Lagrange and Hamilton equations of hydrodynamics for anisotropic superfluid He^3-A liquid

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The Lagrangian method of obtaining the equations of hydrodynamics for He II is generalized to the case of the quantum anisotropic liquid He^3-A. A complete set of He^3-A orbital hydrodynamic equations is derived on the basis of the Lagrange equations. It is shown how the set of Lagrange equations can be reduced to the Hamilton form. The spin subsystem parameters are included in the Hamiltonian formalism The He^3-A spin equations of hydrodynamics are found by taking into account the presence of an external magnetic field. The general form of the conservation laws for a system obeying the Hamilton equations is presented. The energy, momentum, angular momentum and total spin conservation laws are found on this basis. The form of the dissipative corrections to the He^3-A equations of hydrodynamics is considered. The kinetic coefficients are enumerated and the requirements are found which they should satisfy in the case of a combined analysis of the spin and orbital subsystems as well as in the case of an analysis of only the latter subsystem.

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INTRODUCTION

The problem of the Lagrangian or Hamiltonian form of the equations of hydrodynamics has long been of interest. Such a form of the equations was found by Lamb for the classical liquid.[1] Davydov[2] found the Lagrangian form of the equations for a barotropic liquid.[2] The problem was solved by Khalatnikov for helium II.[3] We shall briefly review this latter work.

The basic characteristics of the superfluid are the mass density ρ, the entropy density s, the normal and the superfluid velocities $\mathbf{v}_n$ and $\mathbf{v}_s$, and the momentum density $\mathbf{j}$. The equations of hydrodynamics are formulated with the help of the Lagrangian

$$L=-{}^1/_2\rho v_s^2+\mathbf{j}\mathbf{v}_s-\bar\varepsilon(\rho,\ s,\ \mathbf{v}_n-\mathbf{v}_s)+\alpha(\dot\rho+\nabla\mathbf{j})+\beta(\dot s+\nabla(s\mathbf{v}_n))+\gamma(\dot f+\nabla(f\mathbf{v}_n)),\tag{1}$$

where $\bar\varepsilon=\varepsilon-\mathbf{g}(\mathbf{v}_n-\mathbf{v}_s)$ is obtained by a Legendre transformation from the thermodynamic energy density ε, $\mathbf{g}=\partial\bar\varepsilon/\partial\mathbf{v}_s$ is the normal momentum density. The variables α, β and γ are the Lagrangian multipliers in the laws of conservation of mass, entropy, and one more quantity, the reason for whose introduction will be explained when the Lagrangian (1) is varied with respect to $\mathbf{v}_n$. This yields

$$\mathbf{g}=s\nabla\beta+f\nabla\gamma.\tag{2}$$

Thus, thanks to the introduction of the Clebsch variables f and γ, we have obtained (together with β) three arbitrary functions that are necessary for the description of the three independent components of $\mathbf{g}$.

As was shown in Refs. 4, the Lagrangian formalism can be reformulated in Hamiltonian language in standard fashion. The Hamiltonian here has the usual form

$$\mathscr{H}=\int d^3xE,\quad E={}^1/_2\rho v_s^2+\mathbf{g}\mathbf{v}_s+\varepsilon.\tag{3}$$

Here $\varepsilon=\varepsilon(\rho,s,g)$; it is also necessary to substitute (2) and $\mathbf{v}_s=\nabla\alpha$. After this, the equations of hydrodynamics can be obtained as the Hamiltonian equations if the pairs (ρ,α), (s,β), (f,γ) are considered as pairs of canonically conjugate variables (q,p), for which

$$\begin{aligned}\frac{\partial}{\partial t}q&=\frac{\delta\mathscr{H}}{\delta p}\equiv\frac{\partial E}{\partial p}-\nabla\frac{\partial E}{\partial\nabla p},\\ \frac{\partial}{\partial t}p&=-\frac{\delta\mathscr{H}}{\delta q}\equiv-\frac{\partial E}{\partial q}+\nabla\frac{\partial E}{\partial\nabla q}.\end{aligned}\tag{4}$$

Recently, the problem of the Lagrangian and Hamiltonian formulation of the hydrodynamical equations of a quantum liquid were considered in Ref. 5.

In an anisotropic quantum liquid He^3-A (see, for example, the review of Ref. 6) there is an order parameter $\hat\Delta$ (the physical meaning of which is the energy gap in the excitation spectrum)

$$\hat\Delta=i(\mathbf{n}\hat\sigma)\sigma_y\Delta_0(\Phi\mathbf{k}/k),\quad\Phi_\alpha\Phi_\beta=\delta_{\alpha\beta},$$

where $\hat\sigma$ are Pauli matrices, $\mathbf{k}$ is the wave vector of the excitations, and Δ_0 is the maximum value of the gap. The vector $\mathbf{\Phi}=\mathbf{\Phi}_1+i\mathbf{\Phi}_2$ characterizes the orbital part of the order parameter, such that the vector $\mathbf{l}=\mathbf{\Phi}_1\times\mathbf{\Phi}_2$ indicates the direction of the orbital momentum of the pair and $\mathbf{n}$ determines the direction of the spin momentum of the pair.

The superfluid velocity is expressed as follows in terms of the order parameter

$$\mathbf{v}_s=-\frac{\hbar}{2m}\Phi_{2i}\nabla\Phi_{1i}.$$

In Ref. 7, the authors assumed the Lagrangian formulation of the equations of orbital hydrodynamics of He^3-A. In the present paper, along with the Lagrangian, we consider the Hamiltonian formalism, and in comparison with Ref. 7 we also consider the spin hydrodynamics of He^3-A, in addition, we take into account the

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dependence of the energy $\tilde{\varepsilon}$ on the time derivative of $\mathbf{l}$; moreover, the equations are formulated from the very beginning such that the stress tensor is symmetric.

