Conformal symmetry in a two-dimensional Heisenberg ferromagnet

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Conformal symmetry in a system with a finite correlation length r_c can be restored if a transformation of r_c is made simultaneously with the coordinate transformation. Ward identities for the many-point correlators of the energy density of a two-dimensional ferromagnet are a consequence of this. Conformal invariance in the new formulation is proved to first order in $1/N$ (N is the number of spin components). It is shown how the proof can be carried through to any order in $1/N$. A phenomenological Hamiltonian $F_c[r_c]$ is found for a system in weakly nonuniform external conditions.

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1. CONFORMAL SYMMETRY IN A SYSTEM WITH FINITE CORRELATION LENGTH. THE 1/N EXPANSION METHOD

We consider a two-dimensional Heisenberg ferromagnet with the Hamiltonian

$$H=\frac{1}{2}\int J(\partial_\mu \mathbf{n})^2 d^2x, \tag{1}$$

where $J>0$ is the exchange integral and $\mathbf{n}$ is an N-component vector; $|\mathbf{n}|=1$. The Hamiltonian possesses scale and conformal symmetry, i.e., it is covariant under the transformations

$$\mathbf{x}\to\lambda\mathbf{x}, \tag{2}$$

$$\mathbf{x}\to 1/\mathbf{x}=\mathbf{x}/x^2. \tag{3}$$

Conformal symmetry at the phase-transition point has been studied by Polyakov[1] (see also Ref. 2, Chap. 2, Sec. 4). Equations for the many-point correlators, analogous to the Gross-Wess equations, were found. Away from the transition point it is not possible to repeat the same arguments word-for-word, since there is a length-scale—the correlation length. In a two-dimensional spin system,[3]

$$r_c\sim\Lambda^{-1}\exp\left\{-\int\limits_T^\infty\frac{dT'}{W_0(T')}\right\}, \tag{4}$$

where $W_0(T)$ is the Gell-Mann–Low function, Λ is the cutoff parameter—the momentum at the Brillouin-zone boundary, and T is the temperature in units of J. In the first logarithmic approximation,[4]

$$W_0(T)=-\frac{N-2}{2\pi}T^2+O(T^3),$$
$$r_c=\Lambda^{-1}\exp\{2\pi/(N-2)T\}. \tag{5}$$

It is natural to expect that the symmetry (2), (3) could be restored at least partially, if, simultaneously with the coordinate transformations, we make the corresponding change in the correlation length:

$$\mathbf{x}\to\lambda\mathbf{x},\ r_c\to\lambda r_c; \tag{2$'$}$$

$$\mathbf{x}\to 1/\mathbf{x},\ r_c\to r_c/x^2. \tag{3$'$}$$

We shall study the requirements that the symmetry (2′), (3′) imposes on the free energy as a function of temperature.

For this it is most convenient to use the $1/N$ expansion method,[3] since in this case r_c appears explicitly in the Hamiltonian. The free energy has the form

$$F=\ln\int D\mathbf{n}\,\delta(|\mathbf{n}|-1)\exp\left\{-{}^1/_2\beta\int(\nabla\mathbf{n})^2d^2x\right\}. \tag{6}$$

Here $\beta=1/T$, and for brevity the factor $-T$ has been omitted in F. We represent F in the form of the sum of a regular part F_r and a fluctuational part F_c, such that F_c depends only on r_c and therefore possesses the symmetry (2′), (3′).

We make the change of variables $\mathbf{n}\to\beta^{1/2}\mathbf{n}$ and use an integral representation of the δ-function:

$$F=-\frac{N-2}{2}\int\Lambda^2d^2x\ln\beta+\ln\int D\mathbf{n}\,\delta(n^2-1)\exp\left\{-\frac{1}{2}\int(\nabla\mathbf{n})^2d^2x\right\}$$
$$=-\frac{N-2}{2}\int\Lambda^2d^2x\ln\beta+\ln\int D\mathbf{n}\int\limits_{-i\infty}^{i\infty}\frac{D\varphi}{2\pi i}\exp\left\{-\frac{1}{2}\int[(\nabla\mathbf{n})^2+\varphi n^2-\beta\varphi]d^2x\right\}. \tag{7}$$

The first term corresponds to the energy of $N-2$ noninteracting spin waves. It is obvious that the field φ has the meaning of the energy density ($\langle\varphi\rangle\sim\delta F/\delta\beta$), or, to be more exact (cf. (9)), its fluctuational component.

