it is violated. On the other hand, for $r_D \ll l_e$ the distribution function depends only on the total energy $E$. It is thus necessary to satisfy the chain of inequalities

$$1 \ll r_D/l \ll \delta^{-1}, \quad l = \frac{\pi\hbar^4\rho s^2}{m^2 E_c^2 kT}, \tag{25}$$

which leads to a lower temperature bound on the range of applicability of the results obtained.

Let us note that in finding the potential of a charged dislocation we regarded the donors as fully ionized and in doing so restricted ourselves to temperatures such that $kT \gg \varepsilon_d$, where $\varepsilon_d$ is the activation energy of the donors. However, our results hold qualitatively even at lower temperatures, when screening is provided by electrons moving among the donors without their prior activation. This occurs until quasicontinuity of the donor distribution in the volume of the semiconductor is ensured, i.e., until the distance between the donors $n_d^{-1/3}$ is much less than the characteristic scale of variation of the electrostatic potential, given by the Debye radius, i.e.

$$T > \frac{4\pi e^2}{\varkappa k} n_d^{1/3}. \tag{26}$$

For germanium in which $m \sim 10^{-28}$ g, $\varkappa = 16$, $n_d \sim 10^{13}$ cm$^{-3}$, and $10^{-2} < \zeta < 10^{-1}$, a combined analysis of the inequalities (25) and (26) shows that the results obtained in this work are qualitatively valid down to temperatures $T \sim 30$ K. For the temperature at which the dependence of the cross section on temperature goes over from a power function to an exponential one we obtain $T_0 \approx 130$ K. The absolute magnitude of the capture cross section (radius) in the temperature range $30\ \text{K} < T < 300\ \text{K}$ proves to be of the order of $3 \cdot 10^{-7}\ \text{cm} < \sigma < 2 \cdot 10^{-4}\ \text{cm}$, which is in agreement with the experimentally observed[2] large cross sections.

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# The electronic thermal conductivity of clean superconductors

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The temperature dependence of the electron-phonon thermal conductivity is obtained by numerical solution of the kinetic equation and compared with experiment.

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1. In very clean superconductors the electronic thermal conductivity $\varkappa_e$ is determined entirely by the scattering of electrons by phonons ($\varkappa_{eph}$) and by the crystal boundaries, while the phonon thermal conductivity $\varkappa_{ph}$ is determined entirely by the scattering of the phonons by electrons ($\varkappa_{phe}$) and also by the crystal boundaries. The solution for the nonequilibrium correction $\varphi$ to the electron distribution function $f$ has the form[1,2]

$$\varphi = -\frac{f-f_0}{f_0(1-f_0)} = \left[\varphi_2(\varepsilon, T) + \frac{\xi}{|\xi|}\varphi_3(\varepsilon, T)\right]\cos(\widehat{\mathbf{p}\nabla T}), \tag{1}$$

where $\varepsilon = (\xi^2 + \Delta^2)^{1/2}$. Under the condition[2]

$$\left(\frac{T}{\Theta_D}\right)^4 \frac{e_F}{\Theta_D} e^{\Delta/T} \ll 1 \tag{2}$$

the decisive role in the calculation of the electronic thermal conductivity, as was noted in the work of Gurevich and Krylov,[1] is played by the function $\varphi_3$. In Ref. 3 an integral equation for $\varphi_3$ was obtained. In this article we report the results of a numerical calculation of the electronic thermal conductivity, based on the solution of this equation, and compare the results with the experimental data. We neglect the influence of the nonequilibrium character of the phonons on the electron distribution function, which is permissible for temperatures that are not too low[1,2]:

$$\frac{T^2}{\varepsilon_F \Theta_D} e^{\Delta/T} \ll 1.$$

2. For the electronic thermal conductivity in a normal metal we have, using Matthiessen's rule,

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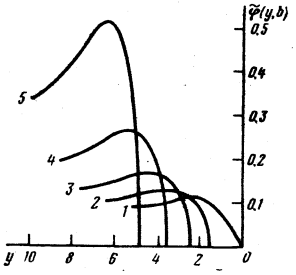


FIG. 1. Plots of the function $\tilde{\varphi}(y, b)$ for different values of $b$: 1) $t=1$, $b=0$; 2) $t=0.8$, $b=1.55$; 3) $t=0.64$, $d=2.41$; 4) $t=0.48$, $b=3.50$; 5) $t=0.36$, $b=4.81$.

$$\frac{1}{\varkappa_e^n}=\frac{1}{\varkappa_{eph}^n}+\frac{1}{\varkappa_{ei}^n}=\alpha T^2+\frac{\beta}{T}. \tag{3}$$

The first term in (3) is due to scattering by phonons and the second to scattering by impurities or the crystal boundaries. As a rule, even in the case of a pure metal it is necessary to keep the second term, since it becomes essential in the low-temperature limit. In superconductors the thermal conductivity is usually represented in the form

$$\frac{1}{\varkappa_e^s}=\frac{\alpha T^2}{g(t)}+\frac{\beta}{Tf(t)} \tag{4}$$

(here and below, $t\equiv T/T_c$). The function $f(t)$ has been calculated repeatedly[3-5] and agrees well with the experiments.

