ics of the USSR and combines all fields of the physical sciences. It publishes experimental and theoretical work of general interest to physics, whereas more specialized material is now sent to the appropriate journal. Experience has shown that there really is a continuing need for a journal of this kind to cover all fields in physics, despite the trend towards narrow specialization seen in physics and in the other sciences. Table I contains data for 1975 and includes most journals of physics published by the Academy of Sciences of the USSR. We see that JETP is still the biggest journal of physics. We see also that whereas the average length of articles for most journals is 4–5 pages, for JETP this figure is considerably higher—about 10 pages.

Foreign policy plays a major role in the Soviet state and is geared towards détente and the prevention of war. International scientific cooperation and particularly the exchange of scientific information is part of that policy. For more than twenty years now this exchange has been facilitated by the fact that most of our journals of physics are translated into English and published in the USA. Accordingly, VINITI reproduces a large number of foreign journals. The last column in the table gives the number of English copies of journals published by the American Institute of Physics. Over the sixty years since the founding of the Soviet state, Soviet physics has explored many avenues and now enjoys a leading place in world science. We can be sure that in years to come Soviet science will remain equal to the tasks facing our country.

Translated by J. Mitchell

Distribution of stars in the neighborhood of a massive compact body

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The steady-state distribution of the stars in the neighborhood of a massive compact body (black hole, supermassive star, etc.) at the center of a spherically symmetric star cluster is investigated. A two-dimensional Fokker-Planck equation is used, without additional assumptions about the form of the distribution function, to find the flow rate (swallowing rate) of stars through the tidal radius r_t. In the region between r_t and the critical radius $r_{\rm crit}$, in which diffusion of the star orbits to low angular momentum is important, the spatial density of the stars varies as $n(r)\propto r^{-1/2}$, and the distribution function deviates strongly from the power-law from $f\propto|E|^p$ with $p=\text{const}$ considered previously in the literature. A power-law section of the distribution function with $p=1/4$ is restricted to the region in which diffusion of the star orbits to different energies is decisive, with the result that the $n(r)\propto r^{-7/4}$ cusp in the star density corresponding to this function must be restricted to the region $r_{\rm crit}<r<r_h$, where r_h is the capture radius of the massive body. Some astrophysical consequences of the star distribution are discussed. In particular, it is pointed out that in the globular clusters with x-ray bursters (taken by many to be massive black holes with $M\sim 10^2$–$10^3 M_\odot$) the excess star density to be expected around the black hole will be only very slight.

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1. INTRODUCTION

The possible existence in nature of massive ($M>10^2 M_\odot$) compact bodies—whether collapsed (black holes) or not (supermassive stars, magnetoids)—poses serious problems for theory in connection with the detection of such objects. As they will be formed with greatest probability at the centers of spherical star systems (nuclei of galaxies, globular clusters), it is of considerable interest to investigate the interaction of such bodies with the surrounding stars.

The gravitational field of a massive body can significantly influence the motion of the surrounding stars if their orbits approach closer than the distance known as the capture radius r_h of the hole:

$$r_h=\frac{2GM}{\langle v^2\rangle}\approx 2.7\cdot 10^{17}\left(\frac{M}{10^3 M_\odot}\right)\left(\frac{\langle v^2\rangle^{1/2}}{10\,\text{km/sec}}\right)^{-2}\ [\text{cm}],\qquad (1)$$

where $\langle v^2\rangle^{1/2}$ is the velocity dispersion of the stars in the cluster. If a star approaches sufficiently close to the massive body, the consequence is even more drastic: The star may be disrupted by the tidal forces. For a star with mean density ρ, the radius at which tidal forces are sufficient for its disruption (tidal radius) is given by an expression that was already found by Jeans:

$$r_t\approx\left(\frac{6}{\pi}\frac{M}{\rho}\right)^{1/3}\approx 1.4\cdot 10^{12}\left(\frac{M}{10^3 M_\odot}\right)^{1/3}\left(\frac{\rho}{\rho_\odot}\right)^{-1/3}\ [\text{cm}].\qquad (2)$$

The tidal radius is small compared with r_h but for a star like the Sun it is still much greater than the gravitational radius of the black hole $r_g=2GM/c^2=3.10^8(M/10^3 M_\odot)$ cm if $M<10^8 M_\odot$, or the radius of stabilized rotation of a supermassive star $R\approx 10^{16}(M/10^8 M_\odot)$ cm if $M<5\cdot 10^4 M_\odot$. Under these circumstances, the compact object does not swallow the star in a single gulp, but disrupts it by tidal

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forces. In the case of a massive black hole, the disruption of stars, and also the accretion of gas liberated by the disruption are especially interesting since they may be observed and used to detect black holes.