Following the standard procedure for expanding in $1/N$,[3] we integrate over $D\mathbf{n}$ and obtain, to within constants,

$$F=-\frac{N-2}{2}\int\Lambda^2d^2x\ln\beta$$
$$+\ln\int\limits_{-i\infty}^{i\infty}D\varphi\exp\left\{-\frac{1}{2}\int\left[\Lambda^2N\,\mathrm{Tr}\ln\left(\frac{-\Delta+\varphi}{\Lambda^2}\right)-\beta\varphi\right]d^2x\right\}. \tag{8}$$

By expanding the second term in a series at the saddle point[1)]

$$\varphi_0=\Lambda^2e^{-4\pi\beta/N}, \tag{9}$$

we obtain finally

$$F=-\frac{N-2}{2}\int\Lambda^2 d^2x\ln\beta+\frac{N}{8\pi}\int\Lambda^2 d^2x-\frac{N}{8\pi}\int\varphi_s d^2x$$
$$+\ln\int_{-i\infty}^{i\infty}D\varphi\exp\left\{\frac{N}{2}\int\sum_{n=2}^{\infty}\frac{(-1)^n}{n}\mathrm{Tr}\left(\frac{1}{-\Delta+\varphi_s}\right)\varphi^n d^2x\right\}. \qquad (10)$$

In Sec. 3 it will be proved that to order $1/N$ the first two terms are the regular part of the free energy and the others are the fluctuational part.

We see that $\langle\varphi\rangle=\varphi_s\neq 0$. The presence of the nonzero average violates the conformal invariance of the Hamiltonian in (7). If, simultaneously with the coordinate transformation (2), (3), we carry out the corresponding transformation of φ_s, the symmetry can be restored. The fact that $\varphi_s^{-1/2}$ is the correlation length for fluctuations of n leads us in a natural way to the formulation (2′), (3′).

2. CONFORMAL WARD IDENTITIES

In this section we shall consider the consequences of the invariance of the free energy F_c of the fluctuations under the scale and conformal transformations (2′), (3′) for the functions

$$\Gamma^n(\mathbf{x}_1\ldots\mathbf{x}_n,\beta,\Lambda)=\langle\langle\varphi(\mathbf{x}_1)\ldots\varphi(\mathbf{x}_n)\rangle\rangle=\frac{\delta^n F_c[\beta,\Lambda]}{\delta\beta(\mathbf{x}_1)\ldots\delta\beta(\mathbf{x}_n)}, \qquad (11)$$

which have the meaning of many-point correlations of the energy density. By its definition, F_c depends on β and Λ only in the combination

$$r_c(\beta,\Lambda)=\Lambda^{-1}\exp\left\{-\int\frac{dT}{W_0(T)}\right\},\quad T=\frac{1}{\beta}.$$

Therefore, we investigate first the auxiliary functions

$$\mathcal{T}^n(\mathbf{x}_1\ldots\mathbf{x}_n,r_c)=\frac{\delta^n F_c[r_c]}{\delta\ln r_c(\mathbf{x}_1)\ldots\delta\ln r_c(\mathbf{x}_n)}=\frac{1}{r_c^{2n}}\bar{\mathcal{T}}^n\left(\frac{\mathbf{x}_1}{r_c}\ldots\frac{\mathbf{x}_n}{r_c}\right) \qquad (12)$$

($\bar{\mathcal{T}}^n$ is a certain universal function). Here we have expressed explicitly the fact that the scaling dimension of $\mathcal{T}^n$ is equal to $2n$,[2)] while the only dimensional parameter is r_c. By virtue of our choice of it, $F_c[r_c]$ should remain invariant under the transformations (2′), (3′). This implies that the $\mathcal{T}^n$ are also invariant under these transformations, to within a factor determined by the scaling dimension. From this follow Ward identities for the $\mathcal{T}^n$. We make the transformation (2′) with $\lambda=1+\alpha$, $|\alpha|\ll 1$; as a result,

$$\delta\mathcal{T}^n=-2n\alpha\mathcal{T}^n=\alpha\left[\sum_{i=1}^n x_{i\mu}\frac{\partial\mathcal{T}^n}{\partial x_{i\mu}}+r_c\frac{\partial\mathcal{T}^n}{\partial r_c}\right]. \qquad (13)$$