ORBITAL HYDRODYNAMICS

A number of papers have recently appeared devoted to the orbital hydrodynamics of He^3-A (see Refs. 8 and 9). This calls for taking into account the dependence of $\tilde{\varepsilon}$ on $\mathbf{l}$, and also on its derivatives. To the Lagrangian (1) we add the structure condition on the reference frame $(\Phi_1, \Phi_2, \mathbf{l})$ with the Lagrangian multiplier $\mathbf{\Omega}$, and also a term which describes the transport of the order parameter. It should be so chosen that variation with respect to $\mathbf{j}$ yields the correct expression for the superfluid velocity. The term $(\hbar/2m)\Phi_2(\mathbf{j}\cdot\nabla)\Phi_1$ satisfies this requirement but is not a Galilean invariant. Therefore, it is necessary to introduce the "new" derivative

$$\frac{d}{d\tau}\Phi_i=\frac{\hbar}{2m}\left(\left(\rho\frac{\partial}{\partial t}+\mathbf{j}\nabla\right)\Phi_i-\frac{1}{2}\rho[\operatorname{rot}\mathbf{v}_n\times\Phi_1]\right).$$

The component $c\,\operatorname{curl}\mathbf{v}_n$ is necessary in order that we obtain ultimately a symmetric stress tensor. Similar considerations lead to the result that $\tilde{\varepsilon}$ should depend on the total derivative

$$\frac{d}{dt}\mathbf{l}=\left(\frac{\partial}{\partial t}+\mathbf{v}_n\nabla\right)\mathbf{l}-\frac{1}{2}[\operatorname{rot}\times\mathbf{v}_n\mathbf{l}].$$

As a result, we obtain the Lagrangian

$$L=-\frac{1}{2}\rho v_s^2+\mathbf{j}\mathbf{v}_s-\tilde{\varepsilon}\left(\rho,s,\mathbf{v}_n-\mathbf{v}_s,\mathbf{l},\nabla_i\mathbf{l},\frac{d\mathbf{l}}{dt}\right)+\Phi_2\frac{d}{d\tau}\Phi_1+\Omega(\mathbf{l}-[\Phi_1\times\Phi_2])$$
$$+\alpha(\dot\rho+\nabla\mathbf{j})+\beta(\dot s+\nabla(s\mathbf{v}_n))+\gamma(\dot f+\nabla(f\mathbf{v}_n)).\qquad(5)$$

This Lagrangian leads to the following set of equations (on the left we have the variables the variation with respect to which yields the corresponding equations):

$$\Omega:\quad \mathbf{l}=[\Phi_1\times\Phi_2],\qquad(6)$$

$$\mathbf{j}:\quad \mathbf{v}_s=\nabla\alpha-\frac{\hbar}{2m}\Phi_{2i}\nabla\Phi_{1i},\qquad(7)$$

$$\mathbf{v}_n:\quad \mathbf{g}=\frac{\partial\tilde{\varepsilon}}{\partial\mathbf{v}_s}=s\nabla\beta+f\nabla\gamma-G_i\nabla l_i+\frac{1}{2}\operatorname{rot}([\mathbf{l}\times\mathbf{G}]+[\Phi_1\times\mathbf{F}])\qquad(8)$$

$$\mathbf{v}_s:\quad \mathbf{j}=\rho\mathbf{v}_s+\mathbf{g},\qquad(9)$$

$$\alpha,\beta,\gamma:\quad \dot\rho+\nabla\mathbf{j}=0,\quad \dot s+\nabla(s\mathbf{v}_n)=0,\quad \dot f+\nabla(f\mathbf{v}_n)=0,\qquad(10)$$

$$\rho:\quad \rho\dot\alpha+\mathbf{j}\nabla\alpha+{}^1\!/_2\rho v_s^2-\mathbf{j}\mathbf{v}_s-\Omega\mathbf{l}+\rho\mu=0,\qquad(11)$$

$$s,f:\quad \dot\beta+\mathbf{v}_n\nabla\beta+T=0,\quad \dot\gamma+\mathbf{v}_n\nabla\gamma=0,$$

$$\Phi_\alpha:\quad \frac{d}{d\tau}\Phi_\alpha=[\Omega\times\Phi_\alpha],\qquad(12)$$

$$l:\quad \frac{d}{dt}\mathbf{G}+\mathbf{G}(\nabla\mathbf{v}_n)=\Omega-\frac{\partial\tilde{\varepsilon}}{\partial\mathbf{l}}+\nabla_i\frac{\partial\tilde{\varepsilon}}{\partial\nabla_i\mathbf{l}},\qquad(13)$$

where

$$\mathbf{G}=-\frac{\partial\tilde{\varepsilon}}{\partial\dot{\mathbf{l}}},\qquad \mathbf{F}=\frac{\hbar\rho}{2m}\Phi_2,\qquad(14)$$

and $T=\partial\tilde{\varepsilon}/\partial s$ is the temperature, $\mu=\partial\tilde{\varepsilon}/\partial\rho$ is the chemical potential.

It follows from (12) that

$$\left(\rho\frac{\partial}{\partial t}+\mathbf{j}\nabla\right)(\Phi_\alpha\Phi_\beta)=0,$$

i.e., the relation $\Phi_\alpha\Phi_\beta=\delta_{\alpha\beta}$ can be considered as the boundary conditions for the given system of equations. With account of these relations, we can find from (6) and (7) the expression for the curl of the superfluid velocity[10]:

$$\nabla_i v_{sj}-\nabla_j v_{si}=\frac{\hbar}{2m}\mathbf{l}[\nabla_i\mathbf{l}\times\nabla_j\mathbf{l}].\qquad(15)$$

Moreover, we note that the term $\nabla\alpha$ in the expression (7) for the superfluid velocity can always be eliminated through the gauge transformation $\Phi\to\exp(-2im\alpha/\hbar)\Phi$.