In the present paper, we concentrate our attention on a different important aspect—the influence of the compact massive body on the distribution of the stars surrounding it.[1–5] We assume that the body (which for brevity we shall henceforth call a black hole) is at the center of a spherically symmetric cluster containing a large number of stars (in globular clusters $N\sim 10^5-10^6$, in the nucleus of a galaxy like ours, $N\sim 10^7-10^8$). The mass of the black hole is usually assumed to be much less than the mass of the complete cluster: $m \ll M \ll NM$, where m is the mass of an individual star (for simplicity, we assume that the stars are identical and have mass $m \approx M$). Peebles,[1] who was the first to consider the distribution of stars around a black hole, assumed that the distribution function of the stars bound by the black hole, i.e., the stars for which the energy

$$E=mv^2/2-GMm/r \tag{3}$$

is negative, has a power-law form: $f \propto |E|^p$, $p=\text{const}$. From the condition that the flow rate of the stars is independent of the energy E, he found $p=\frac{3}{4}$. This leads to an $n(r)\propto r^{-9/4}$ cusp in the spatial density of the stars near the black hole. A more rigorous treatment[2,5] using a one-dimensional Fokker-Planck equation to describe the diffusion of the star orbits to different energies showed that a power-law distribution function with $p=\frac{3}{4}$ is an unacceptable solution because, although the flow rate is independent of the energy E, it implies a very rapid rate of diffusion of stars away from the black hole. In[2,5], the exponent $p=\frac{1}{4}$ was obtained by requiring formally a vanishing flow rate of stars into the sphere of radius r_t. In this case, the density of stars near the black hole is given by $n(r)\propto r^{-7/4}$. It is interesting to note that as early as 1964 Gurevich[6] obtained an analogous solution for the distribution of electrons in the neighborhood of a positively charged Coulomb center.

In reality, the flow rate of stars through the sphere of radius r_t is finite, which leads to a nonequilibrium distribution function. The flow rate is governed by the diffusion of the star orbits to different angular momenta,[4,5] i.e., it is related to anisotropy of the distribution function. In[4], Frank and Rees found the flow rate on the basis of the qualitative analogy with heat conduction in a hemispherical shell; in[5] it was found by analyzing a two-dimensional Fokker-Planck equation derived under the assumption that the energy dependence of the distribution function has a power-law form.

In the present paper, the steady-state distribution of the stars around a black hole is analyzed by means of the Fokker-Planck equation, but in deriving the basic equation we do not assume a power-law distribution function, which turns out to be very important. The paper is divided up as follows. In Sec. 2, we derive the Boltzmann equation for the distribution of the stars with allowance for their two-body interactions. In Sec. 3, we analyze the steady-state solution of this equation neglecting the anisotropy of the distribution function and, therefore, the diffusion of the star orbits to different angular momenta. We find the corrections to the above power-law solution for the distribution function associated with the finite rate of flow of stars through the sphere defined by the tidal radius. In Sec. 4, the Boltzmann equation with allowance for the anisotropy of the distribution function is derived. It enables one to find the steady-state flow rate of stars through the tidal radius resulting from the diffusion of the star orbits to lower angular momenta. It is shown that this diffusion leads to an appreciable deviation of the distribution function from the power-law dependence $f\propto |E|^{1/4}$ for $E<E_{\rm crit}$ and to an effective cutoff of $f(E)$ at $E\approx LE_{\rm crit}$, where

$$E_{\rm crit}=-GMm/2r_{\rm crit},\quad L=\ln(r_{\rm crit}/r_t)^{1/2},$$

and $r_{\rm crit}$ is the radius of the sphere within which the diffusion of the star orbits to different angular momenta becomes important. In Sec. 5, we consider the spatial distribution of the star density around a black hole. We show that in the case $r_{\rm crit}<r_h$ the distribution of the bound stars in the region $r<r_{\rm crit}$ is given by $n(r)\propto r^{-1/2}$, so that the $n(r)\propto r^{-7/4}$ cusp obtained for the star density without allowance for angular momentum diffusion is restricted to the region $r_{\rm crit}<r<r_h$. If $r_{\rm crit}\gtrsim r_h$, there is no such cusp at all around a black hole. In Sec. 6, we discuss the results and some of their applications.

2. THE BOLTZMANN EQUATION

The distribution function of the stars is described by the Boltzmann equation[7]

$$\frac{\partial f}{\partial t}+\{H,f\}=-\frac{\partial}{\partial p^i}j^i, \tag{4}$$

where

$$j^i=2\pi(Gm^2)^2\Lambda\int d^3p'w^{ik}\left(\frac{\partial f}{\partial p'^k}f-\frac{\partial f}{\partial p^k}f'\right), \tag{5}$$

$$w^{ik}=\frac{1}{u^3}(u^2\delta^{ik}-u^iu^k); \tag{6}$$

$u^i=m^{-1}(p'^i-p^i)$ is the relative velocity of two stars, and $\Lambda=\ln(N/2)$ is the gravitational Coulomb logarithm. The collisional term in Eq. (4) takes into account only two-body interactions (the Fokker-Planck approximation). The Coulomb logarithm Λ appears in (5) because it is the distant encounters, in which the star trajectories change only slightly, that are decisive for stars interacting through the Coulomb law. The orbit of an individual star is changed appreciably over the relaxation time T_R (the characteristic time of variation of the star's angular momentum), which is given by[8]

$$T_R=\frac{2^{1/2}v_c^3}{\pi G^2m^2n_c\Lambda}, \tag{7}$$

where $v_c=(\frac{1}{3}\langle v^2\rangle)^{1/2}$ is the velocity dispersion of the stars in the cluster along the line of sight and n_c is the star density in the central part of the cluster (the core). The mean free path of stars in the cluster, $l\sim v_cT_R$, is much greater than the capture radius r_h given by Eq. (1), namely[2]

$$l/r_h \sim (Nm)^2/mM \gg 1,$$

so that the distribution function f of the stars, which in the general case is a function of seven variables, can in a first approximation be assumed to depend on only the energy E, the angular momentum J, and the time. We shall assume that the distribution function of the stars for $E>0$ is Maxwellian, i.e.