Making use of the trivial relation

$$\frac{\partial}{\partial r_c}=\int d^2x\frac{\delta}{\delta r_c(\mathbf{x})},$$

we obtain the first set of Ward identities:

$$\sum_{i=1}^n\left(x_{i\mu}\frac{\partial\mathcal{T}^n}{\partial x_{i\mu}}+2\mathcal{T}^n\right)=-\int d^2x\,\mathcal{T}^{n+1}(\mathbf{x}_1\ldots\mathbf{x}_n,\mathbf{x}). \qquad (14)$$

The second set, corresponding to the conformal transformations, has the form

$$\sum_{i=1}^n x_{i\mu}x_{i\nu}\frac{\partial\mathcal{T}^n}{\partial x_{i\mu}}-\frac{x_\mu^2}{2}\frac{\partial\mathcal{T}^n}{\partial x_{i\nu}}+2x_{i\nu}\mathcal{T}^n=-\int d^2x\,x_\nu\mathcal{T}^{n+1}(\mathbf{x}_1\ldots\mathbf{x}_n,\mathbf{x}). \qquad (15)$$

It is obtained analogously to (14) from the infinitesimal conformal transformation

$$x_\mu\to x_\mu+x_\mu\alpha\mathbf{x}-{}^1/_2\alpha_\mu x^2,\quad r_c\to r_c(1+\alpha\mathbf{x}), \qquad (3'')$$

which is a superposition of the inversion (3′), translation through the vector $\boldsymbol{\alpha}$ ($|\alpha|\ll 1$) and the reverse inversion.[1]

To transform to the correlators Γ^n (11), we make use of the formula (4) connecting r_c and T. We introduce the function

$$W(\beta)=\frac{d\beta}{dT}W_0(T)=-\beta^2 W_0\left(\frac{1}{\beta}\right). \qquad (16)$$

Then $r_c\delta/\delta r_c=W(\beta)\delta/\delta\beta$, and, to within terms of second order in T, we obtain

$$\mathcal{T}^n=\left(r_c\frac{\delta}{\delta r_c}\right)^n F_c=\left(W(\beta)\frac{\delta}{\delta\beta}\right)^n F_c$$
$$=W^n(\beta)\Gamma^n+W^{n-1}(\beta)\frac{\partial W(\beta)}{\partial\beta}\sum_{i\neq j}\delta(\mathbf{x}_i-\mathbf{x}_j)\Gamma^{n-1}(\mathbf{x}_1\ldots\mathbf{x}_n|\mathbf{x}_i).$$

The notation $\Gamma^{n-1}(\mathbf{x}_1\ldots\mathbf{x}_n|\mathbf{x}_i)$ implies that the point $\mathbf{x}_i$ is excluded from the set of arguments $\mathbf{x}_1,\ldots,\mathbf{x}_n$.

If we substitute these expressions for $\mathcal{T}^n$ into the identities (14) and (15) we obtain the following relations for Γ^n:

$$\sum_{i=1}^n x_{i\mu}\frac{\partial\Gamma^n}{\partial x_{i\mu}}+(2+W'(\beta))\Gamma^n=-W(\beta)\int\Gamma^{n+1}(\mathbf{x}_1\ldots\mathbf{x}_n,\mathbf{x})d^2x, \qquad (17)$$

$$\sum_{i=1}^n x_{i\mu}x_{i\nu}\frac{\partial\Gamma^n}{\partial x_{i\mu}}-\frac{x_{i\mu}^2}{2}\frac{\partial\Gamma^n}{\partial x_{i\nu}}+x_{i\nu}(2+W'(\beta))\Gamma^n$$
$$=-W(\beta)\int x_\nu\Gamma^{n+1}(\mathbf{x}_1\ldots\mathbf{x}_n,\mathbf{x})d^2x, \qquad (18)$$

which are correct to $O(\beta^{-2})$.

