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Asymptotic estimates of high order perturbation theory approximations in scalar electrodynamics

A. P. Bukhvostov and L. N. Lipatov

B. P. Konstantinov Nuclear Physics Institute, USSR Academy of Sciences, Leningrad
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The asymptotic behavior of the expansion coefficients of the Green's functions in a perturbation theory series is found with an accuracy up to a factor preceding the exponential. In order to obtain the form of the solutions of classical equations with finite action for which a saddle exists in the functional integral for the Green's functions a specialized perturbation theory is used, the parameter of which is the ratio of the orders of perturbation theory in terms of the coupling constants e and g.

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1. INTRODUCTION

In earlier papers of one of the authors (L. N. L.) a method was proposed for estimating high orders of perturbation theory for the Green's functions based on the saddle-point method of calculating corresponding functional integrals.[1] Although in these papers only scalar field theory models were considered, with the interaction $H_{\text{int}}=g\varphi^n/n!$ in $D=2n/(n-2)$-dimensional Euclidean space-time, it was shown that the method of calculating can be transferred to other more interesting field models. In the present paper we obtain asymptotic formulas for high orders of perturbation theory in scalar electrodynamics. We regard this result as an intermediate stage preceding the main problem—obtaining asymptotic estimates in spinor electrodynamics. The solution of this problem would certainly be of interest both from the point of view of an approximate calculation of higher radiative corrections to observable quantities of the type of the anomalous magnetic moment of the electron,[2] and also from a purely theoretical point of view as a method of going beyond the framework of perturbation theory in the problem of internal selfconsistency of quantum electrodynamics.[3]

The first indications of the divergence of the perturbation theory series in quantum electrodynamics were obtained by Dyson.[4] He gave qualitative arguments in favor of the Green's functions having a singularity at the origin in terms of the fine structure constant α. This singularity arises as a consequence of the instability of the theory for $\alpha<0$. Within the framework of quantum mechanics of the anharmonic oscillator the nature of the singularity in terms of the coupling constant was investigated in the papers by Vaĭnshteĭn[5] and Langer.[6] The exact asymptotic formula for the expansion coefficient for the energy of the ground state of an anharmonic oscillator was obtained in the paper by Bender and Wu.[7] Until very recently, there existed in quantum field theory only rough estimates of high orders of perturbation theory (cf., for example, Ref. 8). The saddle-point method of calculating the functional integral utilized in Ref. 1 enables one to construct asymptotic formulas in principle of arbitrary accuracy in the form of an expansion in inverse powers of the order of perturbation theory. This method was generalized by the authors of Ref. 9 to the case when the scalar field has an internal symmetry group. They also showed that an analogous approach can be formulated for the problem of estimating high orders in the ε-expansion due to Wilson. This provided the possibility of calculating with greater confidence critical indexes in the theory of second-order phase transitions[9] and in Regge theory.[10]

Asymptotic estimates in perturbation theory enable one to calculate the nature of the singularity in the case of small coupling constants.[11] In Ref. 12 the question is discussed of the summability in the Borel sense of the perturbation theory series under the condition that the theory does not contain solutions of classical equations appropriate to the physical sign of the coupling constant. Summation of the perturbation theory series according to Borel is equivalent to replacing it by a Watson-Sommerfeld integral under the conditions that the coefficients are analytic in the index. The evaluation of the sum of the perturbation theory series with the aid of the Watson-Sommerfeld transformation was utilized in Refs. 1 and 13.

Bogomol'nyĭ[14] and Parisi[15] generalized to field the-

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ory the formalism of Langer[6] for evaluating the discontinuity in the amplitudes at the singularity in terms of the coupling constant g. They showed that the discontinuity for $g<0$ is determined by the probability of the wave function of the ground state of the system leaking through the potential barrier. The corresponding probability can be found in the quasiclassical approximation by means of solving the classical equations for imaginary time, i.e., in Euclidean theory. From the technical point of view it is apparently simpler to seek the discontinuity starting from the asymptotic formulas (cf., Ref. 11).

After the present work had been completed there appeared in print a note by Itzykson *et al.*[16] in which similar questions are discussed. As we shall show the solution of classical equations obtained in Ref. 16 does not realize the maximally highest saddle-point in evaluating the functional integral in a high order of perturbation theory. Nevertheless certain results of the cited paper are of considerable interest. Thus, for example, the use of the insertion theorems due to Sobolev[17] could give a rigorous mathematical basis for the method of Ref. 1.

2. FIVE-DIMENSIONAL FORMULATION OF SCALAR ELECTRODYNAMICS

The action for scalar electrodynamics has the following form in Euclidean four-dimensional space:

$$S(A \text{ or } e, g) = \int H d^4x = \int d^4x [{}^1/_4 (\partial_\nu A_\mu - \partial_\mu A_\nu)^2 + |(\partial_\nu - ieA_\nu)\varphi|^2 + {}^1/_2 g|\varphi|^4], \quad (1)$$

where φ is a complex scalar field. We have added to the Lagrangian H the term $g|\varphi|^4/2$ since it inevitably arises as a result of renormalizations. The mass term for the field φ is not included in the action (1) since we shall be interested in the behavior of the Green's functions for large momenta, with the momentum μ at the point of normalization of the invariant charges $e^2(p^2)|_{p^2=\mu^2} = e_\mu^2$, $g(p^2)|_{p^2=\mu^2} = g_\mu$ being chosen much greater than the mass of the scalar particle:

$$\mu \gg m. \quad (2)$$

An arbitrary Green's function $G(x_1, \ldots, x_M; y_1, \ldots, y_N; z_1, \ldots, z_N)$ in scalar electrodynamics can be found in the form of a perturbation theory series in terms of the renormalized charges e_μ and g_μ:

$$G(x_1, \ldots, z_N) = \sum_{km} (e_\mu^2)^m (g_\mu^2)^k G_{km}(x_1, \ldots, z_N), \quad (3)$$

where $G_{km}(x_1, \ldots, z_N)$ is given in the Lorentz gauge by the functional integral

$$G_{km}(x_1, \ldots, x_M; y_1, \ldots, y_N; z_1, \ldots, z_N) = Z_0^{-1} \int \prod_{x,\nu} [dA_\nu(x) d\varphi(x) d\varphi^*(x) \delta(\partial_\rho A_\rho(x))] \prod_{r=1}^{M} A_{\nu_r}(x_r) \times \prod_{s=1}^{N} \varphi(y_s)\varphi^*(z_s) \int \frac{de_\mu^2}{2\pi i (e_\mu^2)^{m+1}} \int \frac{dg_\mu}{2\pi i (g_\mu)^{k+1}} \times \exp\left\{-S(A_\nu \varphi, e_\mu, g_\mu) - \int d^4x H'(A_\nu, \varphi, e_\mu, g_\mu)\right\}. \quad (4)$$