It follows directly from (6) and (12) that

$$\frac{d}{d\tau}\mathbf{l}=[\Omega\times\mathbf{l}].\qquad(16)$$

We then find for the superfluid velocity

$$\frac{\partial v_{si}}{\partial t}=-\nabla_i\left(\mu+\frac{1}{2}v_s^2+\frac{1}{2}(\operatorname{rot}\mathbf{v}_n)\frac{\hbar\mathbf{l}}{2m}\right)+\frac{\hbar}{2m}\dot{\mathbf{l}}[\nabla_i\mathbf{l}\times\mathbf{l}].\qquad(17)$$

We shall show how the obtained equations can reduce to Hamiltonian form. Forst, for the variables $q-(\rho,s,f,\mathbf{l},\Phi_1)$, on the time derivatives of which the Lagrangian (5) depends, we can introduce the canonically conjugate variables $p=\partial L/\partial\dot q$. We find the following (q,p) pairs:

$$(\rho,\alpha),\quad (s,\beta),\quad (f,\gamma),\quad (\mathbf{l},\mathbf{G}),\quad (\Phi_1,\mathbf{F}).\qquad(18)$$

We construct the Routh function $R(q,p,\mathbf{v}_n,\mathbf{v}_s,\mathbf{j},\Omega)=p\dot q-L$, which gives the Hamiltonian equation (4) for the variables (q,p). Equations (7)–(9), which are specified by the Lagrangian variables $\mathbf{v}_n$, $\mathbf{v}_s$, and $\mathbf{j}$ of R can be regarded as expressions of the corresponding quantities in terms of the Hamiltonian variables (q,p). After substitution of these expressions in the Routh function R, it should be reduced to the energy density E with the additional structural condition

$$\tilde R(q,p,\Omega)=\frac{\rho v_s^2}{2}+\mathbf{g}\mathbf{v}_s+\varepsilon(\rho,s,\mathbf{g},\mathbf{l},\nabla_i\mathbf{l},\mathbf{G})-\Omega\left(\mathbf{l}-\frac{2m}{\hbar\rho}[\Phi_1\times\mathbf{F}]\right).\qquad(19)$$

In particular, the following Hamiltonian equation exists for l:

$$\frac{d\mathbf{l}}{dt}=\frac{\partial\varepsilon}{\partial\mathbf{G}}.\qquad(20)$$

Generally speaking, ε depends on $\mathbf{G}$ and also $\mathbf{g}$ (for this reason the total derivative appears in (20)); however, the derivative in (20) is at constant $\mathbf{g}$.

The function $\tilde R$ actually is identical with R, with accuracy to total divergence of the function of (q,p) (which is not reflected in the form of Eqs. (4)), if we assume

$$\tilde{\varepsilon}=\varepsilon-\mathbf{g}(\mathbf{v}_n-\mathbf{v}_s)-\mathbf{G}\frac{\partial\varepsilon}{\partial\mathbf{G}}.\qquad(21)$$

Thus we see (with account of $\mathbf{v}_n-\mathbf{v}_s=\partial\varepsilon/\partial g$), that $\tilde{\varepsilon}$ is obtained from ε by the Legendre transformation with a change from the variables $\mathbf{G}$ and $\mathbf{g}$ to their conjugates $d\mathbf{l}/dt$ and $\mathbf{v}_n-\mathbf{v}_s$, the expression $T=\partial\tilde{\varepsilon}/\partial s=\partial\varepsilon/\partial s$ is substantiated, and so on.

Hamilton's equations, given by (19) are equivalent to the system (10)–(13), (20). We note that although there are two equations for $\mathbf{l}$ [(16) and (20)], the unknown vector function $\mathbf{\Omega}$ enters into the system, the choice of which also assures the consistency of these equations.

SPIN HYDRODYNAMICS

The spin hydrodynamics of He^3-A has been considered in the linear approximation both microscopically[11] and phenomenologically.[12] The Hamiltonian and Lagrangian forms of the nonlinear equations of spin hydrodynamics will be formulated below.

Here we must take into account the dependence of the thermodynamic energy density ε on the magnetization $\mathbf{M}$, and also on the vector $\mathbf{n}$ and its spatial derivatives. We shall also assume that the system is located in the external field $\mathbf{H}$, so that we must add to the internal energy density ε_i the term that is due to this field: $\varepsilon = \varepsilon_i - \mathbf{M}\cdot\mathbf{H}$.

With the help of the Hamiltonian (3) or the Routh function (19), we can obtain the usual canonical equations if we introduce pairs of canonically conjugate variables $(\mathbf{n}, \boldsymbol{\zeta})$, $(\boldsymbol{\eta}, \boldsymbol{\xi})$, the meaning of which is that

$$\mathbf{M}=\Gamma([\boldsymbol{\zeta}\times\mathbf{n}]+[\boldsymbol{\xi}\times\boldsymbol{\eta}]), \tag{22}$$

where Γ is the gyromagnetic ratio. The variables $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ are necessary since it is impossible to describe the longitudinal $\mathbf{n}$ part of $\mathbf{M}$ by means of $\boldsymbol{\zeta}$. It is also necessary to take it into account that a contribution g_M is made to the momentum density by the spin subsystem,

$$g_M=-\zeta_i\nabla n_i-\xi_i\nabla\eta_i-\frac{1}{2\Gamma}\operatorname{rot}\mathbf{M}, \tag{23}$$

and must now be substituted in (8).