By replacing

$$x_\mu\to i\frac{\partial}{\partial q_\mu},\quad -i\frac{\partial}{\partial x_\mu}\to q_\mu,$$

we go over to the Fourier transforms in the equalities (17) and (18):

$$\sum_{i=1}q_{i\mu}\frac{\partial\Gamma^n}{\partial q_{i\mu}}-W'(\beta)\Gamma^n=W(\beta)\Gamma^{n+1}(\mathbf{q}_1\ldots\mathbf{q}_n,0), \qquad (17')$$

$$\sum_{i=1}^n q_{i\mu}\frac{\partial^2\Gamma^n}{\partial q_{i\mu}\partial q_{i\nu}}-\frac{q_{i\nu}}{2}\frac{\partial^2\Gamma^n}{\partial q_{i\mu}^2}-W'(\beta)\frac{\partial\Gamma^n}{\partial q_{i\nu}}$$
$$=W(\beta)\frac{\partial\Gamma^{n+1}}{\partial q_\nu}(\mathbf{q}_1\ldots\mathbf{q}_n,\mathbf{q})|_{\mathbf{q}=0}. \qquad (18')$$

3. WARD IDENTITIES IN PERTURBATION THEORY IN 1/*N*

In this section we verify that the Ward identities are fulfilled to first order in $1/N$. Thereby, it will be proved that the fluctuational and regular parts of the free energy (8) were found correctly to $O(N^{-2})$. It will then be explained how we can carry out the analogous proof in any order in $1/N$.

In first order the Ward identities in the momentum representation will be verified by direct differentiation. Since the computations are cumbersome, we shall prove a certain analog of (17) and (18) rather than (17) and (18) directly. The point is that the correlators Γ^n in perturbation theory are determined by the complete set of one-particle-reducible graphs with n external lines, obtained from the Hamiltonian for φ (see the last two terms of (10)). Therefore, it is natural to make a Legendre transformation and change from F to a new thermodynamic potential $\tilde{F}$, analogous to the energy.[3] This enables us to confine ourselves to considering only the one-particle-irreducible vertices of the Hamiltonian (see the Appendix of the English translation of Ref. 2).

In accordance with (8) and (10), we write

$$F_c[\beta+\delta\beta]=\ln\int_{-i\infty}^{i\infty}D\varphi\exp\Big\{-\frac{1}{2}\int\Big[\Lambda^2 N\,\mathrm{Tr}\ln\Big(\frac{-\Delta+\varphi}{\Lambda^2}\Big)-(\beta+\delta\beta)\varphi\Big]d^2x\Big\}-\frac{N}{4\pi}\int\Lambda^2 d^2x \tag{19}$$

and define

$$\tilde{F}_c[\varphi]=F_c[\beta+\delta\beta(\varphi)]-\int\delta\beta\frac{\delta F_c}{\delta\beta}d^2x. \tag{20}$$

We shall prove the Ward identities for the quantities

$$P^n(\mathbf{x}_1\ldots\mathbf{x}_n)=\frac{\delta\tilde{F}_c}{\delta\varphi(\mathbf{x}_1)\ldots\delta\varphi(\mathbf{x}_n)}=\langle\langle\beta(\mathbf{x}_1)\ldots\beta(\mathbf{x}_n)\rangle\rangle \tag{21}$$

(β and φ are canonically conjugate variables). Fulfillment of the identities for P^n will imply the conformal invariance (2′), (3′) for $\tilde{F}_c$ and, consequently, for F_c.

In the $\mathbf{k}$-representation the identities for P^n have the following appearance:

$$\sum_{i=1}^{n}q_{i\mu}\frac{\partial P^n}{\partial q_{i\mu}}+2P^n=-2\varphi_s P^{n+1}(\mathbf{q}_1\ldots\mathbf{q}_n,0), \tag{22}$$

$$\sum_{i=1}^{n}q_{i\mu}\frac{\partial^2 P^n}{\partial q_{i\mu}\partial q_{i\nu}}-\frac{q_{i\nu}}{2}\frac{\partial^2 P^n}{\partial q_{i\mu}^2}+2\frac{\partial P^n}{\partial q_{i\nu}}=-2\varphi_s\frac{\partial P^{n+1}(\mathbf{q}_1\ldots\mathbf{q}_n,\mathbf{q})}{\partial q_\nu}\Big|_{\mathbf{q}=0}. \tag{23}$$