Here H' is a counterterm corresponding to the renormalization of the masses, the wave functions and the vertex functions. In order to evaluate the asymptotic behavior of G_{km} at large k and m it is sufficient to know the lowest terms in the expansion of H' in terms of g_μ and e_μ^2 (they are contained, for example, in the paper by Vladimirov[18]). The value of Z_0 in formula (4) is chosen by imposing the condition that for $e_\mu = g_\mu = 0$ the function $G(x_1, \ldots, z_N)$ would be given by a product of free Green's functions. We note that formula (4) contains a contribution from disconnected diagrams and from diagrams with vacuum loops, but for large k and m this contribution is not significant.[1]

In accordance with Ref. 1 we evaluate the integral (4) by the saddle-point method. For this it is necessary to obtain a solution of the Euler-Lagrange equations for the functional (1) for small and, generally speaking, complex values of the constants e and g. For these solutions the action must be finite:

$$|\bar{\varphi}(x)||_{x^2\to\infty} < \text{const}/(x^2)^{1/2}, \quad |\bar{A}(x)||_{x^2\to\infty} < \text{const}/(x^2)^{1/2}. \quad (5)$$

Thus, the solutions must be bounded in space, i.e., they must have a definite center x_0 and a scale λ. In virtue of the translational and scale invariance of the functional (1) the set of such solutions is expressed in terms of the solution $\bar{\varphi}(x), \bar{A}_\nu(x)$ with the center at the origin and with a unit scale by means of the following formulas:

$$\bar{\varphi}^{(x_0,\lambda)}(x) = \frac{1}{\lambda}\bar{\varphi}\left(\frac{x-x_0}{\lambda}\right), \quad \bar{A}_\nu^{(x_0,\lambda)}(x) = \frac{1}{\lambda}\bar{A}_\nu\left(\frac{x-x_0}{\lambda}\right). \quad (6)$$

Below we consider the solution with the center at the origin and with unity scale. This means that from the general 15-parameter group of transformations under which the action (1) is invariant (the Lorentz transformation, a displacement, a stretch, a conformal transformation) we can hope to retain the invariance of the solution only with respect to the 10-parameter group. In order to make the invariance of the action (1) with respect to this group with ten parameters explicit it is necessary to go over, following Adler,[19] to a five-dimensional formulation of scalar electrodynamics.[1)] We introduce the five-coordinates of a point on a unit sphere in five-dimensional space according to the following formulas

$$z_i = \frac{2x_i}{1+x^2} \quad (i=1,2,3,4), \quad z_5 = \frac{x^2-1}{x^2+1}, \quad \sum_{i=1}^{5} z_i^2 = 1; \quad (7)$$
$$\int dS_5 = \int d^5z\,\delta((\Sigma z_i^2)^{1/2}-1) = \int d^4x\,(2/(1+x^2))^4.$$

In place of the fields φ and A_ν it is convenient to introduce new fields Y and A_i according to the following formulas

$$\varphi(x) = \frac{2}{1+x^2}Y(z), \quad A_\nu = \frac{\partial z_i}{\partial x_\nu}A_i;$$
$$\frac{\partial z_i}{\partial x_\nu}\frac{\partial z_j}{\partial x_\nu} = \left(\frac{2}{1+x^2}\right)^2 (\delta_{ij} - z_i z_j),$$
$$F_{\nu\lambda} = \frac{\partial z_i}{\partial x_\nu}\frac{\partial z_j}{\partial x_\lambda}F_{ij}, \quad F_{ij} = \partial_j A_i - \partial_i A_j. \quad (8)$$

We impose an additional condition on the five-dimensional vector A_i:

$$z_iA_i=0. \tag{9}$$

Using formulas (7) and (8) we can rewrite the action (1) in the following form:

$$S=\int dS_5[{}^1/_{12}(L_{ij}A_k+L_{jk}A_i+L_{ki}A_j)^2+{}^1/_2|(L_{ij}-ieA_{ij})Y|^2 +2|Y|^2+{}^1/_2g|Y|^4], \quad A_{ij}=z_iA_j-z_jA_i, \tag{10}$$

where

$$L_{ij}=z_i\partial_j-z_j\partial_i \tag{11}$$

is an antihermitean infinitesimal operator for the rotation of a five-dimensional sphere in the (ij) plane. As can be seen from formulas (10) and (11), the action depends only on the values of the fields and of their derivatives along the surface of the sphere. The invariance of the action (10) and of the auxiliary conditions (9) with respect to the 10-parameter group of rotations of the five-dimensional sphere is obvious.

The functional (10) is also invariant with respect to gauge transformations of the form

$$Y\to Ye^{ie\rho}, \quad A_i\to A_i+(\partial_i-z_i(z\partial))\rho \tag{12}$$

with an arbitrary function ρ depending on the angles on the five-dimensional sphere. We shall utilize this invariance in such a manner that the analog of the Lorentz gauge would be satisfied (cf., Ref. 19):

$$(\partial_i-z_i(z\partial))A_i=0. \tag{13}$$

Under the condition (9) relation (13) is equivalent to

$$L_{ij}A_j=A_i. \tag{14}$$

The Euler-Lagrange equations for the functional (10) together with the auxiliary conditions (9) and (14) have the following form

$$(-{}^1/_2L_{ij}^2+2)A_k=ie_\mu z_i[Y(L_{ik}+ie_\mu A_{ik})Y^*-Y^*(L_{ik}-ie_\mu A_{ik})Y], \quad [-{}^1/_2(L_{ij}-ie_\mu A_{ij})^2+2+g_\mu|Y|^2]Y=0, \quad [-{}^1/_2(L_{ij}+ie_\mu A_{ij})^2+2+g_\mu|Y|^2]Y^*=0; \tag{15}$$

$$z_iA_i=0, \quad L_{ij}A_j=A_i. \tag{16}$$

In order to take into account the translational and scale invariances we introduce into the functional integral (4) the following expansion of unity (cf., Ref. 1):

$$1=\int d^4x_0\int_0^\infty\frac{\partial\lambda}{\lambda^5}\delta^5\left(\int d^4x\,Hz_i^{(x_0,\lambda)}\right)\det\left|\int d^4x\,H(z_i^{(x_0,\lambda)}z_j^{(x_0,\lambda)}-\delta_{ij})\right|, \tag{17}$$

where

$$z_i^{(x_0,\lambda)}=\frac{2\lambda(x-x_0)_i}{\lambda^2+(x-x_0)^2} \quad (i=1,2,3,4), \quad z_5^{(x_0,\lambda)}=\frac{(x-x_0)^2-\lambda^2}{(x-x_0)^2+\lambda^2}. \tag{18}$$

