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Effect of nonlinear dissipation on plasma heating in the strong Langmuir turbulence regime

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We consider the one- and three-dimensional problems of plasma heating taking nonlinear effects into account. We study in the one-dimensional case the buildup of Langmuir solitons due to dissipative slowingdown, we obtain the way the spectrum develops in time in the constant pumping regime, and we investigate the self-similar electron heating regime. We consider in the three-dimensional case the effect of nonlinear conversion of the Langmuir oscillations into sound on the plasma heating when acoustic collapse takes place. We estimate the maximum extent of the inertial range corresponding to such a regime. We obtain the self-similar electron distribution for heating due to nonlinear conversion.

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In connection with the problem of the heating of a plasma target by powerful beams of light or of relativistic electrons, the heating of a plasma under strong Langmuir turbulence conditions has recently been studied intensively (see, for instance, Refs. 1 to 5). In those papers the main subject of the study was the resonance mechanism for the formation of hot electron "tails" and the influence of non-linear effects was not taken into account, although the role of non-linear dissipation in the dynamics of Langmuir solitons had been studied earlier.[1,6] Only the recent paper by Galeev *et al.*[7] drew attention to the non-linear conversion process of Langmuir waves into sound which under conditions of constant pumping is generated when collapsing solitons are damped.[5,8] This process is the main one in a typical three-dimensional problem; in the one-dimensional case, and also under conditions of adiabatically slow damping of a collapsing soliton,[3] the sound produced cannot guarantee conversion—in that case the process of the slowing-down of the solitons by trapped particles[1] comes into play and it can, in particular, determine completely the structure of the wave spectrum.[6]

The aim of the present paper is the study of the details of the heating of the particles under conditions when the dynamics of strong Langmuir turbulence is essentially determined by the non-linear dissipation, as well as the determination of the conditions for the existence of such regimes. We consider the one-dimensional case in the framework of the soliton-gas model, and the three-dimensional one in the acoustic collapse approximation, which to some extent distinguishes our considerations from the ones in ref. 7.

1. STRONG ONE-DIMENSIONAL TURBULENCE. SOLITON BUILD-UP REGIME

We consider a one-dimensional model of strong Langmuir turbulence which is a set of Langmuir solitons—localized non-linear Langmuir waves.[9] The frequency of the oscillations of the solitons is close to ω_{pe}. Characteristic parameters are: amplitude E, reciprocal of the width $k_0=eE/\sqrt{6}T$, and velocity v, $0\leqslant v<c_s$. The spectral expansion of the Langmuir soliton field has the form

$$E_k=E\left(k_0\,\mathrm{ch}\frac{\pi k}{2k_0}\right)^{-1}\approx\sqrt{6}\frac{T}{e}\vartheta\left(1-\frac{\pi k}{2k_0}\right). \quad (1)$$

It is well known that an isolated Langmuir soliton is a stationary and stable structure. Therefore, the transfer of energy from large to small dimensions which is characteristic for the strong turbulence regime proceeds in the case of not too powerful pumping through the fusion of solitons which are close in size.[10] As solitons with appreciably different amplitudes do not interact, the transfer can only take place in relays. In the hydrodynamic approximation a steady-state soliton amplitude distribution is established

$$F(E)\approx\mathrm{const}\cdot E^{-3}, \quad (2)$$

and in the model with discrete levels $E_{n+1}=2E_n$ we have accordingly for the occupation numbers $N(E)$

$$N(E)\approx\mathrm{const}\cdot E^{-2}, \quad (3)$$

which gives the spectral energy density $|E_k|^2\propto k^{-2}$.

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The soliton-gas model with a relay transfer along the sizes is not unique. In particular, under the conditions of a numerical experiment with simulation of plasma heating by an external oscillating field, two facts which distinguish the real regime from the model proposed in ref. 9 are very often encountered.

a) Whereas we assume that in the above mentioned model[9] that the phases of the solitons which constitute the strong turbulence are random, in numerical experiments one sometimes observes a rigid phase correlation which leads to a more or less regular structure of the turbulent pulsations (this regime was called in Ref. 11 "dynamic turbulence"). Such a situation can indeed arise if during the pumping time dissipative effects do not succeed in developing; such effects are weak plasma inhomogeneities or other mechanisms which are sufficiently weak not to destroy the internal soliton structure, but lead to a phase mismatch. Clearly, in the problem of phasma heating this regime, if it is realized at all, characterizes only the initial stage when one can neglect the interaction of the waves with the plasma particles.