$$f(E)=n_c(2\pi m^2 v_c^2)^{-3/2}\exp(-E/mv_c^2). \tag{8}$$

The problem of finding the function $f(E, J, t)$ thus reduces to solving Eq. (4) in the energy region $E<0$, i.e., for the stars that are bound by the black hole.

3. DIFFUSION OF THE STAR ORBITS TO DIFFERENT ENERGIES

In the steady-state case, Eq. (4) under neglect of the anisotropy of the distribution function (i.e., the dependence of f on J) reduces to the following equation, which was obtained for the first time in[6] and later independently in[2]:

$$i=\int_{-\infty}^{\infty} dx'\left[\frac{dy(x')}{dx'}y(x)-\frac{dy(x)}{dx}y(x')\right]\max(x,x')^{-5/2}\quad (x>0), \tag{9}$$

where the dimensionless variables x, y, and i are defined by

$$x=-E/mv_c^2,\quad y=(2\pi m^2 v_c^2)^{3/2} n_c^{-1} f(E), \tag{10}$$

$$x_0=-E_0/mv_c^2,\quad E_0=-GMm/2r_t, \tag{11}$$

$$\dot{N}_{\text{bnd}}=i\tilde{N}, \tag{12}$$

$$\tilde{N}=\frac{9}{\sqrt{2}}\pi^2 r_h^3 (Gm)^2 n_c^2 v_c^{-3}\Lambda. \tag{13}$$

Here, $\dot{N}_{\text{bnd}}$ is the rate of flow of the bound stars through the sphere defined by the tidal radius r_t. Stars that pass within the tidal radius are disrupted there, so that we can take $y(x_0)=0$ as a boundary condition for Eq. (9). Equation (9) describes the diffusion of the star orbits to different energies in the field of the black hole resulting from their two-body interactions, and this leads to a nonvanishing swallowing rate $\dot{N}_{\text{bnd}}$ by the hole.

Under the formal condition $i=0$, the approximate solution of the equation in the energy range $1 \ll x \ll x_0$ has the simple power-law form[6]

$$y=Cx^{1/4},\quad C\approx 1.53. \tag{14}$$

Then the spatial density of the stars determined from the dimensionless distribution function (14) has the radial dependence $n(r) \propto r^{-7/4}$ ([2,6]).

But in reality the swallowing rate is finite, which leads to a deviation of the distribution function from the power law $y(x) \propto x^{1/4}$ and a corresponding restriction on the applicability of the solution (14). This must then change the dependence $n(r)$.

We find the deviation of the solution of Eq. (9) in the region $1 \ll x \ll x_0$ from the solution (14) with allowance for the finite flow rate, i.e., $i \neq 0$. Differentiating thrice we can reduce Eq. (9) to a differential equation whose solution under the condition $ix \ll 1$ can, as can be shown, be represented in the form

$$y=Cx^{1/4}\left(1+\alpha_1\frac{i}{C^2}x+\alpha_2\frac{i^2}{C^4}x^2+\ldots\right). \tag{15}$$

The coefficients α_1, $\alpha_2, \ldots$ are found by substitution and are

$$\alpha_1=\frac{45}{64},\quad \alpha_2=\left(\frac{3}{4}\right)^2\left(\frac{5}{8}\right)^2\frac{91}{41}\approx 0.3. \tag{16}$$

The flow rate $\dot{N}_{\text{bnd}}$ (respectively i) is determined, as we show below, by the diffusion of the star orbits to lower angular momenta. To find the flow rate, we must have recourse to the Boltzmann equation that takes into account the anisotropy of the distribution function.

4. DIFFUSION OF STAR ORBITS TO DIFFERENT ANGULAR MOMENTA

The disruption of stars that pass within the tidal radius leads to the appearance in the velocity space of a loss cone,[4] with semi-vertex angle

$$\sin\theta_{lc}=\frac{r_t}{r}\left[1+\frac{2GM}{r_t v^2(r)}\right]^{1/2}. \tag{17}$$