After changing the parameters and the variables of integration according to the following formulas

$$x\to x_0+\lambda x, \quad \varphi(x_0+\lambda x)\to\frac{1}{\lambda}\varphi(x), \quad A_\nu(x_0+\lambda x)\to\frac{1}{\lambda}A_\nu(x) \tag{19}$$

and after transition to the five-dimensional variables (7) and (8) we obtain in place of (4)

$$G_{km}(x_1,\ldots,x_M;\,y_1,\ldots,y_N;\,z_1,\ldots,z_N)=Z_0^{-1}\int d^4x_0\int_0^\infty\frac{d\lambda}{\lambda^5} \times\int\prod_{z,i}[dA_i(z)\,dY(z)\,dY^*(z)\,\delta(z_jA_j(z))\,\delta(\partial_jA_j-z_j(z\partial)A_j)] \times\prod_{r=1}^M\frac{1}{\lambda}A_{\nu_r}\left(\frac{x_r-x_0}{\lambda}\right)\prod_{s=1}^N\frac{1}{\lambda^2}\varphi\left(\frac{y_s-x_0}{\lambda}\right)\varphi^*\left(\frac{z_s-x_0}{\lambda}\right) \times\int\frac{de_\mu^2}{2\pi i(e_\mu^2)^{m+1}}\int\frac{dg_\mu}{2\pi ig_\mu^{k+1}}\exp\left\{-S-\int d^4x\,H_\lambda'\right\} \times\delta^5\left(\int d^4x\,Hz_i\right)\det\left|\int d^4x\,H(z_iz_j-\delta_{ij})\right|. \tag{20}$$

Here we have utilized the five-dimensional gauge (13) for the potentials A_i. It corresponds to the following gauge in four-dimensional space:

$$\partial_\nu A_\nu=4x_\nu A_\nu/(1+x^2). \tag{21}$$

It is necessary to rewrite the counterterm H_λ' in formula (20) in the same gauge.

3. THE FORM OF SOLUTIONS OF SADDLE-POINT EQUATIONS

The saddle-point values for the fields $\tilde A_i$ and $\tilde Y$ and for the parameters $\tilde e_\mu$ and $\tilde g_\mu$ in the integral (20) are obtained from the condition that the functional be an extremum

$$J(A_i,\,Y,\,Y^*,\,e_\mu,\,g_\mu)=S+m\ln(-e_\mu^2)+k\ln(-g_\mu), \tag{22}$$

where S is given by formula (10). Variation of J with respect of A_i, Y, Y^* leads to the classical equations (15) with the auxiliary conditions (16) and the restriction

$$\int dS_5\,Hz_i=0, \tag{23}$$

arising as a consequence of the δ-functions in formula (20).

The variation of J with respect to g_μ and e_μ leads to the conditions

$$k=-\frac{g_\mu}{2}\int dS_5|\tilde Y|^4, \quad m=-\frac{\tilde e_\mu}{4}\frac{\partial}{\partial\tilde e_\mu}\int dS_5|(L_{ij}-i\tilde e_\mu\tilde A_{ij})\tilde Y|^2, \tag{24}$$

which fix the values of $\tilde g_\mu$ and $\tilde e_\mu$.

In order to guess the form in which one should seek the solution of Eqs. (15) we consider first the special case when the order of perturbation theory in terms of g_μ is considerably greater than the order with respect to e_μ^2:

$$k\gg m\gg 1. \tag{25}$$

In this case one should expect that in the first approximation it is sufficient to define in (20) the saddle for the scalar field in the absence of an interaction with the electromagnetic field. But such a problem has already been solved[1,9] and the corresponding saddle-point field is a constant on the sphere:

$$\tilde Y^{(0)}=(-2/g_\mu)^{1/4}e^{i\chi}, \tag{26}$$

where χ is an arbitrary constant phase (we have a kind of a Higgs mechanism).

The solution (26) can be obtained if in the second and third equations of system (15) we set $A_{ij}=0$. It is important that at the same time the restriction (23) is satisfied. As will be seen below this is sufficient that the restriction (23) would be satisfied also for the exact solution. Just as in Ref. 1 the existence of solution (26) leads to the factorial dependence of the coefficients of expansion of the Green's function in terms of g_μ on the order k of perturbation theory. At the same time the dependence of these coefficients on the order of perturbation theory in terms of e_μ in the region (25) is only a power-law dependence. Indeed, the radius of convergence of the series in terms of e_μ in the region (25) is determined by the value $e_\mu=\tilde{e}_\mu$ minimal with respect to the modulus, for which the first equation of the system has a solution different from zero. Such a value of $\tilde{e}_\mu$ will give a root singularity in terms of e_μ after the evaluation of the functional integral with respect to A_i in formula (20).

Substituting (26) into the first equation of the system (15) it is easy to obtain the solution for the minimal value of $|e_\mu|$:

$$\bar{A}_i \approx \eta_{ij} z_j a, \quad -\tilde{e}_\mu^2 = -3/2\tilde{g}_\mu, \tag{27}$$

where in virtue of the restrictions (16) the matrix η_{ij} is antisymmetric:

$$\eta_{ij}=-\eta_{ji}. \tag{28}$$

We have separated out the factor a in formula (27) in order to normalize η_{ij} in an appropriate manner (cf., below (30)).

As can be seen from equations (24), $a \sim (m/k)^{1/2} \ll 1$, i.e., the neglect of terms proportional to A_{ij} in the second and the third equations of the system (15) in obtaining (26) was justified in the region (25). One can take into account the correction to (26) attributable to the terms $\sim A_i$ (27):

$$\Gamma^{(1)}=\Gamma^{(0)}\left[1-e^2\left(-\frac{a^2}{20}\eta_{ij}^2+\frac{a^2}{6}\eta_{ij}\eta_{ik}\left(z_jz_k-\frac{1}{5}\delta_{jk}\right)\right)\right]. \tag{29}$$

Substituting (29) into (15) and obtaining the correction $\sim a^3$ to A_i taking (24) into account we must, proceeding in the same spirit, obtain the solutions in the form of an expansion in powers of m/k. However this procedure, as can be easily seen, can not be carried out for some arbitrary choices of the matrix η_{ij} in (27). Indeed, on substituting (29) into the right-hand side of the first of equations (15) we obtain terms of two kinds for the first spherical harmonic: $z_j\eta_{ij}$ and $z_j(\eta^3)_{ij}$. Since the left-hand side of the equation does not contain the first harmonic, these terms must be compensated by a small deviation $(-\tilde{e}_\mu^2)$ from its critical value (27). From this requirement emerges the following condition on the form of the matrix η_{ij}:

$$\eta_{ij}\eta_{kj}\eta_{ki}=\eta_{ii} \tag{30}$$

(the choice of normalization, apparently, is not significant and is associated with the normalization of a in formula (27)). It can be easily shown that this condition is sufficient for carrying out the iteration procedure indicated above. Equation (30) means that the symmetric matrix $g_{ik}=\eta_{ij}\eta_{kj}$ satisfies the relation

$$g_{ik}g_{ki}=g_{ii} \text{ or } g^2=g, \tag{31}$$

from which it follows that its eigenvalues can be equal only to zero or to unity. Thus, this matrix must coincide with the unit matrix in a certain subspace $E_{n_\perp}$ of the five-dimensional Euclidean space E_5; in other words, the matrix η_{ij} in virtue of the conditions (28) and (31) is an antisymmetric orthogonal matrix in this subspace:

$$\eta_{ij}=-\eta_{ji}, \quad \eta_{ij}\eta_{kj}=\delta_{ik}^\perp, \quad \delta_{ik}^\perp\delta_{kl}^\perp=\delta_{il}^\perp, \quad \delta_{ii}^\perp=n_\perp. \tag{32}$$