We note also that the unavoidable idealization of the problem when we set up a numerical experiment is frequently connected with the introduction of additional regularizing mechanisms in consequence of which one must sometimes allow a re-evaluation of the universality of the results. For instance, in the paper by Sudan *et al.*[12] one finds a statement that for a not too strong pumping field $E_0^2 \ll 4\pi nT$ the modulational instability leads to the formation of a system of standing and non-interacting solitons which is valid only for the case of a standing excitation wave and zero mismatch $\omega_0 - \omega_{pe}$, where ω_0 is the pumping frequency. If, however, we consider the regime where weakly turbulent noise is excited while later on solitons are formed from the plasmon condensate and, in particular, pumping due to the beam instabilities, there is very little probability in that case for a regular structure of soliton turbulence.

b) Another important difference from the model of Ref. 9 occurs in the problem of pumping with a rather strong external field so that the interaction with the latter directly leads to the appearance of solitons with an amplitude $E \sim (nT)^{1/2}$ and, hence, resonance damping is switched on ("physical collapse"[13] or "quasi-collapse"[14]). For weak pumping fields such an interaction leads merely to oscillations in the soliton amplitude,[4] but in the opposite case of a large amplitude

$$E_0^2/16\pi nT \gtrsim (m/M)^{1/2} \tag{4}$$

the solitons transfer, indeed, efficiently the energy to thermal particles. Condition (4) corresponds to a very considerable pumping power Q

$$Q \sim nT\omega_{pi}. \tag{4'}$$

Even, if inequality (4) is not satisfied, as the electrons are heated the phase velocity corresponding to the dissipation region grows and "physical collapse" will be an efficient dissipation channel, at least in the final stage of the heating process. In the case of a beam instability the picture is somewhat different: when the soliton amplitude increases the pumping is automatically switched off,[15] but we shall not make more precise in what follows the noise source, since we are interested in the case which is the opposite of (4′)—the regime of slow pumping (see below). Since we consider moreover especially the heating process, effects connected with the correlations of soliton phases may also be neglected. Apparently, under such conditions the soliton-gas model with a relay transfer along the sizes is the best analytical description of strong Langmuir turbulence.

If the non-linear kinetic effects are taken into account in the framework of this model, the turbulence spectra can differ appreciably from (2),(3), as was shown in Ref. 6.

The main non-linear effect corresponds to the interaction between a soliton and particles which are in resonance with the soliton velocity v, which leads to a slowing down of the soliton over a finite length.[1] In particular, the interaction between ions and a soliton with a large amplitude ($E^2/8\pi nT > m/M$) and a small velocity $v < v_{Ti}$ leads to its slowing down over a length less than its width k_0^{-1}. Near the dissipation region ($k_0 r_D \to 1, r_D = v_{Te}/\omega_{pe}$) the Langmuir solitons can thus exist either with $v = 0$ or with $v \gg v_{Ti}$, which excludes the interaction with resonant ions. In the latter case the soliton velocity is clearly of the order of the ion sound speed c_s. Just these fast moving solitons guarantee the energy flux along the spectrum. However, they also undergo a non-linear slowing down, but now due to resonance with the electrons; the characteristic mean free path is then

$$\lambda_e(E) = \frac{\zeta}{k_0}\left(\frac{M}{m}\right)^{1/2}, \quad \zeta \sim 1.$$