By means of (22), (23), we find the Hamilton's equations

$$\frac{d\mathbf{n}}{dt}=\Gamma[\mathbf{n}\times(\mathbf{h}-\mathbf{H})], \tag{24}$$

$$\frac{d\boldsymbol{\eta}}{dt}=\Gamma[\boldsymbol{\eta}\times(\mathbf{h}-\mathbf{H})], \tag{25}$$

$$\frac{d\boldsymbol{\zeta}}{dt}+\boldsymbol{\zeta}(\nabla\mathbf{v}_n)=\Gamma[\boldsymbol{\zeta}\times(\mathbf{h}-\mathbf{H})]+\boldsymbol{\psi}, \tag{26}$$

$$\frac{d\boldsymbol{\xi}}{dt}+\boldsymbol{\xi}(\nabla\mathbf{v}_n)=\Gamma[\boldsymbol{\xi}\times(\mathbf{h}-\mathbf{H})], \tag{27}$$

where

$$\mathbf{h}=\partial\varepsilon_0/\partial\mathbf{M},\quad \boldsymbol{\psi}=-\partial\varepsilon_0/\partial\mathbf{n}+\nabla_i\partial\varepsilon_0/\partial(\nabla_i\mathbf{n}).$$

For $\mathbf{M}$ we find the equation

$$\frac{d\mathbf{M}}{dt}+\mathbf{M}(\nabla\mathbf{v}_n)=\Gamma[\mathbf{M}\times(\mathbf{h}-\mathbf{H})]+\Gamma[\boldsymbol{\psi}\times\mathbf{n}]. \tag{28}$$

It follows from Eqs. (24) and (28) that

$$\left(\frac{\partial}{\partial t}+\nabla\mathbf{v}_n\right)(\mathbf{M}\mathbf{n})=0. \tag{29}$$

The pairwise scalar products of the vectors $\mathbf{n}$, $\boldsymbol{\eta}$, $\boldsymbol{\xi}$ are conserved in this same sense (or with the replacement of the derivative by $\partial/\partial t+\mathbf{v}_n\nabla$). Therefore, we can choose the orthogonality of this basis as the boundary condition; in addition, we assume $\mathbf{n}^2=\eta^2=1$; here $\Gamma\xi=(\mathbf{M}\cdot\mathbf{n})\eta\times\mathbf{n}$. Starting out from these conditions, we find the following for the momentum flux:

$$g_{Mi}=\Gamma^{-1}(\mathbf{M}[\mathbf{n}\times(\nabla_i\mathbf{n})]+(\mathbf{n}\mathbf{M})\mathbf{n}[\boldsymbol{\eta}\times(\nabla_i\boldsymbol{\eta})]-{}^1/_2[\nabla\times\mathbf{M}]_i). \tag{30}$$

To change over to the Lagrangian, we must carry out an operation that is inverse to that given in the previous section. We construct $\tilde{L}=p\dot{q}-R$, taking it into account that now we must also include $(\mathbf{n}, \boldsymbol{\zeta})$, $(\boldsymbol{\eta}, \boldsymbol{\xi})$ in the set (q, p). Separating the total divergence, we get the Lagrangian (5), except that now

$$\tilde{\varepsilon}=\varepsilon-\mathbf{g}(\mathbf{v}_n-\mathbf{v}_s)-\mathbf{G}\frac{\partial\varepsilon}{\partial\mathbf{G}}-(\mathbf{h}-\mathbf{H})\mathbf{M}, \tag{31}$$

and $\boldsymbol{\zeta}$, $\boldsymbol{\xi}$ (which determine $\mathbf{M}$ in accord with (22)) are assumed to be expressed in terms of $\mathbf{n}$, η and their derivatives from (24), (25). The Lagrange equations for $(\mathbf{n}, \boldsymbol{\eta})$ are equivalent here to (26), (27).

Recognizing that the term g_M (23) is now included in the momentum density, we find

$$\mathbf{j}=\rho\nabla\alpha+s\nabla\beta+f\nabla\gamma-F_i\nabla\Phi_{1i}-G_i\nabla l_i-\zeta_i\nabla n_i -\xi_i\nabla\eta_i+{}^1/_2\operatorname{rot}([\mathbf{l}\times\mathbf{G}]+[\boldsymbol{\Phi}_1\times\mathbf{F}]-\Gamma^{-1}\mathbf{M}). \tag{32}$$

CONSERVATION LAWS

Before proceeding to consideration of He^3-A, we formulate some general premises that are valid for a system described by Hamilton's equations (4). Let S be the generator of the group acting on (q, p). Writing out this action on E and using (4), we find

$$S(E+p\dot{q})=\frac{\partial}{\partial t}(qSp)+\nabla\left(\frac{\partial E}{\partial\nabla p}Sp+\frac{\partial E}{\partial\nabla q}Sq\right). \tag{33}$$

Substituting $S=\partial/\partial t$, we obtain (assuming that the energy E does not depend explicitly on the time) the energy conservation law

$$\dot{E}+\nabla Q=0,\qquad Q=-\dot{q}\frac{\partial E}{\partial(\nabla q)}-\dot{p}\frac{\partial E}{\partial(\nabla p)}. \tag{34}$$

Substituting $S=\nabla$, we obtain (assuming that E does not depend explicitly on the spatial variables) the law of momentum conservation

$$\frac{\partial}{\partial t}(-p\nabla_i q)+\nabla_k\Pi_{ik}=0,$$
$$\Pi_{ik}=\delta_{ik}(p\dot{q}-E)+\frac{\partial E}{\partial\nabla_k q}\nabla_i q+\frac{\partial E}{\partial\nabla_k p}\nabla_i p. \tag{35}$$

The generator $\mathbf{J}$ of a group of rotations in space presents a special case of S, since it does not commute with the derivatives of ∇, as was tacitly assumed in the derivation of (33). Let $\mathbf{J}$ be the generator of rotations about that spatial point at which the values of q and p[1)] are taken, then E and $p\dot{q}$ should be invariant to this rotation (in the absence of an external field), and instead of (33) we obtain

$$e_{ikn}\Pi_{kn} = -\frac{\partial}{\partial t}(qJ_i p) - \nabla_k B_{ik},$$
$$B_{ik} = \frac{\partial E}{\partial \nabla_k p} J_i p + \frac{\partial E}{\partial \nabla_k q} J_i q. \quad (36)$$

Thus, we see that the antisymmetric part of the stress tensor Π_{ik} contains the term

$$-\frac{1}{2} e_{ikn} \frac{\partial}{\partial t}(qJ_n p),$$

which we use for a redefinition of the momentum density. Starting from (35)

$$\mathbf{I} = -p\nabla q - {}^1/_2 \operatorname{rot}(q\mathbf{J}p). \quad (37)$$