The function $g(t)$ is connected with the correction to the electron distribution function by the relations[2]

$$g(t)=\frac{\varkappa_{eph}^s(T)}{\varkappa_{eph}^n(T)}=\frac{\Phi(b)}{\Phi(0)},$$
$$\Phi(b)=\int_b^\infty f_0(1-f_0)\tilde{\varphi}(y,b)\,y\,dy, \tag{5}$$
$$b=\frac{\Delta}{T},\quad f_0=(e^\nu+1)^{-1},\quad y=\frac{\varepsilon}{T}.$$

The function $\tilde{\varphi}$ is connected with the function $\varphi_3$ in (1) by the relation

$$\tilde{\varphi}=T^4\alpha_0\varphi_3 \tag{6}$$

($\alpha_0$ depends on the matrix element of the electron-phonon interaction, the sound velocity and the Fermi velocity[2]) and obeys the integral equation[2]

$$f_0f_0'e^\nu y=\tilde{\varphi}(y,b)\left(\int_b^\infty K(y,y')dy'-\int_b^\infty K(y,-y')dy'\right)$$
$$+\int_b^\infty K(y,-y')\tilde{\varphi}(y',b)dy'-\int_b^\infty K(y,y')\tilde{\varphi}(y',b)dy', \tag{7}$$
$$K(y,y')=\left(1-\frac{b^2}{yy'}\right)\frac{yy'}{(y^2-b^2)^{1/2}(y'^2-b^2)^{1/2}}K(y,y'),$$
$$K(y,y')=\frac{f_0f_0'e^\nu e^{\nu'}(y'-y)^2}{|e^\nu-e^{\nu'}|}.$$

As follows from (5)–(7), for the BCS model the function $g(t)$ is a universal function of $t$. The integral equation (7) was solved by reducing it to a linear equation of large rank. The mesh frequency was chosen everywhere to be such that the accuracy of the calculations was not worse than 1%. The explicit form of the function $\tilde{\varphi}(y,b)$ for several values of $b$ is shown in Fig. 1. The dependence of $b$ on $t$ in the BCS model has been calculated and tabulated in detail by Mühlschlegel.[6] Since the function $g(t)$ describes a whole class of superconductors close to the BCS model, we also give its values in the form of a table. It is interesting to note that, despite the rapid decrease, associated with the gap, in the number of electronic excitations, the absolute value of the thermal conductivity $\varkappa_{eph}$ increases right down to extremely low temperatures (see Table I). Below temperatures such that

$$\left(\frac{T}{\Theta_D}\right)^4\exp\left(\frac{2\Delta}{T}\right)\gg 1,$$

i.e., for $t\lesssim 0.4$,[1,2] it is necessary to take into account the contribution of the phonon thermal conduction.[5]

TABLE I.

| $t$ | $g(t)$ | $\frac{\varkappa_{eph}^s(T)}{\varkappa_{eph}^n(T_c)}$ | $t$ | $g(t)$ | $\frac{\varkappa_{eph}^s(T)}{\varkappa_{eph}^n(T_c)}$ | $t$ | $g(t)$ | $\frac{\varkappa_{eph}^s(T)}{\varkappa_{eph}^n(T_c)}$ |
|---|---|---|---|---|---|---|---|---|
| 0.96 | 0.956 | 1.04 | 0.68 | 0.624 | 1.36 | 0.40 | 0.350 | 2.19 |
| 0.92 | 0.896 | 1.06 | 0.64 | 0.584 | 1.43 | 0.36 | 0.296 | 2.28 |
| 0.88 | 0.849 | 1.10 | 0.60 | 0.544 | 1.51 | 0.32 | 0.246 | 2.40 |
| 0.84 | 0.799 | 1.13 | 0.56 | 0.505 | 1.61 | 0.28 | 0.191 | 2.44 |
| 0.80 | 0.754 | 1.18 | 0.52 | 0.474 | 1.75 | 0.24 | 0.129 | 2.25 |
| 0.76 | 0.711 | 1.23 | 0.48 | 0.427 | 1.85 | 0.20 | 0.0628 | 1.57 |
| 0.72 | 0.668 | 1.29 | 0.44 | 0.382 | 1.97 | 0.16 | 0.0144 | 0.564 |

A comparison of the dependence $g(t)$ calculated for the BCS model with the experimental data for tin[7] and indium[7,8] is given in Fig. 2. Here the thermal conductivity of tin has been taken as an average over different directions.

