}
&\frac{Z^2k_F^2}{3\pi^2c_F}\sum_p [c_{ia}^+(p)c_{ia}(p)+{}^1/_{10}(2\delta_{ai}\delta_{bj}\\
&+2\delta_{aj}\delta_{bi}-3\delta_{ab}\delta_{ij})(c_{ia}(p)c_{jb}(-p)+c_{ia}^+(p)c_{jb}^+(-p))].
\end{aligned}
\tag{4.15}
$$

We consider the term with $p=0$, in which we put $c_{ia}=u_{ia}+iv_{ia}$, $c_{ia}^+=u_{ia}-iv_{ia}$. We obtain a sum of two forms:

$$
{}^2/_5(u_{ia}u_{ia}+u_{aa}u_{bb}+u_{ia}u_{ai})+{}^3/_5(4v_{ia}v_{ia}-v_{aa}v_{bb}-v_{ia}v_{ai}).
\tag{4.16}
$$

the u-form has three null eigenvectors corresponding to the variables $u_{12}-u_{21}$, $u_{23}-u_{32}$, $u_{31}-u_{13}$, while the v form has one null vector corresponding to the variable $v_{11}+v_{22}+v_{33}$. Therefore at small p, after expanding $A(p)-A(0)$ and $B(p)-B(0)$ in powers of ω^2 and k^2, there will be no free terms for the corresponding "phonon" variables. The condition that the terms proportional to ω^2 and k^2 be cancelled out after the analytic continuation $i\omega\to E$ yields the equation $E=ck$ and yields the speeds of the sounds. The terms proportional to ω^2 and k^2 are of the form

$$
\begin{aligned}
&\frac{k_F^2}{\pi^2c_Fc^2}\Big\{\omega^2\left(\frac{13}{90}u_{ia}u_{ia}-\frac{1}{45}(u_{aa})^2-\frac{1}{45}u_{ia}u_{ai}\right)\\
&+\frac{c_F^2}{45}(u_{ia}u_{ia}k^2+2k_ik_ju_{ia}u_{ja})-\frac{c_F^2}{630}(4k_ak_bu_{ia}u_{ib}+8k_ak_bu_{ia}u_{bi}+2k^2u_{ia}u_{ai}\\
&+8k_ik_au_{ia}u_{bb}+2k^2(u_{aa})^2-5u_{ia}u_{ia}k^2-10k_ik_ju_{ia}u_{ja})\\
&+\omega^2\left(\frac{7}{90}v_{ia}v_{ia}+\frac{1}{45}(v_{aa})^2+\frac{1}{45}v_{ia}v_{ai}\right)+\frac{c_F^2}{45}(v_{ia}v_{ia}k^2+2k_ik_jv_{ia}v_{ja})\\
&+\frac{c_F^2}{630}(4k_ak_bv_{ia}v_{ib}+8k_ak_bv_{ia}v_{bi}+2k^2v_{ia}v_{ai}+8k_ik_av_{ia}v_{bb}+2k^2(v_{aa})^2\\
&-5v_{ia}v_{ia}k^2-10k_ik_jv_{ia}v_{ja})\Big\}.
\end{aligned}
\tag{4.17}
$$

Let the excitation propagate along the third axis, so that $k_1=k_2=0$ and $k_3=k$. Putting $u_{ia}=0$ and $v_{ia}=v\delta_{ia}$, we obtain that the coefficient of v^2 is proportional to $(\omega^2+c_F^2k^2/3)$, and we get the phonon branch

$$
E=c_Fk/\sqrt{3},
\tag{4.18}
$$

corresponding to the acoustic mode.

For the u modes we have

$$
\begin{aligned}
&E_{(u_{12}-u_{21})}=c_Fk/\sqrt{5},\\
&E_{(u_{23}-u_{32})}=E_{(u_{31}-u_{13})}=\sqrt{2/5}\,c_Fk.
\end{aligned}
\tag{4.19}
$$

The excitations (4.19) correspond to spin waves. The results (4.18) and (4.19) agree with those obtained by other methods.[11–15]

Carrying out analogous manipulations for the two-dimensional 2D phase, we obtain the following results.

The "phonon" variables are

$$
u_{12}-u_{21},\quad v_{11}+v_{22},\quad u_{31},\quad u_{32},\quad v_{31},\quad v_{32}.
$$

The first two of them correspond to the acoustic phonon branch

$$
E_{(u_{12}-u_{21})}=E_{(v_{11}+v_{22})}=c_Fk/\sqrt{3},
\tag{4.20}
$$

and the last four to a branch of the form

$$
E_{u_{31}}=E_{u_{32}}=E_{v_{31}}=E_{v_{32}}=c_Fk_\parallel,
\tag{4.21}
$$

where $k_\parallel$ is the component of the vector $\mathbf{k}$ along the magnetic-field direction.