The divergence of the remaining term in the antisymmetric part of the stress tensor reduces to the divergence of the symmetric tensor. We finally obtain the law of momentum conservation with the symmetric stress tensor

$$\frac{\partial I_i}{\partial t} + \nabla_k \tilde{\Pi}_{ik} = 0,$$
$$\tilde{\Pi}_{ik} = {}^1/_2(\Pi_{ik} + \Pi_{ki}) - \frac{1}{2} e_{ikm} \nabla_m B_{ji} - \frac{1}{2} e_{jim} \nabla_m B_{jk}. \quad (38)$$

In connection with the symmetry of the stress tensor, we obtain the possibility of formulating the law of the conservation of angular momentum in the usual form:

$$\frac{\partial}{\partial t}(e_{ikn} x_k I_n) + \nabla_m (e_{ikn} x_k \tilde{\Pi}_{nm}) = 0. \quad (39)$$

Although our equations are given by the Routh function (19), the latter differs from E by a term which does not depend on the derivatives of (q, p) and which vanishes upon substitution of the equations of motion. Therefore, all the given formulas are applicable.

If we choose as S the generator of those rotations in spin space relative to which E_i is invariant upon neglect of the spit-orbital interaction, then, upon substitution in (33), we obtain the conservation law

$$\frac{\partial}{\partial t}\left(\frac{M_k}{\Gamma}\right) + \nabla_i \left(\left[\mathbf{n} \times \frac{\partial \epsilon}{\partial \nabla_i \mathbf{n}}\right]_k + \frac{v_{ni}}{\Gamma} M_m + \frac{1}{2\Gamma}(\delta_{ik}(\mathbf{v}_n \mathbf{M}) - v_{nk} M_i) \right)$$
$$+ [\mathbf{M} \times \mathbf{H}]_h = 0. \quad (40)$$

Equations (24) and (40), with neglect of terms with velocity $\mathbf{v}_n$, transform into the equations formulated in the linear approximation by Graham and Pleiner.

Substituting all the pairs of canonically conjugate variables that were introduced into consideration by us, we find

$$q\mathbf{J}p = [\mathbf{G} \times \mathbf{l}] - \frac{\hbar \rho \mathbf{l}}{2m} + \frac{1}{\Gamma}\mathbf{M}. \quad (41)$$

Substituting these pairs in (34), and using the equations of motion, we find

$$\mathbf{Q} = \mathbf{j}\left(\mu + \frac{1}{2} v_s^2\right) + \mathbf{v}_n \left(\mathbf{v}_n \mathbf{g} + Ts + \mathbf{G}\frac{\partial \varepsilon}{\partial \mathbf{G}} + (\mathbf{h} - \mathbf{H})\mathbf{M}\right)$$
$$+ \frac{1}{2}\mathbf{j}_0 \left(\operatorname{rot} \mathbf{v}_n \frac{\hbar \mathbf{l}}{2m}\right) - \dot{l}_i \frac{\partial \varepsilon}{\partial \nabla l_i} - \dot{n}_i \frac{\partial \varepsilon}{\partial \nabla n_i} -$$
$$-\frac{1}{2}\mathbf{v}_n \times \operatorname{div}[\mathbf{v}_n \times (q\mathbf{J}p)] - \frac{1}{2}\left[\mathbf{v}_n \times \frac{\partial}{\partial t}(q\mathbf{J}p)\right], \quad (42)$$

where $\mathbf{j}_0 = \mathbf{j} - \rho \mathbf{v}_n$.

Our expression for the current (32) differs from (37) by the total gradient, i.e., we can obtain the previous formulas by the redefinition

$$\Pi_{ik}' = \Pi_{ik} + \delta_{ik} \frac{\partial}{\partial t}(\rho\alpha + s\beta + f\gamma).$$

Substituting, we find from (35),

$$\Pi_{ik}' = \delta_{ik} P + j_k v_{si} + v_{nk} g_i + \nabla_i \mathbf{l} \frac{\partial \varepsilon}{\partial \nabla_k \mathbf{l}} + \nabla_i \mathbf{n} \frac{\partial \varepsilon}{\partial \nabla_k \mathbf{n}} + {}^1/_2 v_{nm} e_{kml} \nabla_i (qJ_l p)$$
$$+ {}^1/_2[\nabla \times (q\mathbf{J}p)]_i v_{nk} - {}^1/_2 \delta_{ik} \operatorname{div}[\mathbf{v}_n \times (q\mathbf{J}p)], \quad (43)$$

where the pressure is

$$P = \rho\mu + Ts + \mathbf{g}(\mathbf{v}_n - \mathbf{v}_s) + \mathbf{G}\frac{\partial \varepsilon}{\partial \mathbf{G}} + \mathbf{M}(\mathbf{h} - \mathbf{H}) - \varepsilon. \quad (44)$$

Finally, with account of the presence of the external field that which violates the conservation laws, we find

$$\frac{\partial}{\partial t} j_i + \nabla_k \pi_{ik} - \frac{1}{2}[\nabla \times [\mathbf{M} \times \mathbf{H}]]_i = 0. \quad (45)$$

Here π_{ik} is obtained from Π_{ik}' according to the rules (38), and

$$B_{ik} = \left[\mathbf{l} \times \frac{\partial \varepsilon}{\partial \nabla_k \mathbf{l}}\right]_i + \left[\mathbf{n} \times \frac{\partial \varepsilon}{\partial \nabla_k \mathbf{n}}\right]_i - \frac{\hbar l_i}{2m} j_{0k} + v_{nk} qJ_i p - \frac{1}{2}(\delta_{ik}(\mathbf{v}_n q\mathbf{J}p) - v_{ni} qJ_k p). \quad (46)$$