3. It is natural to ascribe the slight deviation of $g(t)$ from the experimental points to strong-coupling effects, which manifest themselves, in particular, in the different temperature dependence of $b(t)$ from that in the BCS model. To take these into account one sometimes uses the semi-empirical formula

$$b(t)=\left(\frac{\Delta C_{exp}}{\Delta C_{BCS}}\right)^{1/2}b^{BCS}(t), \tag{8}$$

where $\Delta C$ is the difference in the specific heats above and below the transition point. For the BCS model, $\Delta C_{BCS}/C_n=1.43$; for tin, $\Delta C/C_n=1.6$. As a comparison with the experimental data of Ref. 9 shows, formula (8) is not a bad description of the temperature dependence $b(t)$ in the temperature region under consideration. The theoretical curve calculated from formula (8) is given by the dashed line in Fig. 2.

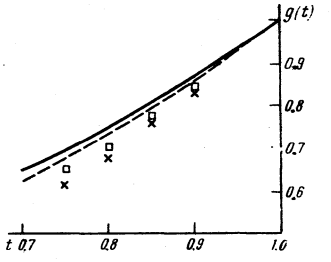


FIG. 2. Plot of the function $g(t)$. The solid curve corresponds to the BCS theory, and the dashed curve is calculated from formula (8) for Sn with corrections for strong-coupling effects; □—data for Sn[7]; ×—data for In.[7,8]

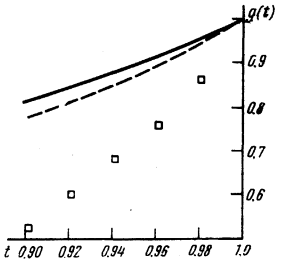


FIG. 3. Plot of the function $g(t)$ for lead. The solid curve is calculated from formula (9) and the dashed curve is calculated with the coefficient 4.5 in (9); □—experimental data.[12,13]

To conclude this section we make comparisons with the calculations in other papers.[4,10] In Ref. 4 the function $g(t)$ was determined by means of a variational method. The form of the trial function differed substantially from that given in Fig. 1. In Ref. 4 the function $g(t)$ was found to increase below the transition point, owing to an unfortunate choice of the trial function.

In Refs. 10 the method of Landau and Pomeranchuk[11] was used to solve the electron kinetic equation. In this case, a correction even in $\xi$ to the distribution function was used to determine the temperature dependence of the thermal conductivity. Because the correction is even, in the calculation of the heat flux **Q** it is necessary to take into account the dependence of the momentun on $\xi(p=p_0(1+\xi/2\varepsilon_F))$. Then the integrand in the expression for **Q** turns out to be close to the integrand in (5), and this leads to good agreement with experiment.[10] However, the good quantitative agreement between the expressions for the ratio of the thermal conductivities not only in the region $(y-b)/b \ll 1$ but also in the broader region $y \sim b$ can scarcely be regarded as constituting a rule. We note also that, in view of the fact that the solution of the momentum type is a solution of a homogeneous equation,[14] when the nonequilibrium character of the phonons is taken into account the equation for the electron distribution function, despite the fact that the drag effect is small, will have a different mathematical form.[2]

4. In order to examine the question of the applicability of the calculations of Ref. 2 in the case of superconductors with strong coupling we have calculated the temperature dependence of the thermal conductivity of lead near $T_c$. In this region, for the temperature dependence $b(t)$ we can use the formula

$$b(t)=4.0(1-t)^{1/2}. \tag{9}$$

Figure 3 shows the results of the calculation (the solid curve) and the experimental data of Refs. 12 and 13. Even with the factor 4.5 in formula (9) the theoretical curve is much higher (the dashed curve in Fig. 2). Finally, calculations using the formula (8) (for lead, $\Delta C/C_n=2.65$) would lead to a curve coinciding with the solid curve in Fig. 3 near $T_c$ and coinciding with the dashed curve near $t=0.9$. Thus, for lead the difference between the theoretical curve and the experimental data turns out to be rather substantial, indicating the need to use a more refined model.

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