Near the tidal radius, there are no stars within the loss cone, but with increasing distance from the black hole the boundaries of the loss cone are gradually smeared because of the diffusion of the star orbits between different angular momenta, so that outside a certain r_{crit} the loss cone is completely filled in with stars. The radius r_{crit} within which diffusion of the star orbits does not replenish the loss cone with stars is, in the case $r_{\text{crit}}<r_h$, approximately equal to[4]

$$r_{\text{crit}}/r_h \approx (T_R v_c r_t/r_h^2)^{4/9}. \tag{18}$$

The distribution function of the stars with $E<E_{\text{crit}}$ ($E_{\text{crit}}=-GMm/2r_{\text{crit}}$) becomes anisotropic since there are no stars whose orbits lie within the loss cone (17) or, which is the same thing, stars whose angular momenta satisfy $J<J_0$, where

$$J_0 \approx m(2GMr_t)^{1/2}. \tag{19}$$

We now find the equation which describes the diffusion of the star orbits between angular momenta for $E<E_{\text{crit}}$. We proceed from Eq. (4), in which, recalling that $f=f(E, J, t)$, we go over to the variables $P^i=(E, J, J_z)$, i.e., to integrals of the motion. This transition can be made conveniently by taking as intermediate variables the action variables I^α and the canonically conjugate angle variables ϑ^α ($\alpha=1, 2, 3$) ([9]). For the gravitational field with potential $U=-GM/r$, the action variables, as in the Coulomb case, are

$$I^1=GMm(m/2|E|)^{1/2}-J,\quad I^2=J-J_z,\quad I^3=J_z. \tag{20}$$

We shall regard the flux j^β defined by Eq. (5) formally as a six-dimensional vector ($\beta=1, 2, \ldots, 6$), assuming that the spatial components of the flux are zero. Then Eq. (4) in these canonical variables takes the form

$$\frac{\partial f}{\partial t}+\frac{\partial}{\partial \vartheta^\alpha}\left(\frac{\partial H}{\partial I^\alpha}f\right)-\frac{\partial}{\partial I^\alpha}\left(\frac{\partial H}{\partial \vartheta^\alpha}f\right)=-\frac{\partial j_\alpha^{(I)}}{\partial I^\alpha}-\frac{\partial j_\alpha^{(\vartheta)}}{\partial \vartheta^\alpha}. \tag{21}$$

Averaging Eq. (21) with respect to the periodic variables ϑ^α and then going over from the action variables to the integrals of the motion P^i, we obtain the equation

$$\frac{\partial f}{\partial t}=-2\pi(Gm^2)^2\Lambda\frac{1}{g^{1/2}}\frac{\partial}{\partial P^i}\left[g^{1/2}\left(R_1{}^i f-R_2{}^{ik}\frac{\partial f}{\partial P^k}\right)\right], \tag{22}$$

where

$$g^{1/2}=\frac{\partial(I^1,I^2,I^3)}{\partial(E,J,J_z)}=GM\left(\frac{m}{2|E|}\right)^{3/2}, \tag{23}$$

$$R_1{}^i=\frac{1}{(2\pi)^3}\int d^3\vartheta\frac{\partial P^i}{\partial p^l}\int d^3p'w^{lm}\frac{\partial f'}{\partial p'^m}, \tag{24}$$

$$R_2{}^{ik}=\frac{1}{(2\pi)^3}\int d^3\vartheta\frac{\partial P^i}{\partial p^l}\frac{\partial P^k}{\partial p^m}\int d^3p'\,w^{lm}f'. \tag{25}$$

As one of the boundary conditions for Eq. (22), we choose

$$f(E,J_0)=0, \tag{26}$$

(where J_0 is determined by the expression (19)), which corresponds to disruption (swallowing) of stars which stray into the loss cone.

In the range of energies $E<E_{\text{crit}}$ the characteristic time τ_J of diffusion of the star orbits through the angular momenta until they pass within the tidal radius (we shall show below that $\tau_J\sim LT_R$) is much shorter than the characteristic time of diffusion of the star orbits between different energies, which is $\tau_E\sim T_R r_{\text{crit}}/r_t$ ([4]). Therefore, in this range of energies, the main contribution to the flow of stars through the tidal sphere comes about by the diffusion of the star orbits to lower angular momenta. From Eq. (22), we find that the flow rate is

$$\dot N_{\text{bnd}}=-(2\pi)^4(Gm^2)^2\Lambda\int\limits_{E_{\min}}^{E_{\text{crit}}}g^{1/2}\left[R_1{}^2f-R_2{}^{12}\frac{\partial f}{\partial E}-R_2{}^{22}\frac{\partial f}{\partial J}\right]2J\,dE. \tag{27}$$

In the steady-state case, $\dot N_{\text{bnd}}$ is constant, and therefore the integrand in (27) does not depend on J. Further, by virtue of the boundary condition (26), $f\to 0$ as $J\to J_0$, and in the expression (27) at angular momenta J near J_0 we can ignore the contribution of the first two terms compared with the third. Neglecting them, we obtain

$$\dot N_{\text{bnd}}=2(2\pi)^4(Gm^2)^2\Lambda\int\limits_{E_{\min}}^{E_{\text{crit}}}g^{1/2}R_2{}^{22}J\frac{\partial f}{\partial J}dE. \tag{28}$$

The coefficient R_2^{22} in (28) is calculated in the Appendix, and for $J_0<J\ll J_{\max}(E)$, where

$$J_{\max}(E)=GMm(m/2|E|)^{1/2}, \tag{29}$$

it is given by (see A.9)

$$R_2{}^{22}=\frac{5\pi}{3}m^2\left(\frac{GMm}{|E|}\right)^2\int\limits_E^\infty f(E')\,dE'. \tag{30}$$