If the average distance between solitons in the hydrodynamic model is less than λ_e, the interaction is impossible. It was shown in Ref. 6 that such a situation arises for a sufficiently weak specific pumping power

$$Q/nT\omega_{pi} \ll m/M. \tag{5}$$

In that case there occurs in the plasma, due to the pumping, a build-up of solitons with a given amplitude until their density exceeds λ_e^{-1} and only after this does the occupation of the next level in amplitude start. As a result we get the following stationary distribution of occupation numbers in the discrete level model:

$$N(E) = \beta \frac{eE}{T}\left(\frac{m}{M}\right)^{1/2}, \quad \beta \sim 1, \tag{6}$$

and a soliton-amplitude distribution function of the form

$$F(E) = \beta \frac{e}{T}\left(\frac{m}{M}\right)^{1/2}, \tag{6'}$$

which yields the following expression for the spectral density of the energy:

$$W_k=\beta\frac{T}{e}\left(24\pi nT\frac{m}{M}\right)^{1/2}r_D(k_{max}-k);\qquad(7)$$

while the total energy built-up in the standing solitons equals

$$W=\frac{T}{e}\left(6\pi nT\frac{m}{M}\right)^{1/2}k_{max}^2r_D.\qquad(8)$$

Here and henceforth we omit the numerical coefficient $\beta\sim1$ which is unimportant for our estimates and we also assume that $T_i\ll T_e$, which is quite natural in the strong Langmuir turbulence regime.

As we have already mentioned above, under weak pumping conditions [inequality (5)] the transfer of energy to the small-size region is possible only when the levels with a given amplitude are saturated. Hence it follows that the noise spectrum at each time will have the form (7) and the rate at which the boundary of the spectrum moves can be found from Eq. (8):

$$k_{max}\approx r_D^{-1}(Qt/2\pi nT\sqrt{6})^{1/2}.\qquad(9)$$

When, finally, the spectrum shifts into the small-dimensions region to such an extent that an efficient particle acceleration starts, the build-up of a large quantity of standing solitons in the plasma will cause the rate at which energy is acquired by the electron "tail" to be much larger than the one which might be guaranteed by the pumping power Q.

We trace how the noise spectrum in that case will evolve. For the sake of clarity we give the discussion in the framework of the discrete level model, as we have already done.[10,6] The generalization to the cause of a continuous soliton distribution $F(E)$ does not present any particular difficulties.

Let, thus, the maximum amplitude of the solitons at a given time equal E_{max}, let there further occur the levels $\frac{1}{2}E_{max}$, $\frac{1}{4}E_{max}$, and so on. The damping of Langmuir solitons when they interact with resonant particles corresponds to their "shifting" to a lower amplitude.

Let, after some characteristic damping time, the solitons move from the level E_{max} to the level $\frac{1}{2}E_{max}$, and next in turn to the level $\frac{1}{4}E_{max}$, and so on. One can see from Eq. (6) that there occurs then a two-fold supersaturation of each level and as a result the interaction between solitons is switched on and there occurs a fast "shedding" of the solitons—already in a smaller amount—back to the level E_{max}. If we moreover take it into account that for a monotonically decreasing electron distribution function $f(v)$ the characteristic damping time decreases with increasing soliton amplitude, this gives additional arguments in favor of the above-described picture of the effect.

In the heating process there occurs thus a continuous redistribution of energy between the levels so that the distribution described by Eqs. (6), (6′), and (7) is preserved also in that case. The particle heating proceeds exclusively via a decrease with time of the number of solitons with maximum amplitude or, what is the same, via the motion—now already in the long-wavelength region—of the limit of the spectrum k_{max}. Hence it follows that the heating process can be described analytically.

We denote by $v_0=\omega_{pe}/k_{max}$ the lower limit of the resonance region in velocity space. From the equation

$$\frac{dW}{dt}=\int_0^{k_{max}}2\gamma_kW_k\,dk=\pi\omega_{pe}^2\int_{v_0}^{\infty}W_k(v)\frac{\partial f}{\partial v}dv\qquad(10)$$

and Eqs. (7), (8) we get

$$\frac{\partial}{\partial t}\frac{1}{v_0^2}=\frac{\pi\omega_{pe}}{2}\int_{v_0}^{\infty}\frac{f(v)dv}{v^2}.\qquad(11)$$

The second equation which we shall use is the equation of quasi-linear diffusion, which can, as the result of substituting in it the spectral density of the energy (7), be reduced to the form

$$\frac{\partial f}{\partial t}=4\pi^2\sqrt{6}\,\omega_{pi}\frac{v_{Te}^4}{v_0}\frac{\partial}{\partial v}\frac{v-v_0}{v^2}\frac{\partial f}{\partial v}.\qquad(12)$$