KINETIC COEFFICIENTS

The dissipative terms should be introduced as corrections to the obtained set of hydrodynamic equations. By virtue of the condition (15), the generalization (7) is achieved through the substitution $\mu \to \mu + z$, by adding $\mathbf{K}$ and $\mathbf{A}$ respectively to the expressions for the time derivatives of $\mathbf{G}$ and $\mathbf{M}$, and by adding to the tensor π_{ik} the dissipative contribution τ_{ik}. By virtue of the condition $l^2 = n^2 - 1$, the corrections to the derivative of $\mathbf{l}$ and $\mathbf{n}$ with respect to time should have the forms $\mathbf{u} \times \mathbf{l}$ and $\mathbf{y} \times \mathbf{n}$ respectively. Finally, we obtain the set of hydrodynamic equations

$$\frac{\partial}{\partial t}\rho + \nabla \mathbf{j} = 0, \quad (47)$$

$$\frac{\partial}{\partial t} j_i = -\nabla_k(\pi_{ik} + \tau_{ik}) + \frac{1}{2}[\nabla[\mathbf{M} \times \mathbf{H}]]_i, \quad (48)$$

$$\frac{\partial}{\partial t} v_{si} = -\nabla_i\left(\mu + \frac{1}{2} v_s^2 + \frac{1}{2} \operatorname{rot} \mathbf{v}_n \frac{\hbar \mathbf{l}}{2m} + z\right) + [(\nabla \times_i \mathbf{l}) \times \mathbf{l}]\frac{\hbar \dot{\mathbf{l}}}{2m}, \quad (49)$$

$$\frac{d}{d\tau}\mathbf{l} = \left[\left(\mathbf{\Omega} + \frac{\hbar\rho}{2m}\mathbf{u}\right) \times \mathbf{l}\right], \quad (50)$$

$$\frac{d}{dt}\mathbf{l} = \frac{\partial \varepsilon}{\partial \mathbf{G}} + [\mathbf{u} \times \mathbf{l}], \quad (51)$$

$$\frac{d}{dt}\mathbf{G} + \mathbf{G}(\nabla \mathbf{v}_n) = \mathbf{\Omega} - \frac{\partial \varepsilon}{\partial \mathbf{l}} + \nabla_i \frac{\partial \varepsilon}{\partial \nabla_i \mathbf{l}} + \mathbf{K}, \quad (52)$$

$$\frac{d}{dt}\mathbf{n} = \Gamma[\mathbf{n}(\mathbf{h} - \mathbf{H} - \mathbf{y}/\Gamma)], \quad (53)$$

$$\frac{d}{dt}\mathbf{M} + \mathbf{M}(\nabla \mathbf{v}_n) = \Gamma[[\mathbf{M} \times (\mathbf{h} - \mathbf{H})] + \Gamma[\psi \times \mathbf{n}] + \mathbf{A}. \quad (54)$$

The quantity $\mathbf{\Omega}$ is obtained from this same set of equations.

In the absence of a dependence of $\tilde{\varepsilon}$ on $d\mathbf{l}/dt$, we obtain (assuming $\mathbf{G}=\mathbf{K}=0$) for the orbital hydrodynamics a set of equations coinciding with the set Xo, [9] with the exception of the expression for the stress tensor.

For the entropy, the generalization of (10) takes the form

$$T\left(\frac{\partial}{\partial t}s+\nabla(s\mathbf{v}_n+\boldsymbol{\sigma})\right)=D, \tag{55}$$

where $\boldsymbol{\sigma}$ is the dissipative entropy flux, and D is the dissipation function, which determines the rate of increase of entropy. Requiring that the law of energy conservation have the form[13]

$$\dot{E}+\nabla(\mathbf{Q}+\mathbf{Q}')=0, \tag{56}$$

we find, separating the total divergence,

$$Q'=T\boldsymbol{\sigma}+\mathbf{j}_0 z+v_{ni}\tau_i+\frac{\partial\varepsilon}{\partial\nabla l_i}[\mathbf{l}\times\mathbf{u}]_i+\frac{\partial\varepsilon}{\partial\nabla n_i}[\mathbf{n}\times\mathbf{y}]_i, \tag{57}$$

$$D=-\boldsymbol{\sigma}\nabla T-z\nabla\mathbf{j}_0-\tau_{ik}w_{ik}-\mathbf{u}[\Xi\times\mathbf{l}]-\mathbf{y}[\boldsymbol{\psi}\times\mathbf{n}]-\mathbf{K}\frac{\partial\varepsilon}{\partial\mathbf{G}}-\mathbf{A}(\mathbf{h}-\mathbf{H}), \tag{58}$$

where

$$\Xi=-\frac{\partial\varepsilon}{\partial\mathbf{l}}+\nabla_i\frac{\partial\varepsilon}{\partial\nabla_i\mathbf{l}}+\left[\frac{\hbar\mathbf{l}}{2m}\times((\mathbf{j}_0\nabla)\mathbf{l})\right],$$

$$w_{ik}=\tfrac{1}{2}(\nabla_i v_{nk}+\nabla_k v_{ni}).$$

In Eq. (58), we have the sum of the products of the generalized forces and generalized flux densities. They should be connected with one another through the kinetic coefficients. We have at our disposal a triad of orthogonal axial vectors $\nu_{(1)}\propto\mathbf{l}+\mathbf{n}$, $\nu_{(2)}\propto\mathbf{l}-\mathbf{n}$, and $\nu_{(3)}\propto\mathbf{l}\times\mathbf{n}$ (we assume them to be normalized), with the help of which we can construct an arbitrary polar tensor of fourth rank and an arbitrary axial tensor of odd rank.

Thus, in the linear approximation, the generalized forces and flux densities are divided into two groups, in the first of which are $-\nabla T$ and $\boldsymbol{\sigma}$ (polar vectors), in the second ($-\nabla\mathbf{j}_0$, $-w_{ik}$, $[\mathbf{l}\times\Xi]$, $[\mathbf{n}\times\psi]$, $-\partial\varepsilon/\partial\mathbf{G}$, $\mathbf{H}-\mathbf{h}$) and (z, τ_{ik}, $\mathbf{u}$, $\mathbf{y}$, $\mathbf{K}$, $\mathbf{A}$) (scalars, pseudovectors and polar tensors of the second rank), and the generalized forces and flux densities are expressed in terms of one another only within these groups.