From (28), (30), the condition that the flow rate be independent of the angular momentum, and with allowance for the boundary condition (26), we now find that the distribution function for $J_0<J\ll J_{\max}(E)$ has the form

$$f(E,J)=\Phi(E)\ln(J/J_0). \tag{31}$$

By virtue of the weak (logarithmic) dependence of the distribution function on the angular momentum, to calculate $\dot N_{\text{bnd}}$ we shall assume that (31) is also true for large angular momenta. Then (31) can be represented in the form

$$f(E,J)=f(E)L^{-1}\ln(J/J_0), \tag{32}$$

where

$$L=\ln[J_{\max}(E)/J_0], \tag{33}$$

and $f(E)$ is the isotropic part of the distribution function.

Substituting (23), (30), and (32) in (28) and going over to the dimensionless variables, we obtain

$$\dot N_{\text{bnd}}=\dot N\frac{5}{4}\int\limits_{x_{\text{crit}}}^{x_{\max}}\frac{1}{L}\frac{1}{x^{7/2}}y(x)\,dx\int\limits_{-\infty}^{x}y(x')\,dx'. \tag{34}$$

The dependence of the distribution function $y(x)$ in the region $x>x_{\text{crit}}$ must be determined by solving Eq. (22), but because the denominator of the integrand in (34) contains the $x^{7/2}$ the main contribution to the integral is made by the region near x_{crit}. Therefore, to estimate the flow rate, we can use the distribution function (14) obtained from the solution of the problem of the energy diffusion of the star orbits. Substituting (14) in (34), we obtain

$$\dot N_{\text{bnd}}\approx\frac{C^2}{Lx_{\text{crit}}}\dot N. \tag{35}$$

Here, $L=\frac{1}{2}\ln(r_{\text{crit}}/r_t)$. Using (13) and (14), we find the numerical value of the flow rate:

$$\dot N_{\text{bnd}}\approx 0.7\cdot 10^{-7}\left(\frac{M}{10^3M_\odot}\right)^3\left(\frac{n_c}{5\cdot 10^4\ \text{pc}^{-3}}\right)^{-3/2}\left(\frac{r_c}{1\ \text{pc}}\right)^{-9}\cdot\times\left(\frac{\Lambda}{10}\right)\left(\frac{L}{5}\right)^{-1}\left(\frac{r_{\text{crit}}}{r_h}\right)\ \text{yr}^{-1}, \tag{36}$$

where r_c is the radius of the cluster core. The radius r_{crit} calculated in accordance with (18) is

$$\frac{r_{\text{crit}}}{r_h}=0.9\left(\frac{M}{10^3M_\odot}\right)^{-20/27}\left(\frac{r_c}{1\ \text{pc}}\right)^{32/9}\left(\frac{n_c}{5\cdot 10^4\ \text{pc}^{-3}}\right)^{4/3}\left(\frac{\Lambda}{10}\right)^{-4/9}. \tag{37}$$

Substituting (37) in (36), we finally obtain

$$\dot N_{\text{bnd}}\approx 0.7\cdot 10^{-7}\left(\frac{M}{10^3M_\odot}\right)^{61/27}\left(\frac{n_c}{5\cdot 10^4\ \text{pc}^{-3}}\right)^{-7/6}\left(\frac{r_c}{1\ \text{pc}}\right)^{49/9}\cdot\times\left(\frac{L}{5}\right)^{-1}\left(\frac{\Lambda}{10}\right)^{5/9}\text{yr}^{-1}\approx 2\cdot 10^{-7}\left(\frac{M}{10^3M_\odot}\right)^{61/27}\left(\frac{n_c}{5\cdot 10^4\ \text{pc}^{-3}}\right)^{14/9}\cdot\left(\frac{v_0}{10^6\ \text{cm/sec}}\right)^{-49/9}\left(\frac{L}{5}\right)^{-1}\left(\frac{\Lambda}{10}\right)^{5/9}\text{yr}^{-1}. \tag{38}$$

This expression for the flow rate (38) coincides to within a numerical factor of order 2 with the value obtained by Bahcall and Wolf[3] on the basis of numerical calculations; it is approximately five times greater than the flow rate obtained by Lightman and Shapiro,[5] who assumed a power-law form of the distribution function when deriving the Fokker–Planck equation; and it is almost 30 times greater than Frank and Rees's estimate.[4]

Comparing (35) and (12), we find

$$i=\frac{C^2}{L}\frac{1}{x_{crit}}\ll 1, \quad (39)$$

so that, as can be seen from (15), the simple power-law solution $f(x)\propto x^{1/4}$ for the distribution function holds only in the region $1\ll x\ll x_{crit}$.