Such matrices, which are called symplectic, can exist only in the case of an even number of dimensions, and from this it follows that the dimensionality of the subspace $E_{n_\perp}$ must be one of two possible ones:

$$\text{a) } n_\perp=2, \quad \text{b) } n_\perp=4. \tag{33}$$

For these two cases one can respectively exhibit matrices of a special form satisfying conditions (32):

$$\text{a) } \eta_{ij}^{(1)}=\begin{pmatrix} 0&1&0&0&0\\ -1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}, \quad \text{b) } \eta_{ij}^{(2)}=\begin{pmatrix} 0&1&0&0&0\\ -1&0&0&0&0\\ 0&0&0&1&0\\ 0&0&-1&0&0\\ 0&0&0&0&0 \end{pmatrix}. \tag{34}$$

All the remaining matrices of this type can be obtained from the matrices (34) by five-dimensional orthogonal transformations and form two 6-parameter families:

$$\eta_{ij}=U_{ik}U_{jl}\eta_{kl}^{(1,2)}, \quad U_{ik}U_{ki}=\delta_{kl}. \tag{35}$$

By a further iteration of Eqs. (15) in the region (25) one can easily verify that their solutions should be sought in the form

$$\bar{A}_i=\frac{1}{(-\tilde{e}_\mu^2)^{1/2}}\eta_{ij}z_ja(s), \quad \Gamma=\frac{e^{i\chi}}{(-2\tilde{e}_\mu^2)^{1/2}}\Phi(s), \quad s=(z^\perp)^2=\eta_{ij}\eta_{kj}z_iz_k, \tag{36}$$

where $a(s)$ and $\Phi(s)$ satisfy the equations

$$\begin{aligned} &-4s(1-s)a''+2(7s-2-n_\perp)a'+(6-\Phi^2)a=0,\\ &-4s(1-s)\Phi''+2(5s-n_\perp)\Phi'+(2-sa^2-\varkappa\Phi^2)\Phi=0,\\ &\varkappa=\tilde{g}_\mu/2\tilde{e}_\mu^2. \end{aligned} \tag{37}$$

These equations can be obtained from the action (10) which in the case of solutions (36) has the form

$$S=-\frac{8\pi^2}{\tilde{e}_\mu^2}\int_0^1\frac{ds}{n_\perp}(1-s)^{1/2}\left(\frac{s}{1-s}\right)^{n_\perp/2-1}\left[2s^2(1-s)a'^2+3sa^2+2s(1-s)\Phi'^2+\Phi^2-\frac{s}{2}a^2\Phi^2-\frac{\varkappa}{4}\Phi^4\right]. \tag{38}$$

We must select the solutions of the system of equations (37) which are regular at $s=0$ and $s=1$:

$$a(s)|_{s\to0}=a_1+a_2s+\ldots, \quad \Phi(s)|_{s\to0}=\Phi_1+\Phi_2s+\ldots, \tag{39a}$$

$$a(s)|_{s\to1}=a_3+a_4(1-s)+\ldots, \quad \Phi(s)|_{s\to1}=\Phi_3+\Phi_4(1-s)+\ldots \tag{39b}$$

The greater part of the boundary conditions (39) follows from the requirement that the action (38) be finite. Only the condition (39b) for $n_\perp=4$ requires the use of formula

(23) for its justification. Choosing the matrices η in the form (34b) and setting $i=5$ in (23) we satisfy this relation in the case when H is an even function of z_5, i.e., $a(s)$ and $\Phi(s)$ do not have a singularity $\sim(1-s)^{1/2}$ which can be present in the general solution of Eqs. (37).

One can qualitatively demonstrate the existence of solutions satisfying the boundary conditions (39) if in formula (38) one makes the replacement:

$$a=u/s, \quad \Phi=v/s^{1/4}, \quad s=1/\mathrm{ch}^2\xi, \tag{40}$$

as a result of which it assumes the form

$$S=\int_0^\infty d\xi(\mathrm{sh}\,\xi)^{4-n_\perp}\left[\left(\frac{du}{d\xi}\right)^2+2(n_\perp-2)u^2+\left(\frac{dv}{d\xi}\right)^2 +(n_\perp-3)v^2-u^2v^2-\frac{\varkappa}{2}v^4\right] \tag{41}$$

(the constant factor is omitted). In this case conditions (39a) mean that as $\xi\to\infty$ the functions u and v must decrease:

$$u|_{\xi\to\infty}\to a_1e^{-2\xi}, \quad v|_{\xi\to\infty}\to\Phi_1e^{-\xi}, \tag{42}$$

while for $\xi\to 0$ they are bounded and, moreover, are even functions of ξ.

For $n_\perp=4$ (this case with $\varkappa=0$ is discussed in Ref. 16) the action (41) describes the Newtonian motion of a material point in a two-dimensional potential

$$V(u,v)=-4u^2-v^2+u^2v^2+\varkappa v^4/2, \tag{43}$$

which does not explicitly depend on the time ξ. This potential has the form of a hump near the origin surrounded by "walls" for large $|u|$ and $|v|$. The desired solution corresponds to such a motion of the point when it is at rest at the origin for $\xi\to-\infty$, then rolls down the hump, is reflected from the wall at the instant $\xi=0$ and returns over an infinite time over the same trajectory. For large values of $\varkappa$ the walls, roughly speaking, are parallel to the u axis, and therefore the trajectory which has the described properties must be directed along the v axis (the potential is even with respect to u and v and increases as $|v|\to\infty$; consequently, such a trajectory always exists as long as $\varkappa>0$). In other words, the desired solution of equations (37) in this case has the form $a=0$, $\Phi=\mathrm{const}$. As $\varkappa$ decreases the walls begin to become more strongly curved and other trajectories arise for which $u\neq0$ (i.e., $a\neq0$). The first such trajectory, as can be seen from equations (37) appears near the value $\varkappa=1/3$.

In the case $n_\perp=2$ the Euler-Lagrange equations

$$\ddot{u}+2\,\mathrm{cth}\,\xi\dot{u}+uv^2=0, \quad \ddot{v}+2\,\mathrm{cth}\,\xi\dot{v}+v+vu^2+\varkappa v^3=0 \tag{44}$$

describe the motion of the point in the potential

$$V(u,v)=v^2+u^2v^2+\varkappa v^4/2 \tag{45}$$

with damping proportional to $\coth\xi$. The form of the potential (45) shows that an arbitrary solution of equations (44) for $\xi\to\infty$ has the form $u\to\mathrm{const}$, $v\to0$. Since for $\xi\to0$ the conditions $\dot{u}(0)=\dot{v}(0)=0$, which follow from the fact that u and v are even in ξ, must be satisfied we have at our disposal two constants: $u(0)$ and $v(0)$ by the choice of which one can satisfy the condition $u(\infty)=0$. This condition determines a certain set of initial positions of the point from which it arrives at the origin. In such a case, as can be seen from equations (44), we have

$$u|_{\xi\to\infty}\to Ce^{-2\xi}, \quad v|_{\xi\to\infty}\to(C_1+C_2\xi)e^{-\xi}. \tag{46}$$

In order that conditions (42) be satisfied one must also require that $C_2=0$ in formula (46). This can be accomplished utilizing the remaining arbitrariness in the choice of the initial position of the point. Just as in the case $n_\perp=4$, one can verify that for $\varkappa\geq1/3$ the only desired solution is $u=0$, while for $\varkappa<1/3$ a solution with $u\neq0$ appears. For still smaller values of $\varkappa$ other solutions also arise; we shall return below to a discussion of them (cf., Sec. 4).