They are derived in the $\mathbf{x}$-representation in analogy with (14) and (15) ($\varphi_s=r_c^{-2}$) and are then transformed to the $\mathbf{k}$-representation by replacing

$$x_\mu\to i\partial/\partial q_\mu,\quad \partial/\partial x_\mu\to iq_\mu.$$

We not convince ourselves by direct calculation that the relations (22) and (23) are fulfilled to first order in $1/N$. In this approximation, the fluctuations are neglected in $F_c[\varphi]$ and the correlators (21) are determined by the vertices of the Hamiltonian for φ

$$(-1)^n n^{-1}\mathrm{Tr}(-\Delta+\varphi_s)^{-n}$$

(see the last term in (10)) at the saddle point. These vertices have the form

$$P^n(\mathbf{q}_1\ldots\mathbf{q}_n)=(-1)^n\delta\Big(\sum_i\mathbf{q}_i\Big)N\sum{}'\int\frac{d^2p}{(2\pi)^2}\prod_{l=1}^{n}\frac{1}{(\mathbf{p}+\mathbf{k}_l)^2+\varphi_s}, \tag{24}$$

where $\sum'$ denotes a sum over the permutations of the $\mathbf{q}_i$, and

$$\mathbf{k}_m=\sum_{j=1}^{m}\mathbf{q}_{i_j}$$

(the index j labels the position of the momentum $\mathbf{q}_i$ in a permutation). Because of the summation over all the permutations of $(\mathbf{q}_1,\ldots,\mathbf{q}_n)$ the vertex P^n is symmetric in all the variables. In the calculation of

$$F_c[\varphi]=\sum_{n=2}^{\infty}\int\frac{1}{n!}P^n\prod_{i=1}^{n}\varphi_{\mathbf{q}_i}d^2q_i$$

the P^n in the form (24) give (10). Everywhere below we shall denote $\mathbf{p}+\mathbf{k}_i=\mathbf{K}_i$ and omit the factor $(2\pi)^{-2}N$ and the summation sign.

We first prove (22):

$$\begin{aligned}
&\sum_{i=1}^{n}q_{i\mu}\frac{\partial}{\partial q_{i\mu}}\Big((-1)^nP^n\Big/\delta\Big(\sum\mathbf{q}_i\Big)\Big)\\
&=\int d^2p\sum_{i=1}^{n}q_{i\mu}\sum_{i\le j}\frac{-2K_{j\mu}}{K_j^2+\varphi_s}\prod_{l=1}^{n}\frac{1}{K_l^2+\varphi_s}\\
&=-\int d^2p\sum_{i=1}^{n}2\frac{K_{i\mu}k_{i\mu}}{K_i^2+\varphi_s}\prod_{l=1}^{n}\cdots\\
&=\int d^2p\Big[\sum_{i=1}^{n}\Big(-2+\frac{2\varphi_s}{K_i^2+\varphi_s}\Big)-p_\mu\frac{\partial}{\partial p_\mu}\Big]\prod_{l=1}^{n}\cdots
\end{aligned} \tag{25}$$

All the integrals appearing in P^n ($n\geq 2$) converge at the upper limit. The only integral that depends substantially on the cutoff parameter Λ appears in P^1, but it need not be considered, since we are at the saddle point and there are no terms linear in φ. In view of this, the last term in (25) can be integrated by parts, taking $\Lambda=\infty$:

$$\int d^2p\,p_\mu\frac{\partial}{\partial p_\mu}\prod_{l=1}^{n}\cdots=-2\int d^2p\prod_{l=1}^{n}\cdots$$

If we add

$$\Sigma q_{i\mu}\frac{\partial}{\partial q_{i\mu}}\delta(\Sigma\mathbf{q}_i)=-2\delta(\Sigma\mathbf{q}_i)$$

to (25) and reduce the corresponding terms, we obtain (22). The summation over i in the second term in (25) symmetrizes $P^{n+1}(\mathbf{q}_1\ldots\mathbf{q}_n,0)$ in all the variables.