The flow rate $\dot{N}_{bnd}$ given by Eq. (38) can be written in the form

$$\dot{N}_{bnd}\approx N(r_{crit})/LT_R(r_{crit}), \quad (40)$$

where $N(r_{crit})$ is the total number of stars below the critical radius, and $T_R(r_{crit})$ is the relaxation time near r_{crit}. This means that in a time $\tau_J\sim LT_R$ all stars that at a given time have energies $E<E_{crit}$ come within the tidal radius and are then disrupted. In other words, all stars whose orbits are bounded by the radius r_{crit} are replenished in the time τ_J. On the other hand, during this same time the energy of an individual star is reduced as a result of energy diffusion to the value $E_{min}\sim LE_{crit}$. The disruption of stars within the tidal radius resulting from their angular momentum diffusion restricts the time stars remain in the energy range $E<E_{crit}$ to τ_J and leads to an effective cutoff of the distribution of the stars at $E_{min}\sim LE_{crit}$ (Fig. 1). The value of E_{min} differs from E_{crit} by only a few times. For example, for globular clusters $L\sim 5$. In the case of the stars with energies $E_{crit}<E<0$, they will accumulate near the black hole by energy diffusion and form a density cusp.

All the foregoing applied to the case $r_{crit}\ll r_h$. But if $r_{crit}>r_h$, then as a result of the angular momentum diffusion of the stars right down to tidal disruption, which takes place appreciably faster than the energy diffusion, stars will not accumulate at all on orbits at a finite distance from the black hole. In this case, the flow rate of stars through the tidal sphere will be determined by the unbound stars. If they have a Maxwellian distribution function, their flow rate is

$$\dot{N}_{unb}=2(2\pi)^{1/2}r_t^2v_cn_c\left(1+\frac{3}{2}\frac{r_h}{r_t}\right)\approx 3(2\pi)^{1/2}(r_tr_h)n_cv_c$$

$$\approx 4.8\cdot 10^{-8}\left(\frac{M}{10^3M_\odot}\right)^{4/3}\left(\frac{n_c}{10^4\ \text{pc}^{-3}}\right)^{1/3}\left(\frac{r_c}{1\ \text{pc}}\right)^{-1}\ \text{yr}^{-1}, \quad (41)$$

which agrees with the expression found in[10]. It follows from (38) and (41) that in the case $r_{crit}\ll r_h$

$$\dot{N}_{bnd}/\dot{N}_{unb}\approx(r_h/r_{crit})^{5/4}\gg 1, \quad (42)$$

i.e., the flow of the unbound stars is much less than that of the bound ones.

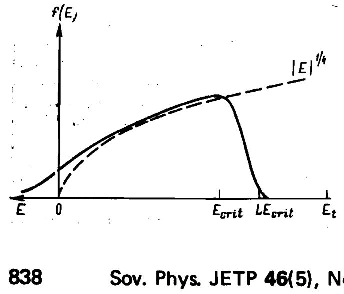

FIG. 1. Distribution function of the stars in the gravitational field of a massive black hole.

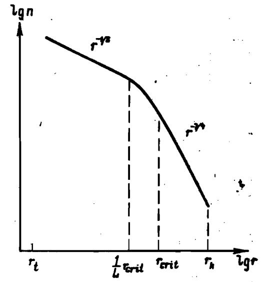

FIG. 2. Distribution of the spatial density of stars near a massive black hole in the case $r_{crit}\ll r_h$.

5. DISTRIBUTION OF THE SPATIAL DENSITY OF STARS

The spatial density of the stars around a massive black hole in the region $r\ll r_h$ when $r_{crit}\ll r_h$ found from the distribution function (14) is[2,6]

$$n_{bnd}(r)\approx 1.8n_c(r_h/r)^{7/4}. \quad (43)$$

This cusp in the density of the stars is due to their accumulation in bound orbits. Allowance for the cutoff of the distribution function at $E_{min}\sim LE_{crit}$ has the consequence that the density increases in accordance with the law $n(r)\propto r^{-7/4}$ only as far as the radius r_{crit}. In the region $r<r\ll L^{-1}r_{crit}$ the radial dependence of the density becomes weaker. Namely, using the distribution function (32) and ignoring its anisotropy (which leads to an additional logarithmic dependence on r) we obtain for the density of bound stars

$$n_{bnd}(r)\propto\int_0^{|E_{min}|}f(E)\,[GMm/r-|E|]^{1/2}\,dE\propto r^{-1/2}. \quad (44)$$

The distribution of the bound stars near a massive black hole is shown in Fig. 2.

If $r_{crit}\gtrsim r_h$, the density of the bound stars has no cusp at all because of their rapid consumption resulting from the diffusion of the star orbits to lower angular momenta.[4] In this case, the density of stars around the black hole is completely determined by the unbound stars, whose density for $r\ll r_h$ increases towards the center in accordance with the law $n_{unb}(r)\propto r^{-1/2}$, as follows from the theory of spherical accretion.[11]