The parameters $\varkappa$ and $\tilde{e}_\mu^2$ in (37) are determined from the relations (24) which taking (36) into account can be rewritten in the form

$$k=-\frac{8\pi^2}{\tilde{e}_\mu^2}\varkappa\int_0^1\frac{ds}{n_\perp}(1-s)^{1/2}\left(\frac{s}{1-s}\right)^{n_\perp/2-1}\frac{\Phi^4}{4},$$
$$m=-\frac{8\pi^2}{\tilde{e}_\mu^2}\int_0^1\frac{ds}{n_\perp}(1-s)^{1/2}\left(\frac{s}{1-s}\right)^{n_\perp/2-1}\frac{sa^2\Phi^2}{2} \tag{47}$$

whence for the determination of $\varkappa$ we obtain the equation

$$\frac{k}{m}=\varkappa\int_0^1 ds(1-s)^{1/2}\left(\frac{s}{1-s}\right)^{n_\perp/2-1}\Phi^4\Big/2\int_0^1 ds(1-s)^{1/2}\left(\frac{s}{1-s}\right)^{n_\perp/2-1}sa^2\Phi^2. \tag{48}$$

We examine within what limits does $\varkappa$ vary as the ratio m/k is varied. From (48) it follows that

$$\varkappa\to0 \text{ as } m/k\to\infty. \tag{49a}$$

In the opposite limiting case (25) we have in virtue of (27)

$$\varkappa\to1/3 \text{ as } m/k\to0. \tag{49b}$$

Thus we have

$$0<\varkappa<1/3. \tag{50}$$

Under this condition, as we have seen, Eqs. (37) have at least one nontrivial solution.

4. THE GREEN'S FUNCTIONS IN HIGH ORDERS OF PERTURBATION THEORY

With an accuracy up to the factor preceding the exponential the saddle-point contribution to the asymptotic behavior of the coefficients of expansion G_{km} of the Green's functions is determined by the expression $(-1)^{k+m}e^{-\tilde{J}}$, where $\tilde{J}$ is the value of the functional J (22) along that solution of Eqs. (37) and (47) for which this value is minimal. Taking into account the relations

$$\int_0^1 ds(1-s)^{1/2}\left(\frac{s}{1-s}\right)^{n_\perp/2-1}\left[2s^2(1-s)a'^2+3sa^2-\frac{s}{2}a^2\Phi^2\right]=0, \tag{51}$$

$$\int_0^1 ds(1-s)^{1/2}\left(\frac{s}{1-s}\right)^{n_\perp/2-1}\left[2s(1-s)\Phi'^2+\Phi^2-\frac{s}{2}a^2\Phi^2-\frac{\varkappa}{2}\Phi^4\right]=0,$$

which follow from Eqs. (37) and the boundary conditions (39), and taking into account formulas (38) and (47) we obtain

$$G_{km}|_{k,m\to\infty}\sim(-1)^{k+m}\left(\frac{k}{16\pi^2e}\right)^k\left(\frac{m}{16\pi^2e}\right)^m\left[C\left(\frac{m}{k}\right)\right]^{m+k}, \tag{52}$$

where the function $C(m/k)$ has the form

$$C\left(\frac{m}{k}\right)=\left(2\varkappa\frac{k}{m}\right)^{-k/(k+m)}\left[\frac{1}{4n}\int_0^1 ds(1-s)^{1/2}\left(\frac{s}{1-s}\right)^{n_\perp/2-1}s\Phi^2a^2\right]^{-1}. \tag{53}$$

From the two possible types of solutions of (33) one must choose such a solution for which $C(m/k)$ will be the greatest for a given m/k.

It is convenient to obtain an approximate solution of Eqs. (37) by the following method. We expand A_l and Y in terms of spherical harmonics over the five-dimensional sphere (cf., (36)):

$$a(s)=\sum_{l=0}^{\infty}A_{2l+1}P^l(s),\quad \Phi(s)=\sum_{l=0}^{\infty}B_{2l}Q^l(s), \tag{54}$$

where the polynomial P^l and Q^l are determined by the equations

$$\begin{aligned}Q^l(s)&=\delta^{\perp}_{i_1i_2}\delta^{\perp}_{i_3i_4}\dots\delta^{\perp}_{i_{2l-1}i_{2l}}\{z_{i_1}z_{i_2}\dots z_{i_{2l}}\},\\ zP^l(s)&=\delta^{\perp}_{i_1i_2}\delta^{\perp}_{i_3i_4}\dots\delta^{\perp}_{i_{2l-1}i_{2l}}\{zz_{i_1}z_{i_2}\dots z_{i_{2l}}\}.\end{aligned} \tag{55}$$

Here $\{z_{i_1}z_{i_2}\dots z_{i_n}\}$ is the traceless part of the product $z_{i_1}\dots z_{i_n}$; it is an eigenfunction of the operator $-\frac{1}{2}L_{ij}^2$ which coincides with the angular part of the five-dimensional Laplacian[20]:

$$-{}^1/_2L_{ij}^2\{z_{i_1}z_{i_2}\dots z_{i_n}\}=n(n+3)\{z_{i_1}z_{i_2}\dots z_{i_n}\}. \tag{56}$$

Formula (56) means that $Q^l(s)$ and $P^l(s)$ satisfy equations

$$\begin{aligned}\left[-4s(1-s)\frac{d^2}{ds^2}+2(5s-n_\perp)\frac{d}{ds}+2\right]Q^l(s)&=(2l+1)(2l+2)Q^l(s),\\ \left[-4s(1-s)\frac{d^2}{ds^2}+2(7s-2-n_\perp)\frac{d}{ds}+6\right]P^l(s)&=(2l+2)(2l+3)P^l(s).\end{aligned} \tag{57}$$

The solutions (55) of these equations have the form

$$\begin{aligned}Q^l(s)&=s^lF\left(-l,-l+1-\frac{n_\perp}{2};-2l-\frac{1}{2};\frac{1}{s}\right)\\&=(-1)^l\frac{L!\Gamma((n_\perp-1)/2)}{2^L\Gamma(2l+{}^3/_2)}(\cos\theta)^{n_\perp/2-2}C_L^{(n_\perp-1)/2}(\cos\theta);\\ P^l(s)&=s^lF\left(-l,-l-\frac{n_\perp}{2};-2l-\frac{3}{2};\frac{1}{s}\right)\\&=(-1)^l\frac{L!\Gamma((n_\perp+1)/2)}{2^L\Gamma(2l+{}^3/_2)}(\cos\theta)^{n_\perp/2-2}C_L^{(n_\perp+1)/2}(\cos\theta);\\ s&=\sin^2\theta,\quad L=2l+2-n_\perp/2.\end{aligned} \tag{58}$$