For the dissipative flux of the first group, we have

$$\begin{pmatrix}\sigma^D\nu_{(1)}\\ \sigma^D\nu_{(2)}\\ \sigma^D\nu_{(3)}\end{pmatrix}=-\begin{pmatrix}C_{11}&C_{12}&C_{13}\\ C_{21}&C_{22}&C_{23}\\ C_{31}&C_{32}&C_{33}\end{pmatrix}\begin{pmatrix}\nu_{(1)}\nabla T\\ \nu_{(2)}\nabla T\\ \nu_{(3)}\nabla T\end{pmatrix}. \tag{59}$$

The contribution to D due to $-\boldsymbol{\sigma}\cdot\nabla T$ is a quadratic form, constructed over $\nu_{(1)}\cdot\nabla T$, $\nu_{(2)}\cdot\nabla T$, $\nu_{(3)}\cdot\nabla T$ with the help of the matrix C. This quadratic form should be positive-definite, and the matrix C is symmetric by virtue of the Onsager symmetry. In addition to the dissipative part there is a reactive part of the flux density $\boldsymbol{\sigma}$ (the index on ν takes on the values 1, 2, 3 over which summation is implied):

$$\boldsymbol{\sigma}^r=b_{1n}[\nu_{(n)}\times(\nabla T)]. \tag{60}$$

Consideration of the second group is carried out similarly. To obtain the dissipative part of the flux density we should write out all the components of the generalized forces and flux densities[2)] belonging to this group in the basis $\nu_{(1)}$, $\nu_{(2)}$, $\nu_{(3)}$, which have the form of the corresponding products (for example, $\mathbf{u}$ generates $\nu_{(3)}\cdot\mathbf{u}^D$ and $[\nu_{(3)}\times\mathbf{l}]\cdot\mathbf{u}^D$). The contribution to D from this group has the form of the sum of pairwise products of components of forces and fluxes. Thus, if the matrix a interconnects the components of forces and fluxes, then the contribution to D is a quadratic form, constructed over the components of the generalized forces with the help of the matrix a. This quadratic form should be positive definite, and the matrix a is symmetric by virtue of the Onsager symmetry.

This matrix a determines the dissipative part of the flux densities; in addition, there are the reactive terms

$$\begin{gathered}\tilde{\tau}_{ik}{}^r=b_{2m}e_{ijn}\nu_{(m)j}w_{nk}+b_{3m}l_k[\nu_{(m)}(wl)]_i+\\ +b_{4m}n_k[\nu_{(m)}(wn)]_i+b_{5m}\nu_{(3)k}[\nu_{(m)}(w\nu_{(3)})]_i,\\ \mathbf{u}^r=b_6(\Xi-\mathbf{l}(\mathbf{l}\Xi))\ \text{[3)]},\qquad \mathbf{y}^r=b_7(\boldsymbol{\psi}-\mathbf{n}(\mathbf{n}\boldsymbol{\psi})),\\ \mathbf{K}^r=b_{8m}[\boldsymbol{\nu}_{(m)}\times\partial\varepsilon/\partial\mathbf{G}],\quad \mathbf{A}^r=b_{9m}[\boldsymbol{\nu}_{(m)}\times(\mathbf{h}-\mathbf{H})],\end{gathered} \tag{61}$$

τ_{ik} is obtained from $\tilde{\tau}_{ik}$ by symmetrization. We note that, generally speaking, all the kinetic coefficients depend on the angle between $\mathbf{l}$ and $\mathbf{n}$.

We now consider the case in which the spin effects can be neglected. We are left with only the single vector $\mathbf{l}$. However, as before, we can separate the group $\boldsymbol{\sigma}$, $-\nabla T$ for which

$$\boldsymbol{\sigma}=-\chi_1(\nabla T-\mathbf{l}(\mathbf{l}\nabla T))-\chi_2\mathbf{l}(\mathbf{l}\nabla T)-\varphi_1[\mathbf{l}\times(\nabla T)]. \tag{62}$$

The coefficients $\chi_1>0$ and $\chi_2>0$ have a dissipative character, while the coefficient φ_1 has a reactive character.

The second group contains scalars, pseudoscalars and tensors of second rank. However, with account of the presence of the vector $\mathbf{l}$ in the transformation plane, the pseudoscalars break up into scalars (projections on $\mathbf{l}$) and two-dimensional vectors orthogonal to $\mathbf{l}$ (thus, $\mathbf{K}=(\mathbf{l}\cdot\mathbf{K})\mathbf{l}+\mathbf{K}^{\perp}$). So far as the tensors of second rank are concerned, they break up as follows:

$$\begin{gathered}\tau_{ik}=\tau_1(\delta_{ik}-l_il_k)+\tau_2l_il_k+(\lambda_il_k+\lambda_kl_i)+\tau_{ik}{}^{\perp},\\ w_{ik}=\tfrac{1}{2}w_1(\delta_{ik}-l_il_k)+w_2l_il_k+\tfrac{1}{2}(\omega_il_k+\omega_kl_i)+w_{ik}{}^{\perp},\end{gathered} \tag{63}$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\omega}$ are vectors orthogonal to $\mathbf{l}$, $\tau_{ik}^{\perp}$ and $w_{ik}^{\perp}$ are also orthogonal to $\mathbf{l}$ (i.e., $l_k\tau_{ik}^{\perp}=l_kw_{ik}^{\perp}=0$) and irreducible ($\tau_{ii}^{\perp}=w_{ii}^{\perp}=0$). The inverse expressions (say, for w) take the form

$$w_1=(\delta_{ik}-l_il_k)w_{ik},\quad w_2=l_il_kw_{ik},\quad \omega_k=2(l_iw_{ik}-w_2l_k), \tag{64}$$

after which $w_{ik}^{\perp}$ is sought from (63).