Thus, if there is a point mass at the center of a star cluster, then the density of the stars around it, or rather, in the region $r\ll r_h$, $r\ll r_{crit}$, always varies in accordance with the law $n(r)\propto r^{-1/2}$.1) Accordingly, the surface density $\sigma(s)$ of stars as a function of the distance s from the center of the cluster under these conditions is

$$1-\sigma(s)/\sigma(0)\propto s^{1/2}. \quad (45)$$

6. DISCUSSION

The finite flow rate of stars through the tidal radius r_t leads to an essentially nonequilibrium distribution of the stars around a massive black hole. The approximate power-law form $f\propto|E|^{1/4}$ of the distribution function can be obtained by assuming formally that the flow rate is zero.[2,5,6] As we have shown above, the finite flow rate has the consequence that this approximation is valid only in the energy range $E_{crit} \ll E \ll -mv_c^2$. In the region $E < E_{crit}$, in which angular momentum diffusion of the star orbits is decisive, the departure from the dependence $f \propto |E|^{1/4}$ is appreciable, and the distribution function is no longer described by a power law. Therefore, in this range of energies the procedure for finding the distribution function as a power law from the condition of formal vanishing of the star flow rate[5] is inconsistent and does not give the correct energy dependence.

The exact form of the distribution function $f(E, J)$ for $E < E_{crit}$ must be determined by solving the two-dimensional Fokker–Planck equation. However, the flow rate of the stars into the tidal sphere in the case $r_{crit} \ll r_h$ is determined basically, as can be seen from (34), by the value of the distribution function at the energy $E \sim E_{crit}$, which is the boundary of the region of energy diffusion of the star orbits. This flow rate can be found to within a numerical factor of order unity by means of the solution (14) for the distribution function in this region. It is found to be appreciably greater than the flow rate of the unbound stars. Since the number of stars in the region bounded by the radius r_{crit} when $r_{crit} \ll r_h$ is comparatively small, the diffusion approximation in which the flow rate is calculated ceases to be good and a more accurate estimate of the flow rate in this approximation is not worthwhile.

The angular momentum diffusion of the star orbits and the disruption of the stars when they stray within the tidal tidal radius lead to an effective cutoff of the distribution function at the energy $E_{min} \sim LE_{crit}$. In its turn, this leads to a radial dependence of the star density in the region $r_t < r \ll r_{crit}$ of the form $n(r) \propto r^{-1/2}$. Thus, the $n(r) \propto r^{-7/4}$ density cusp is restricted to the region $r_{crit} \ll r \ll r_h$, and becomes flatter for $r < r_{crit}$. In the case $r_{crit} \gtrsim r_h$, there is no such density peak in the stars at all since they do not accumulate on bound orbits. In this case, the flow rate of stars through the tidal sphere is determined by the unbound stars [see (41)].

In real globular clusters with characteristic parameters $n_c \sim 5 \cdot 10^4$ pc^{-3} and $r_c \sim 1$ pc, it follows from (37) that the condition $r_{crit} \ll r_h$ can be satisfied only if $M \gg 10^3 M_\odot$. But the interpretation of the x-ray bursters in globular clusters as black holes leads to the estimate $M \sim 10^2$–$10^3 M_\odot$ of the black hole mass.[12] Thus, even if an $n(r) \propto r^{-7/4}$ cusp is present, it will be weak.[13] This circumstance will make it much harder to find massive black holes in globular clusters on the basis of an excess in the star density in the central parts of the clusters.

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APPENDIX

Calculation of the Coefficient R_2^{22} in Eq. (28)

In accordance with (25),

$$R_2^{22} = \frac{1}{(2\pi)^3} \int d^3\vartheta \frac{\partial J}{\partial p^l} \frac{\partial J}{\partial p^m} \int d^3p' \, w^{lm} f'. \tag{A.1}$$

We find first the tensor

$$W^{lm} = \int d^3p' \, w^{lm} f', \tag{A.2}$$

where w^{lm} are given by the expression (6), in the integrand. We make the calculations in a spherical coordinate system (the z axis is along the momentum $\mathbf{p}$) in which W^{lm} is diagonal. We ignore the anisotropy of f' since W^{lm} depends on f' integrally. Integrating with respect to φ and θ, we obtain

$$W^{11} = W^{22} = a = \frac{8\pi m}{3} \int f' p'^2 \, dp' \cdot \begin{cases} (p')^{-1}, & p' > p \\ \dfrac{3}{2} \dfrac{1}{p} \left(1 - \dfrac{p'^2}{3p^2}\right), & p' < p \end{cases} \tag{A.3}$$

$$W^{33} = a - b = \frac{8\pi m}{3} \int f' p'^2 dp' \cdot \begin{cases} (p')^{-1}, & p' > p \\ (p')^2 p^{-3}, & p' < p \end{cases}. \tag{A.4}$$

In an arbitrary coordinate system, W^{lm} can then be written in the form

$$W^{lm} = a\delta^{lm} - bp^l p^m / p^2. \tag{A.5}$$

Substituting this expression in (A.1) and going over from integration with respect to p' to integration with respect to the energy E', we obtain

$$R_2^{22} = \frac{1}{(2\pi)^3} \int d^3\vartheta \, r^2 \left(a - b \frac{J^2}{p^2 r^2}\right), \tag{A.6}$$

where

$$a - b\frac{J^2}{p^2 r^2} = \frac{8\pi}{3} m^2 \left\{ \int_{\mathscr{E}}^{E} f(E') \, dE' \frac{p'}{p} \frac{3}{2} \left[\left(1 - \frac{1}{3} \frac{p'^2}{p^2}\right) - \left(1 - \frac{p'^2}{p^2}\right) \frac{J^2}{p^2 r^2} \right] + \int_E^\infty f(E') \, dE' \right\}; \quad \mathscr{E} = \max(U, E_{min}). \tag{A.7}$$