Here F is a hypergeometric function, C_n^λ are Gegenbauer polynomials.[21]

Substitution of (54) into (37) with (57) taken into account gives the following system of equations for the determination of the coefficients A_l and B_l:

$$\begin{aligned}(l+1)(l+2)A_l&=\sum_{l_1l_2l_3}A_{l_1}B_{l_2}B_{l_3}C^l_{l_1l_2l_3},\\ (l+1)(l+2)B_l&=\sum_{l_1l_2l_3}(A_{l_1}A_{l_2}B_{l_3}+\varkappa B_{l_1}B_{l_2}B_{l_3})C^l_{l_1l_2l_3},\end{aligned} \tag{59}$$

where the summation is carried out over the integral values of l_1, l_2, l_3, which satisfy the conditions $l_1+l_2+l_3\geqslant l$, with A_l being taken different from zero only for odd, and B_l only for even values of the subscript l. The coefficients $C^l_{l_1l_2l_3}$ which are analogous to the Clebsch-Gordan coefficients for three-dimensional space are determined by the following formulas

$$\begin{aligned}Q^{l_1}Q^{l_2}Q^{l_3}&=\sum_l C^{2l}_{2l_1,2l_2,2l_3}Q^l,\quad Q^{l_1}Q^{l_2}Q^{l_3}=\sum_l C^{2l+1}_{2l_1,2l_2,2l_3+1}P^l,\\ sP^{l_1}P^{l_2}Q^{l_3}&=\sum_l C^{2l}_{2l_1+1\,2l_2+1\,2l_3}Q^l.\end{aligned} \tag{60}$$

The method of evaluating the coefficients $C^l_{l_1l_2l_3}$ is discussed in the Appendix.

Equation (48) for the determination of $\varkappa$ can be rewritten with the aid of Eqs. (51) in the form

$$\begin{aligned}1+\frac{2k}{m}=&\left\{\int ds(1-s)^{1/2}\left(\frac{s}{1-s}\right)^{n_\perp/2-1}[2s(1-s)\Phi'^2+\Phi^2]\right\}\\ &\times\left\{\int ds(1-s)^{1/2}\left(\frac{s}{1-s}\right)^{n_\perp/2-1}[2s^2(1-s)a'^2+3sa^2]\right\}^{-1}.\end{aligned} \tag{61}$$

Substituting into it expressions (54) and utilizing the orthogonality relations for the functions P^l and Q^l (cf., Appendix) we obtain

$$\begin{aligned}1+\frac{2k}{m}=&\left\{\sum_{l=0}^{\infty}B_{2l}{}^2(l+1)(2l+1)\frac{\pi L!(2l+n_\perp/2)!}{2^{4l}\Gamma(2l+{}^3/_2)\Gamma(2l+{}^5/_2)}\right\}\\ &\times\left\{\sum_{l=0}^{\infty}A_{2l+1}(l+1)(2l+3)\frac{\pi L!(2l+2+n_\perp/2)!}{2^{4l+2}\Gamma(2l+{}^5/_2)\Gamma(2l+{}^7/_2)}\right\}^{-1}.\end{aligned} \tag{62}$$

In a similar manner formula (53) for $C(m/k)$ can be written in the form

$$\begin{aligned}C\left(\frac{m}{k}\right)=&\left(2\varkappa\frac{k}{m}\right)^{-k/(k+m)}\left[\frac{1}{2n_\perp}\sum_{l=0}^{\infty}A^2_{2l+1}(l+1)(2l+3)\right.\\ &\left.\times\frac{\pi L!(2l+2+n_\perp/2)!}{2^{4l+4}\Gamma(2l+{}^5/_2)\Gamma(2l+{}^7/_2)}\right]^{-1}.\end{aligned} \tag{63}$$

Thus, for the evaluation of the function $C(m/k)$ we must find the coefficients A_l, B_l as functions of $\varkappa$ from the system (59), and then determine $\varkappa$ from equation (62) and utilize formula (63).

For an approximate solution of the system (59) we assume that the principal contribution is given by the lowest harmonics; this is justified by the subsequent calculation of corrections. In the lowest approximation we retain in the system (59) only A_1 and B_0:

$$6A_1\approx B_0{}^2A_1,\quad 2B_0\approx{}^1/_5n_\perp A_1{}^2B_0+\varkappa B_0{}^3. \tag{64}$$

From this we obtain

$$\begin{aligned}&B_0\approx 6^{1/2},\quad A_2\approx\left[\frac{10}{n_\perp}(1-3\varkappa)\right]^{1/2};\\ &B_{2l}\approx 0,\quad A_{2l+1}\approx 0\quad (l\geqslant 1).\end{aligned} \tag{65}$$

TABLE I. Values of the function $C(m/k)$ in formula (52).

$C(m/k)$	$n_\perp$	m/k						
		0	0.2	0.5	1	2	5	∞
Exact values	2	3	4.376	4.832	4.595	3.780	2.574	1.103
	4	3	4.368	4.796	4.515	3.657	2.439	1.016
According to formula (67)	–	3	4.367	4.789	4.500	3.634	2.414	1

Substitution of (65) into (62) yields

$$1+\frac{2k}{m}=\frac{B_0^2}{A_1^2}\frac{5\cdot 4}{(n_\perp+2)(n_\perp+4)}=\frac{1}{1-3\varkappa}. \tag{66}$$

Here we have utilized the fact that according to (33) $n_\perp$ assumes two values. Thus, in the approximation under consideration $\varkappa$ does not depend on $n_\perp$. From (63) we obtain

$$C\left(\frac{m}{k}\right)\approx\left(1+\frac{2k}{m}\right)^{(2k+m)/(k+m)}\left(\frac{3m^2}{4k^2}\right)^{k/(k+m)}\to\begin{cases}3, & m\ll k\\ 1, & m\gg k\end{cases}. \tag{67}$$

Both saddle points (33) give the same contribution in this approximation. In order to calculate the corrections to expression (67) one must substitute expressions (65) into the right-hand side of (59), the values of A_l and B_l obtained in this step must again be substituted into the right-hand side of (59), etc. Utilizing then formulas (62) and (63) one can calculate $C(m/k)$ with any given prescribed accuracy.

In Table I we have given the function $C(m/k)$ for two cases: $n_\perp=2$ and $n_\perp=4$. As can be seen from Table I

$$C(m/k)\,|_{n_\perp=2}>C(m/k)\,|_{n_\perp=4} \tag{68}$$

for all values of m/k (except for the zero value). This signifies that the asymptotic behavior of the coefficients of the perturbation theory series in scalar electrodynamics is determined by the saddle-point function (36) with $n_\perp=2$. The solutions of the classical equations with $n_\perp=4$ give only exponential corrections to the asymptotic formulas.