The proof of (23) is slightly longer. We have

$$\begin{aligned}
&\sum_{i=1}^{n}\Big(q_{i\mu}\frac{\partial^2}{\partial q_{i\mu}\partial q_{i\nu}}-\frac{q_{i\nu}}{2}\frac{\partial^2}{\partial q_{i\mu}^2}\Big)\int d^2p\prod_{l=1}^{n}\cdots\\
&=4\int d^2p\Big[\sum_{i\le j}^{n}\frac{K_{j\nu}K_i^2}{(K_j^2+\varphi_s)(K_i^2+\varphi_s)}\\
&-4p_\mu\sum_{i\le j}^{n}\frac{K_{i\mu}K_{j\nu}+K_{i\nu}K_{j\mu}-\delta_{\mu\nu}K_{i\lambda}K_{j\lambda}}{(K_i^2+\varphi_s)(K_j^2+\varphi_s)}\Big]\prod_{l=1}^{n}\cdots
\end{aligned}$$

The second term is equal to

$$4\int d^2p\left[p_\mu\frac{\partial^2}{\partial p_\mu\partial p_\nu}-\frac{1}{2}p_\nu\frac{\partial^2}{\partial p_\lambda^2}\right]\prod_{l=1}^{n}\cdots$$

$$=-4\int\frac{d^2p}{2}\frac{\partial}{\partial p_\nu}\prod_{l=1}^{n}\cdots=0$$

(by virtue of the translational invariance). Here we have again integrated by parts and made use of the convergence of the integral at the upper limit.

The first term has the form

$$4\int d^2p\left[\sum_{i<j}^{n}\frac{K_{j\nu}}{K_j^2+\varphi_s}\frac{K_i^2}{K_i^2+\varphi_s}\right]\prod_{l=1}^{n}\cdots$$

$$-4\int d^2p\left[\sum_{i<j}^{n}\frac{K_{j\nu}}{K_j^2+\varphi_s}-\varphi_s\sum_{i<j}^{n}\frac{K_{j\nu}}{K_j^2+\varphi_s}\frac{1}{K_i^2+\varphi_s}\right]\prod_{l=1}^{n}\cdots$$

$$=-2\sum_{i=1}^{n}\frac{\partial \tilde{P}^n}{\partial q_{i\nu}}-2\varphi_s\frac{\partial \tilde{P}^{n+1}(q_1\ldots q_n,q)}{\partial q_\nu}\bigg|_{q=0}, \qquad (26)$$

where

$$\tilde{P}^n=P^n\Big/\delta\left(\sum q_i\right).$$

We now consider the terms containing derivatives of the δ-function. We shall denote

$$\frac{\partial f}{\partial q_{i\mu}}=f_{i\mu}=\partial_{i\mu}f$$

and use the definition

$$\delta_\mu(\mathbf{q})f(\mathbf{q})=-\delta(\mathbf{q})f_\mu(\mathbf{q}).$$

We have

$$\sum_{i=1}^{n}\left[(q_{i\mu}\partial_{i\mu}\partial_{i\nu}-{}^1/_2q_{i\nu}\partial_{i\mu}^2)P^n-\delta(\Sigma q_i)(q_{i\mu}\partial_{i\mu}\partial_{i\nu}-{}^1/_2q_{i\nu}\partial_{i\mu}^2)\tilde{P}^n\right]$$

$$=\sum_{i=1}^{n}\left[q_{i\mu}\delta_{\mu\nu}(\Sigma q)-{}^1/_2q_{i\nu}\delta_{\mu\mu}(\Sigma q)\right]\tilde{P}^n$$

$$+\sum_{i=1}^{n}(q_{i\mu}\delta_\mu\tilde{P}_{i\nu}{}^n+q_{i\mu}\delta_\nu\tilde{P}_{i\mu}{}^n-q_{i\nu}\delta_\mu\tilde{P}_{i\mu}{}^n)$$

$$=-3\delta_\nu\tilde{P}^n+\delta_\nu\tilde{P}^n+\delta_\nu\sum_{i=1}^{n}q_{i\mu}\tilde{P}_{i\mu}{}^n+\delta_\mu\sum_{i=1}^{n}(q_{i\mu}\tilde{P}_{i\nu}{}^n-q_{i\nu}\tilde{P}_{i\mu}{}^n)$$

$$=-2\delta_\nu\tilde{P}^n-2(n-1)\delta_\nu\tilde{P}^n-2\varphi_s\delta_\nu\tilde{P}^{n+1}+0. \qquad (27)$$