The set of Eqs. (11), (12) allows a self-similar separation of variables:

$$f(v,t)=\Phi(\xi)/v_0\tau,\quad \tau=4\pi^2\sqrt{6}\omega_{pi}t,$$
$$\xi=v/v_0,\quad v_0=v_{Te}(4\lambda\tau)^{1/4}.\qquad(13)$$

In these variables Eqs. (11), (12) have the form

$$\frac{d}{d\xi}\frac{\xi-1}{\xi^2}\frac{d\Phi}{d\xi}+\lambda\xi\frac{d\Phi}{d\xi}+5\lambda\Phi=0,\qquad(14)$$

$$\int_1^{\infty}\frac{\Phi(\xi)}{\xi^2}d\xi=4\pi\left(\frac{6m}{M}\right)^{1/2}.\qquad(15)$$

The number of resonant particles decreases with time like $n_R\propto\tau^{-1}$. In the region $v<v_0(\tau)$ the distribution function must be stationary and at the same time satisfy the self-similar separation (13). A similar situation is described in Refs. 1, 3. The only function of this kind which satisfies the joining condition at $v=v_0$ is

$$f_\infty(v)=\Phi(1)\frac{4\lambda v_{Te}^4}{v^5}.\qquad(16)$$

If we let in Eqs. (11), (12) formally the time tend to infinity the solution (16) is established in the whole of velocity space. From the particle-number conservation law

$$\int_{v_0}^{\infty}f_\infty\,dv=\int_{v_0}^{\infty}\frac{\Phi(\xi)}{v_0\tau}dv$$

we get yet another normalization condition:

$$\int_1^{\infty}\Phi(\xi)d\xi=\tfrac{1}{4}\Phi(1).\qquad(17)$$

The solution of Eq. (14) for large ξ has the form $\Phi\sim\exp(-\lambda\xi^3/3)$, and for $(\xi-1)\ll1$ we can get

$$\Phi\sim\exp\left[-\frac{5\lambda}{\lambda+1}(\xi-1)\right],$$

whence, using (17), the estimate $\lambda\approx4$ follows. Further, from condition (15) we find $\Phi(1)=24\pi(6m/M)^{1/2}$. We can write the particle distribution function in the whole of velocity space as follows

$$f(v,t)=64\pi\cdot 6\sqrt{6}\, v_{Te}^{4}(m/M)\times\begin{cases} v^{-5}, & v<v_0\\ v_0^{-5}\exp[-\tfrac{4}{3}(v/v_0)^3], & v\gg v_0,\\ v_0^{-5}\exp[-4(v-v_0)/v_0], & v-v_0\ll v_0\end{cases}\tag{18}$$

where

$$v_0=2v_{Te}(4\pi^2\sqrt{6}\,\omega_{pi}t)^{1/4},\quad k_{max}=\omega_{pe}/v_0.$$

We note that the soliton build-up regime can be realized only when[6]

$$\left(\frac{v_{Te}}{v_0}\right)^3>\frac{Q}{nT\omega_{pi}}\frac{M}{m}.\tag{19}$$

On the other hand, when constructing the solution we used the approximation $dW/dt>Q$, which gives

$$\left(\frac{v_{Te}}{v_0}\right)^6>\frac{Q}{nT\omega_{pi}}\frac{(M/m)^{1/2}}{6(8\pi)^2}.\tag{20}$$

If condition (19) is satisfied, inequality (20) adds a very weak restriction so that we may assume that the approximation used by us is rather good under conditions when non-linear effects are important.

We note yet another characteristic property of the solution obtained. Although the distribution (18) contains a non-Maxwellian energy "tail," the total energy included in it is determined by small velocities and, hence, under slow pumping conditions, one can obtain an efficient energy contribution to thermal particles.