The contribution to D from the second group divides into the sum of products of the generalized flux densities and the scalar forces (z, τ_1, τ_2, $\mathbf{l}\cdot\mathbf{K}$) and ($-\nabla\mathbf{j}_0$, $-w_1$, $-w_2$, $-\mathbf{l}\partial\varepsilon/\partial\mathbf{G}$), the vector ($\boldsymbol{\lambda}$, $\mathbf{u}$, $\mathbf{K}-(\mathbf{l}\cdot\mathbf{K})\mathbf{l}$) and ($-\omega$, $-[\Xi\times\mathbf{l}]$, $-\partial\varepsilon/\partial\mathbf{G}+\mathbf{l}(\mathbf{l}\partial\varepsilon/\partial\mathbf{G})$, and the tensor $\tau_{ik}^{\perp}$ and $-w_{ik}^{\perp}$. The set of scalar flux densities is expressed in terms of the set of scalar forces with the help of a symmetric matrix (according to Onsager) such that the quadratic form constructed on it is positive definite. So far as the tensor quantities are concerned, the following relation holds

$$\tau_{ik}^{\perp}=-\chi_3 w_{ik}^{\perp}-\varphi_2(e_{imn}l_m w_{nk}^{\perp}+e_{kmn}l_m w_{ni}^{\perp}), \tag{65}$$

in which the coefficient $\chi_3>0$ has a dissipative character and φ_2 a reactive one.

For the vectors, we get the following relations:

$$\begin{aligned}
\lambda &= -\chi_4\omega-\chi_5[\Xi\times\mathbf{l}]-\chi_6\left(\frac{\partial\varepsilon}{\partial\mathbf{G}}-\mathbf{l}\left(\mathbf{l}\,\frac{\partial\varepsilon}{\partial\mathbf{G}}\right)\right)\\
&\quad-\varphi_3[\mathbf{l}\times\omega]-\chi_7(\Xi-(\Xi\mathbf{l})\mathbf{l})-\chi_8\left[\mathbf{l}\times\frac{\partial\varepsilon}{\partial\mathbf{G}}\right],\\
\mathbf{u} &= -\chi_9\omega-\chi_{10}[\Xi\times\mathbf{l}]-\chi_{11}\left(\frac{\partial\varepsilon}{\partial\mathbf{G}}-\mathbf{l}\left(\frac{\partial\varepsilon}{\partial\mathbf{G}}\,\mathbf{l}\right)\right)-\chi_{12}[\mathbf{l}\times\omega]\\
&\quad-\varphi_4(\Xi-(\Xi\mathbf{l})\mathbf{l})-\chi_{13}\left[\mathbf{l}\times\frac{\partial\varepsilon}{\partial\mathbf{G}}\right],\\
\mathbf{K}-\mathbf{l}(\mathbf{l}\mathbf{K}) &= -\chi_{14}\omega-\chi_{15}[\Xi\times\mathbf{l}]-\chi_{16}\left(\frac{\partial\varepsilon}{\partial\mathbf{G}}-\mathbf{l}\left(\mathbf{l}\,\frac{\partial\varepsilon}{\partial\mathbf{G}}\right)\right)\\
&\quad-\chi_{17}[\mathbf{l}\times\omega]-\chi_{18}(\Xi-(\Xi\mathbf{l})\mathbf{l})-\varphi_5\left[\mathbf{l}\times\frac{\partial\varepsilon}{\partial\mathbf{G}}\right].
\end{aligned} \tag{66}$$

The coefficients $\varphi_3, \varphi_4, \varphi_5$ have a reactive character. So far as the dissipative parts of the generalized vector densities $(\lambda, \mathbf{u}, \mathbf{K}-\mathbf{l}(\mathbf{l}\cdot\mathbf{K}))$ are concerned, as is seen from (66), their components in the two-dimensional system of coordinates orthogonal to $\mathbf{l}$ are connected with the components of the generalized forces in this same system of coordinates $[-\omega, -\Xi\times\mathbf{l}, -\partial\varepsilon/\partial\mathbf{G}+\mathbf{l}(\mathbf{l}\cdot\partial\varepsilon/\partial\mathbf{G})]$ by means of the matrix

$$\begin{pmatrix}
\chi_4 & 0 & \chi_5 & -\chi_7 & \chi_6 & -\chi_8\\
0 & \chi_4 & \chi_7 & \chi_5 & \chi_8 & \chi_6\\
\chi_9 & -\chi_{12} & \chi_{10} & 0 & \chi_{11} & -\chi_{13}\\
\chi_{12} & \chi_9 & 0 & \chi_{10} & \chi_{13} & \chi_{11}\\
\chi_{14} & -\chi_{17} & \chi_{15} & -\chi_{18} & \chi_{16} & 0\\
\chi_{17} & \chi_{14} & \chi_{18} & \chi_{15} & 0 & \chi_{16}
\end{pmatrix}. \tag{67}$$

Correspondingly, according to Onsager, this matrix should be symmetric and the quadratic form constructed on it should be positive definite.

The situation in which the frequencies are such that the spin subsystem, because of the weakness of the spin-orbit interaction, can be considered independently, requires special consideration. (In particular, the law of conservation of **M** holds here, so that **A** must be a total divergence.) This case was considered by Graham and Pleiner.[12]

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[1]We note that here

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{v}_n\nabla-\frac{1}{2}(\operatorname{rot}\mathbf{v}_n)\mathbf{J}.$$

[2]It is necessary here to take it into account that $\mathbf{u}$, $\mathbf{y}$, $[\mathbf{l}\times\Xi]$, and $[\mathbf{n}\times\psi]$ have in fact two components each while τ_{ik} and w_{ik} have six.

[3]We note that this reactive term actually renormalizes Ω.

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