Because of the symmetry, $\tilde{P}^n$ depends on n invariants of the form

$$\left(\sum_{i=1}^{n}\mathbf{q}_i\right)^2,\quad \sum_{i=1}^{n}\mathbf{q}_i^{2j}\quad (j=1,\ldots,n-1).$$

Therefore,

$$\sum_{i=1}^{n}q_{i\mu}\tilde{P}_{i\nu}{}^n-q_{i\nu}\tilde{P}_{i\mu}{}^n=0.$$

The last two terms of (27) were obtained with the aid of (25).

Combining (27) and (26) gives the required result[4)] (23). Thus, the conformal invariance is proved to first order in $1/N$.

We shall not carry through a rigorous proof in the next orders of perturbation theory, since this would require the calculation of corrections to φ_s and the redetermination of F_r and F_c to the required accuracy. We shall demonstrate a way in which such a proof could be carried through. As before, the important point is that, after we have calculated the corrections to φ_s, the Hamiltonian will not depend on the cutoff parameter Λ but will depend only on the correlation length $r_c=\varphi_s^{-1/2}$ (all the integrals converge at the upper limit).

Any higher-order diagram for $P^n_{(k)}$ comprises first-order vertices $P^n_{(1)}$ and correlators of the fluctuations of φ:

$$\langle\langle\varphi_{\mathbf{q}_1}\varphi_{\mathbf{q}_2}\rangle\rangle=\frac{\delta(\mathbf{q}_1+\mathbf{q}_2)}{P(\mathbf{q}_1,\mathbf{q}_2)}$$

$$=\delta(\mathbf{q}_1+\mathbf{q}_2)\left[N\sum_{\substack{1,2\\2,1}}\int\frac{d^2p}{(2\pi)^2}\prod_{l=1}^{2}\frac{1}{K_l^2+\varphi_s}\right]^{-1}. \qquad (28)$$

We already know how the vertices $P^n_{(1)}$ behave under conformal transformations (see (22) and (23)). Omitting the calculations, which are analogous to those performed above, we give formulas for the transformation of $P^{-1}=\langle\langle\varphi,\varphi\rangle\rangle$:

$$\sum_{i=1,2}q_{i\mu}P_{i\mu}{}^{-1}=2\varphi_sP^{-1}\tilde{P}^3(\mathbf{q}_1,\mathbf{q}_2,0)P^{-1}\delta(\mathbf{q}_1+\mathbf{q}_2), \qquad (29)$$

$$\sum_{i=1,2}q_{i\mu}P^{-1}_{i\mu i\nu}-\frac{1}{2}q_{i\nu}P^{-1}_{i\mu i\mu}$$

$$=2\varphi_s\frac{\partial}{\partial q_\nu}\bigg|_{q=0}P^{-1}\tilde{P}^3_{(1)}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q})P^{-1}\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}). \qquad (30)$$

The proof is carried out most conveniently in the coordinate representation. We make an infinitesimal scale ((2′), $\lambda=1+\alpha$, $|\alpha|\ll1$) or conformal ((3″), $|\alpha|\ll1$) transformation. It follows from the identities (22) and (23) for the vertices $P^n_{(1)}$ and the relations (29) and (30) for the correlators P^{-1} that any diagram for $P^n_{(k)}$ composed of these elements is changed in accordance with (22) and (23). Graphs having the form of loops joined to the basic diagram by one line correspond to the refinement of φ_s in higher orders in $1/N$.

We see that conformal symmetry in the formulation (2′), (3′) can be restored in any order of perturbation theory.