Since we are interested in the energy range $E < E_{crit}$, in which the distribution function decreases rapidly to zero with decreasing E as a result of the angular momentum diffusion of the star orbits, the first term in (A.7) for $E < E_{crit}$ is always less than or at the most of the same order as the second term, and can therefore be neglected in an estimate of the flow rate. Further, instead of integrating with respect to the angular variables ϑ^α, we integrate with respect to r, α, φ. The Jacobian of the transition is

$$\frac{\partial(\vartheta^1, \vartheta^2, \vartheta^3)}{\partial(r, \alpha, \varphi)} = \frac{1}{g^{1/2}} \frac{\sin\alpha}{(\sin^2\alpha - J_z^2/J^2)^{1/2}} \frac{1}{v_r}, \tag{A.8}$$

where v_r is the radial component of the velocity and $g^{1/2}$ is determined by the expression (23). After the integration in (A.6), we finally obtain

$$R_2^{22} \approx \frac{5\pi}{3} m^2 \left(\frac{GMm}{|E|}\right)^2 \int_E^\infty f(E') \, dE' \left[1 - \frac{3}{5} \frac{J^2}{J^2_{max}(E)}\right], \tag{A.9}$$

where $J^2_{max}(E)$ is determined by Eq. (29).

Note added in proof (*September* 23, 1977). 1. The distribution of stars around a black hole in the opposite limiting case (black hole mass large compared with the mass of the surrounding stellar system) has been obtained in our note in Pis'ma Astron. Zh. **3**, 391 (1977) [Sov. Astron. Letters **3**, 209 (1977)].

2. The interaction of stars with extended gas clouds—the products of the disruption of stars in tidal or collisional interactions—leads to an even more rapid filling of the loss cone with stars, i.e., it increases their flow through the tidal sphere. This can be further enhanced by the collective interaction of stars, the importance of which has been pointed out by A. M. Fridman. These questions warrant a separate investigation.

[1)] For bound stars with orbits lying within the radius $r_{coll} \approx r_* M/m$, at which the kinetic energy of a star $\sim GMm/r$ is comparable with its binding energy $\sim Gm^2/r_*$, disruption as a result of direct collisions of stars may become an important process.[2] However, comparison of r_{coll} and r_{crit} (Eq. (37)) shows that in real clusters $r_{coll} < r_{crit}$, i.e., a star remains in the region $r < r_{coll}$ for only a small fraction of the time and collisions of stars, being very rare, will not change the dependence $n(r)$ in the region $r < r_{coll}$.

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Self-similar motions of a photon gas and the Friedmann model

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Self-similar motions of a gas with equation of state $p = \epsilon/3$ are considered. It is shown that all solutions with a weak discontinuity have a nonsingular horizon. The Friedmann solution belongs to a one-parameter family of solutions that are continuous at the symmetry center; these are described. The solutions with strong shock waves correspond to the problem of initial focusing of the gas toward the center in the case of a supercritical intensity of the discontinuity. The results of qualitative and numerical investigations of the corresponding dynamical system are presented. New cosmological solutions with strong and weak discontinuities are obtained.

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In the early stages in the expansion of the Universe, the dominant contribution to the matter energy was made by electromagnetic radiation, which follows almost unambiguously from the discovery of the microwave blackbody radiation with temperature 2.83 °K.[1] The simplest model of a radiation-filled Universe—the Friedmann–Lemaître model with flat comoving space—admits a simple analytic expression for the metric and the energy density in Lagrangian coordinates[2]:

$$ds^2 = d\tau^2 - \tau a_0[dR^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2)],$$
$$\varepsilon = 3c^2/32\pi G\tau^2. \quad (1.1)$$

This is a self-similar (similarity) solution. It is therefore natural to consider the status of the Friedmann solution among the other self-similar spherically symmetric solutions and also study the physical and analytic properties of other self-similar solutions, including those with shock waves.

In the present paper, spherically symmetric self-similar motions are studied in the orthogonal coordinate system of an observer:

$$ds^2 = c^2 e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (1.2)$$

Self-similar spherical motions of a gravitating gas in the framework of general relativity were considered for the first time by Skripkin.[3] Because of a not entirely fortuitous choice of the variables, the correct conditions on the gas-dynamic shock waves in the framework of general relativity lacked the simplicity of these conditions in special relativity,[2] but Skripkin succeeded in deriving a condition on the discontinuity after which the gas goes over into a state of rest: $\varepsilon = 3(7\varkappa r^2)^{-1}$. Skripkin reduced Einstein's equations to an equation of second order with radicals.[3] From the fixed velocity of the shock wave, he calculated the parameters of the gas after the discontinuity and used these data to construct numer-