At the same time the table shows that the difference in the values of $C(m/k)$ for the two cases is not great. This is associated with the fact that the determining factor is the contribution of the first approximation (67), while the difference is determined only by corrections to it. To make the point explicitly, the table contains values of $C(m/k)$ given by formula (67); comparison of them with the exact values shows that the iteration process being used converges fairly quickly. The closeness of the values of $C(m/k)$ for the two cases shows that although in a strictly asymptotic sense ($k, m\to\infty$) the contribution of the saddle-point with $n_\perp=4$ is exponentially small, at the same time there exists a fairly broad region of not very great values of k, m in which the contributions of both saddle-points are comparable. Therefore for a comparison of the asymptotic formulas with the exact calculations of lowest orders of perturbation theory it makes sense to consider the fluctuations near both classical solutions.

In the region $m\ll k$, where $\varkappa\to 1/3$, the difference between the values of $C(m/k)$ in the two cases becomes negligibly small. In this limit one can develop a perturbation theory with respect to the small parameter $(1/3)-\varkappa\sim m/k$. Writing a and Φ in the form of the series

$$a=\sum_{r=0}^{\infty}a_r\left(\frac{1}{3}-\varkappa\right)^{r+1/2},\quad \Phi=\sum_{r=0}^{\infty}\Phi_r\left(\frac{1}{3}-\varkappa\right)^{r},\quad \Phi_0=6^{1/2}, \tag{69}$$

we obtain from (37) the following recurrence equations for the coefficients a_r, Φ_r:

$$-4s(1-s)a_r''+2(7s-2-n_\perp)a_r'=\sum_{p=1}^{r}a_{r-p}\sum_{q=0}^{p}\Phi_q\Phi_{p-q},$$

$$-4s(1-s)\Phi_r''+2(5s-n_\perp)\Phi_r'-4\Phi_r=\Phi_0\sum_{p=1}^{r-1}\Phi_p\Phi_{r-p} \tag{70}$$

$$+\frac{1}{3}\sum_{p=1}^{r-1}\Phi_{r-p}\sum_{q=1}^{p-1}\Phi_q\Phi_{p-q}-\sum_{p=0}^{r-1}\Phi_{r-1-p}\sum_{q=0}^{p}(\Phi_q\Phi_{p-q}-sa_qa_{p-q}).$$

Solving them sequentially taking into account the boundary conditions (39) we obtain

$$\Phi_0=6^{1/2},\quad a_0=3\left(\frac{14}{5n_\perp-4}\right)^{1/2};\quad \Phi_1=\frac{a_0^2}{6^{1/2}}\left(s-\frac{n_\perp+2}{7}\right),$$

$$a_1=\frac{a_0^3}{7}\left(s-1-\frac{32-5n_\perp}{27\cdot 28}\right); \tag{71}$$

$$\Phi_2=\frac{6^{1/2}a_0^4}{9\cdot 49}\left[\frac{91}{8}s^2-\frac{s}{9}\left(\frac{113}{2}n_\perp+92\right)+\frac{1}{18}\left(\frac{241}{2}n_\perp-101\right)\right]$$

etc. From this one can find, using formulas (48) and (53), the expansion for $C(m/k)$:

$$\left[C\left(\frac{m}{k}\right)\right]^{k+m}=\left(\frac{m}{2k}\right)^{-m}3^k\exp\left\{m\left(1+\frac{m}{21k}\left(\frac{17}{4}+\frac{5}{n_\perp}\right)+\ldots\right)\right\}. \tag{72}$$

From this formula it can be seen that also in the limit $m\ll k$ the inequality (68) holds, but the difference between its parts turns out to be of the second order of smallness in terms of m/k.

We note that in the region $m\ll k$ the functional integral (20) contains many other saddle-points the number of which increases without limit as $m/k\to 0$. This is associated with the fact that equations (37) for small $\varkappa$ also have other solutions different from (69). Indeed, near the points $\varkappa_n=1/n(2n+1)$ we can seek the solution in the form of an expansion analogous to (69) in powers of $(\varkappa_n-\varkappa)$, setting $\Phi_0=[2n(2n+1)]^{1/2}$. In this case we obtain

$$a\sim(\varkappa_n-\varkappa)^{1/2}P^{n-1}(s),\quad \varkappa_n-\varkappa\sim m/k,$$
$$[C(m/k)]^{k+m}\approx(m/k)^{-m}3^k e^m(6/n(2n+1))^m. \tag{73}$$

Comparing this with (72) we see that the additional solutions lead to lower values of $C(m/k)$, i.e., the corresponding saddle-points are not significant in the continual integral.

5. CONCLUSION

The principal result of the present paper is formula (52) which gives the asymptotic behavior of the expansion coefficients of the Green's functions in perturbation theory series with an accuracy up to the factor in front

of the exponential. The latter is determined by the quantum fluctuations around the saddle-point values of the fields A_γ, φ and of the parameters e_μ^2, g_μ; we expect to investigate them in the near future. However, one can note from formula (20) that since $\tilde{A}_\nu \sim \tilde{\varphi} \sim k^{1/2}$, $\tilde{e}_\mu^2 \sim \tilde{g}_\mu \sim 1/k$ the factor in front of the exponential must be of the form

$$\text{const} \cdot k^{1/2(M+2N+r)-1},$$

where $M+2N$ is the total number of the tails of the Green's function, while $r=12$ is the number of parameters on which depends the solution of the classical equations (they are the parameters λ, $(x_0)_i$, χ and η_{ij} in formulas (18) and (36)). Thus it is necessary to take quantum fluctuations into account only for the exact determination of the constant (depending on the ratio m/k) and of corrections which diminish as $k \to \infty$.

In the present paper in order to find the form of the solutions of classical equations a method based on a special form of perturbation theory was utilized. In the general form this method consists of the fact that for an interaction containing two constants high orders are considered with respect to one constant and relatively low ones with respect to the other constant (of the type e in scalar electrodynamics). In the lowest approximation the interaction proportional to the second constant is omitted. If the resulting equations have a solution then in the next approximation one finds such a critical value of the second constant for which a nontrivial solution exists for the homogeneous equation for the case of a field interacting only because of the presence of this constant (cf., the first of equations (15)). The solution obtained in this case enables one to determine the form of the solution in the general case.

If one applies this method to the theory of the Yang-Mills fields interacting with a scalar selfacting field solutions are found for which the projection of the vector potential A_i^a on the first five-dimensional harmonic $\eta_{ij}^a \sim \int dS_5 A_i^a z_j$ is a generator of the group of isotopic transformations in a certain subspace of the five-dimensional space. Nontrivial realizations of these real antisymmetric matrices η_{ij}^a are the cases for $n_\perp = 3, 4, 5$ corresponding respectively to the representation with isospin $T=1$, the reducible representation derived from two irreducible representations with isospins $T=\frac{1}{2}$ and, finally, the representation with $T=2$. The form of the matrix η_{ij}^a uniquely characterizes the solution for the fields A_i^a and Y which in these cases can be expressed in terms of functions of a single variable. In order to choose the solution determining the asymptotic behavior of the perturbation theory series it is necessary to intercompare the numerical values of the action along these three solutions.