4. THE PHENOMENOLOGICAL HAMILTONIAN

As we have seen using the example of a two-dimensional ferromagnet, the free energy can be represented in the form of the sum (10) of a regular part F_r and a fluctuational part F_c. The part F_c depends only on the correlation length r_c and, because of this, is invariant under the simultaneous transformations (2′), (3′) of the coordinates and r_c. If we assume that, for some reason, r_c may be varying over the sample, this invariance is possessed by the following functional:

$$F_c[r_c]=\int d^dx\left[C_1r_c^{-d}+C_2r_c^{-d}(\nabla r_c)^2\right.$$

$$\left.+C_3r_c^{2-d}\left(\nabla\left[r_c\Delta r_c+\left(\frac{d}{2}-2\right)(\nabla r_c)^2\right]\right)^2+O((\nabla r_c)^6)\right]. \qquad (31)$$

Here d is the dimensionality of space. It is logical to assume that the property of conformal invariance is possessed not only by the model studied but also by other systems in the critical region. Knowledge of the phenomenological Hamiltonian (31) can turn out to be useful if the system being investigated is in weakly nonuniform external conditions.

The Ward identities (14)–(15), (17)–(18), (17′)–(18′) and (22)–(23) are useful for establishing whether any particular system possesses the conformal symmetry (2′), (3′). In general, the description of strongly fluctuating systems with the aid of a locally defined correlation length may be of interest in the case of weakly nonuniform or slowly relaxing systems.

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1) In the first order in $1/N$ we have $\langle\varphi\rangle = \varphi_s = r_c^{-2}$. This implies that the system can be imagined to be a set of noninteracting regions of volume r_c^2, with energy equal to $-\frac{1}{2}$ (in our notation the factor $-T$ multiplying F has been omitted), which corresponds to the usual ideas about critical fluctuations.

2) This follows from the fact that

$$\delta F_c = \sum_{n=1}^{\infty}\int\frac{1}{n!}\mathcal{F}^n(x_1\ldots x_n, r_c)\prod_{i=1}^{n}\delta\ln r_c(x_i)\,d^2x_i.$$

3) We recall that our definition of the free energy $F[\beta]$ differs from that of the usual $F_0[T]$: $F[\beta] = -\beta F_0[1/\beta]$. Therefore, $\widetilde{F}$ does not coincide with the energy.

4) Because of the symmetry, $\partial P^{n+1}/\partial q_\nu|_{q=0} = \delta_\nu \widetilde{P}^{n+1}$.

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The phase transition in a weakly disordered uniaxial ferromagnet

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By the renormalization-group method the exact temperature dependences of the susceptibility and specific heat (above T_c in zero external field) are found for the four-dimensional Ising model with short-range exchange forces and randomly distributed, rigidly fixed impurities. The stability of the impurity fixed point in ($d = 4-\epsilon$)-dimensional space is demonstrated and the critical exponents are calculated to second order in $\epsilon^{1/2}$.

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One of the few exactly soluble realistic problems is that of the phase transition in a three-dimensional easy-axis ferromagnet (or ferroelectric of the displacement type) with dipolar interaction and randomly distributed fixed impurities, the impurity concentration being considerably below the percolation threshold. Near the transition point the indirect interaction of the critical fluctuations of the order parameter via the impurities becomes important, and as a result the behavior of all the thermodynamic quantities is greatly changed from that in the impurity-free case. Although the interaction via the impurities is attractive in sign it does not violate the stability, and, therefore, a second-order phase transition occurs in the system. The temperature dependences of the uniform susceptibility r^{-1} and specific heat C of an impure easy-axis dipolar ferromagnet (in $d=3$ dimensions) and of the four-dimensional impure Ising model with short-range exchange forces have been obtained by the renormalization-group (RG) method in a paper[1] by Aharoni ($T > T_c$, $h=0$):

$$r\sim\tau\exp\{-(D|\ln\tau|)^{1/2}\}, \quad (1)$$

$$C\sim\exp\{-2(D|\ln\tau|)^{1/2}\}|\ln\tau|^{1/2}, \quad (2)$$

where $\tau = (T-T_c)/T_c$; $D = 9/(81\ln(\frac{4}{3})+53)$ for the impure dipolar ferromagnet ($d=3$) and $D=\frac{6}{53}$ for the impure Ising model ($d=4$). (The equation of state and the dynamics of these systems have been considered in Refs. 2 and 3.)

However, the results (1) and (2) are in need of refinement. It is shown in this paper that the true singularities of the susceptibility and specific heat are described by the formulas (41) and (43), which differ from (1) and (2) by slowly varying logarithmic factors. This refinement is of interest because it can, apparently, be

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