Similar results are obtained for the A phase. The phonon variables are here

$$
u_{11}+v_{21},\quad u_{12}+v_{22},\quad v_{13}-u_{23},\quad u_{31},\quad u_{32},\ u_{33},\quad v_{31},\quad v_{32},\quad v_{33}.
$$

The first three correspond to branches that propagate with the speed of sound:

$$
E_{(u_{11}+v_{21})}=E_{(u_{12}+v_{22})}=E_{(v_{13}-u_{23})}=c_Fk/\sqrt{3},
\tag{4.22}
$$

and the remaining six correspond to the branches

$$
E_{u_{3a}}=E_{v_{3a}}=c_Fk_\parallel\quad(a=1,2,3).
\tag{4.23}
$$

We note that in the calculation of the branches (4.21) and (4.23) the coefficient of $\omega^2+c_F^2k_\parallel^2$ is formally a logarithmically diverging integral in terms of the angle variables $\int d\Omega\,(\sin\theta)^{-2}$. In fact, the integral is equal to $\sim\ln(\Delta/c_Fk)$ (Δ is the energy gap), and the logarithmic divergence is obtained if we forget the non-analyticity of the corresponding coefficient function as $\omega\to 0$ and $k\to 0$. No "logarithmic situation" arises in the calculation of the branches (4.20) and (4.22).

In concluding this section we note that although there are many papers devoted to the calculation of the Bose spectrum in the A phase by other methods, most of them are incorrect (for references see, e.g.,[16,17]).[2)]

5. QUALITATIVE CONCLUSIONS FOR He^3

The considered model system can describe the real He^3 only qualitatively. In addition to the approximations

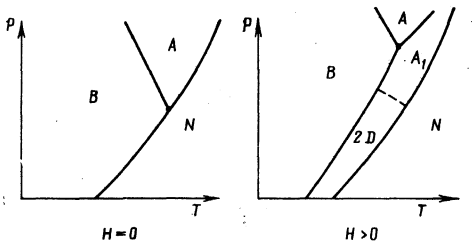

FIG. 1.

$f=0$ and $g=\text{const}$, no account has been taken here of a number of other effects, such as spin-spin interaction.[1] Nonetheless, a number of regularities that have been noticed for the model can hold also for He^3.

In He^3 at low temperatures, in the absence of a magnetic field, the B phase is energywise most favored. The A phase exists in a certain temperature interval at sufficiently high pressures (above the tricritical point). Our model system is close to the gas model and should agree better with the properties of He^3 at low pressures, where only the B phase exists in fact. The existence of the A phase can be regarded as an effect due to high pressure, when He^3 differs substantially from the gas model. The A phase in this model is metastable and is destroyed by an arbitrarily weak magnetic field. In He^3, the A phase has a large margin of strength and continues to exist also when a magnetic field is turned on.

In our model, when the magnetic field is strong enough, the B phase goes over into the two-dimensional $2D$ phase. It seems natural for this to hold true also for He^3 in a sufficiently strong magnetic field. Any magnetic field is "sufficiently strong" in the vicinity of the phase-transition line. It is therefore natural to assume that in a magnetic field at low pressures the normal phase N goes over first into the two-dimensional $2D$ phase, and only then into the B phase, while the phase diagram assumes the form shown in Fig. 1. The dashed line on the phase diagram at $H>0$ marks the boundary between the A_1 phase that exists in the magnetic field at sufficiently high pressure, and the $2D$ phase that exists possibly at low pressure. Transitions between the phases N and $2D$, N and A_1, or B and $2D$ are of second order, while those between A and B, A and A_1, A_1 and B, and A_1 and $2D$ are of first order.

We conclude thus that a $2D$ phase can exist in He^3 in a magnetic field.

As to the Bose spectrum, in the general case of a Fermi system such as He^3 the number of the phonon branches of the spectrum turns out, generally speaking, less than in the considered model system. In the general case there exist four branches in the B and $2D$ phases and five branches in the A phase. This result can be easily obtained if we use for Π an expression that differs from (3.15) by arbitrary coefficients in front of the invariants

$$(\operatorname{tr} A^+A)^2,\ \operatorname{tr} AA^+AA^+,\ \operatorname{tr} AA^+A^*A^T,\ \operatorname{tr} AA^TA^*A^+,\ |\operatorname{tr} AA^T|^2$$

and then obtain the number of eigenvectors of the quadratic form $\delta^2\Pi$, which correspond to zero eigenvalues.

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1)Central University of Venezuela, Caracas.

2)Our formulas (4.22) and (4.23) agree only with the results of[17] in the limit as $T\to 0$.

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