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APPENDIX

The functions $P^l(s)$, $Q^l(s)$ (50) satisfy the recurrence relations

$$P^l(s) = Q^l(s) - b_{2l} P^{l-1}(s), \quad Q^l(s) = sP^{l-1}(s) - b_{2l-1} Q^{l-1}(s), \tag{A.1}$$

where

$$b_{2l} = \frac{2l(2l+3-n_\perp)}{(4l+1)(4l+3)}, \quad b_{2l-1} = \frac{(2l+1)(2l-2+n_\perp)}{(4l-1)(4l+1)}. \tag{A.2}$$

The orthogonality relations for them have the form

$$\int_0^1 ds (1-s)^{1/2} \left(\frac{s}{1-s}\right)^{n_\perp/2-1} Q^l Q^{l_1} = \frac{2}{l(2l+3)} \int_0^1 ds (1-s)^{1/2} \left(\frac{s}{1-s}\right)^{n_\perp/2-1}$$
$$\times s(1-s)(Q^l)'(Q^{l_1})' = \delta_{ll_1} \frac{\pi L!(2l+n_\perp/2)!}{2^{4l+2}\Gamma(2l+3/2)\Gamma(2l+5/2)};$$
$$L = 2l+2-n_\perp/2; \tag{A.3}$$
$$\int_0^1 ds (1-s)^{1/2} \left(\frac{s}{1-s}\right)^{n_\perp/2-1} sP^l P^{l_1} = \frac{2}{l(2l+5)} \int_0^1 ds (1-s)^{1/2}$$
$$\times \left(\frac{s}{1-s}\right)^{n_\perp/2-1} s^2(1-s)(P^l)'(P^{l_1})' = \delta_{ll_1} \frac{\pi L!(2l+2+n_\perp/2)!}{2^{4l+4}\Gamma(2l+5/2)\Gamma(2l+7/2)}.$$

If we denote

$$Q^l(s) = R_{2l}(s), \quad s^{1/2}P^l(s) = R_{2l+1}(s), \tag{A.4}$$

then formulas (52) assume the form

$$R_{l_1} R_{l_2} R_{l_3} = \sum_l C_{l_1 l_2 l_3}^l R_l. \tag{A.5}$$

The coefficients $C_{l_1 l_2 l_3}^l$ are symmetric with respect to a permutation of the lower indices and differ from zero only when the sum of the four indices is even. They can be represented in the form

$$C_{l_1 l_2 l_3}^l = \sum_k C_{l_1 l_2}^k C_{k l_3}^l \tag{A.6}$$

where the coefficients $C_{l_1 l_2}^l$ are determined by the formula

$$R_{l_1} R_{l_2} = \sum_l C_{l_1 l_2}^l R_l. \tag{A.7}$$

The coefficients $C_{l_1 l_2}^l$ are also symmetric with respect to their lower indices and vanish when the sum of all the indices is odd, and moreover when the triangular condition: $|l_1 - l_2| \leq l \leq l_1 + l_2$ is not satisfied.

The explicit form of the coefficients $C_{l_1 l_2}^l$ can be obtained from formulas (50) and (A.3) with the aid of the relations (cf., Ref. 21, 22)

$$\int_{-1}^1 dz (1-z^2)^{\nu-1/2} C_{l_1}^\nu(z) C_{l_2}^\nu(z) C_{l_3}^\nu(z)$$
$$= \frac{2^{1-2\nu}\pi\Gamma(\sigma+2\nu)\Gamma(\sigma-l_1+\nu)\Gamma(\sigma-l_2+\nu)\Gamma(\sigma-l_3+\nu)}{\Gamma^4(\nu)\Gamma(\sigma+\nu+1)(\sigma-l_1)!(\sigma-l_2)!(\sigma-l_3)!},$$
$$2\sigma = l_1+l_2+l_3; \tag{A.8}$$

$$\int_{-1}^1 \frac{dz}{z}(1-z^2)^{\nu-1/2} C_{2l+1}^\nu(z) C_{2l_1}^\nu(z) = \begin{cases} (-1)^{l-l_1} \dfrac{\pi\Gamma(l+\nu+1/2)\Gamma(l_1+\nu)}{\Gamma^2(\nu) l_1! \Gamma(l+3/2)}, & l \geq l_1 \\ 0, & l < l_1 \end{cases} \tag{A.9}$$

We shall not write out the expressions thus obtained since they are quite awkward and not convenient for use.

It is simpler to calculate the coefficients $C^l_{l_1 l_2}$ with the aid of the recurrence relation

$$C^l_{l_1 l_2} = C^{l-1}_{l_1 l_2 - 1} + b_{l+1} C^{l+1}_{l_1 l_2 - 1} - b_{l_2 - 1} C^l_{l_1 l_2 - 2}, \tag{A.10}$$

which follows from formulas (A.1), (A.4) and (A.7); to this one should add the initial condition $C^l_{l_0} = 1$. In particular we obtain from this the following expressions

$$\begin{aligned} &C_{l_1}^{l+1} = 1, \quad C_{l_1}^{l-1} = b_l; \\ &C_{l_2}^{l+2} = 1, \quad C_{l_2}^{l} = b_l + b_{l+1} - b_1, \quad C_{l_2}^{l-2} = b_l b_{l-1}; \\ &C_{l_3}^{l+3} = 1, \quad C_{l_3}^{l+1} = b_l + b_{l+1} + b_{l+2} - b_1 - b_2, \\ &C_{l_3}^{l-1} = b_l (b_{l-1} + b_l + b_{l+1} - b_1 - b_2), \quad C_{l_3}^{l-3} = b_l b_{l-1} b_{l-2} \end{aligned} \tag{A.11}$$

etc.

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Mobility and chemical bond of hydrogen in titanium and palladium hydrides

A. K. Kachalkin, Z. V. Krumshtĕin, V. I. Petrukhin, V. M. Suvorov, D. Horvath, and I. A. Yutlandov

Joint Institute of Nuclear Research
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The probabilities for π^- meson capture by hydrogen are measured at 25, 155 and 200°C in the hydride $TiH_{1.65}$ at 25–120 and −196°C in the hydride $PdH_{0.67}$. An analysis of the results shows that, within the accuracy of the measurement (~10%), a sharp change (up to 10^{12}) in the mobility of hydrogen in the hydrides, induced by temperature changes in the ranges indicated, does not noticeably affect the probabilities for π^- meson capture by bound hydrogen, i.e., does not lead to appreciable changes in the Me–He bond. A comparison of the capture probabilities for palladium hydride and the hydrides of neighboring transition metals shows that there are no pronounced anomalies in the Pd–H bond.

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At the present time there are two opposing hypotheses for explanation of the character of the hydrogen bond in transition metal hydrides[1,2]; the proton hypothesis (the hydrogen in the hydride is principally in the form of the proton H^+) and the hydride hypothesis (existence of hydrogen in the form of the hydride ion H^-). The hydride hypothesis is successfully used for the calculation of crystalline lattices and interionic distances. The proton hypothesis in turn allows us to explain the behavior of the hydrogen in diffusion and relaxation processes.[1] There is no single experiment at the present time which would disprove one hypothesis or the other.[2]

Earlier, it was shown experimentally that the probability of capture of pions by bound hydrogen is sensitive to the features of the chemical bond of the hydrogen in the molecules.[3–5] In the present research, we made an attempt to discover the effect of a change in the hydrogen

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