2. PLASMA PARTICLE HEATING DURING LANGMUIR COLLAPSE IN THE NON-LINEAR DISSIPATION REGIME

We consider the problem of plasma heating during Langmuir collapse. Let the pumping power be constant and equal to Q and let the modulational instability generate N collapsing solitons per unit time:

$$Q_c=N\mathscr{E}_i,\tag{21}$$

where $\mathscr{E}_i$ is the initial energy of a soliton and Q_c the power dissipated through collapse, $Q_c\leq Q$. The problem of the quasi-linear dissipation of collapsing solitons and plasma heating during collapse has been considered before in the approximation of a constant particle flux in the generation region[4] and in the self-similar regime.[3,16] Characteristic for the latter case is the formation of an electron "tail" with a constant total number of particles, a number of resonant particles which decreases with time, and the tendency of the characteristic velocity $v_0(t)\to\infty$. When $v\ll v_0(t)$ the distribution function has in that case a power-law character and is constant in time. The resonant particles carry at $v\geq v_0(t)$ about half the energy and guarantee completely the whole of the energy dissipation at a given time, that is, when their total number $n_R(t)$ tends to zero the quantity

$$\gamma(\omega_{pe}/v_0)\,\tau(v_0/\omega_{pe})\approx 1\tag{22}$$

is conserved, where $\gamma(t)$ is the damping rate and $\tau(\delta)$ the collapse time for a given size τ. These results were obtained for subsonic,[3] supersonic, and sonic[16] collapse modes. In our opinion, the latter is the best analytical model of the collapse when dissipation is present.

In the hydrodynamic approximation the dynamics of the collapse is described by the set of Zakharov equations:[17]

$$\begin{gathered}\operatorname{div}\left(i\frac{\partial\mathbf{E}}{\partial\tau}+\frac{3}{2}\omega_{pe}r_D{}^2\nabla^2\mathbf{E}\right)=\frac{\omega_{pe}}{2n_0}\operatorname{div}\delta n\mathbf{E},\\ \left(\frac{\partial^2}{\partial\tau^2}-c_s{}^2\nabla^2\right)\delta n=\frac{1}{16\pi M}\nabla^2|\mathbf{E}|^2,\end{gathered}\tag{23}$$

where $\tau=t-t_0$, t_0 is formally the time of collapse.

It follows from the set (23) that for large amplitudes $E^2/8\pi nT\gg m/M$ the collapse goes into the supersonic regime, when $E^2/8\pi nT\gg\delta n/n_0$. A collapsing soliton is then characterized by a single size δ, and its main parameters are related through the following self-similar relations:[17]

$$\delta n\sim\delta^{-2},\ E^2/nT\sim\delta^{-3},\ \delta\sim\tau^{2/3}.$$

This character of the collapse is on the whole confirmed by numerical experiments, but it turned out that when dissipation is included the picture changes. It was shown in a numerical experiment[8] that a collapsing soliton, losing altogether 5 to 10% of its total energy, leaves the supersonic regime and its further evolution proceeds in such a way that $\delta n/n_0\sim E^2/8\pi nT$. A correct mathematical description of the dynamics of the collapse in the dissipative regime is of considerable difficulty—there exists only the modification of the self-similar description proposed by Zakharov[8] constructed for a power-law function $\gamma(k)$. Even for this particular case the separation of variables has a rather complicated form which makes it impossible to solve the problem of plasma heating analytically. In the last case one is obliged to adopt simpler models. We consider two of those.

a) One can simply neglect the effect of damping on the dynamics of the collapse—in this way the heating problem was solved in Ref. 4, and in Ref. 16 this model was considered together with the acoustic model in the framework of the time-dependent problem. In any case the model has a region of applicability for adiabatically slow damping of a collapsing soliton and this allows us to describe the asymptotic behavior of the electron "tails" as $v\to\infty$.

b) The "sonic" model, proposed by Rudakov (see, for instance, Ref. 2) explicitly takes into account the fact that $\delta n\propto E^2$ when there is damping. The separation of variables in the set (23) leads exactly to the same as in the supersonic regime, but in the second equation all terms are of the same order, which leads to the following dependence for the modulation scale:

$$\delta=-c_s\tau,\quad r_D/\delta\approx E/(24\pi nT)^{1/2}.\tag{24}$$

The additional condition corresponding to taking all terms into account in the equation for δn leads to the fact that a collapsing soliton turns out to have two scales. The large scale is determined by the time-dependence of the number of quanta $\int|\mathbf{E}(\mathbf{r})|^2 d\mathbf{r}$. When there is no dissipation two-scale solitons are unstable with respect to division into single-scale ones that collapse according to the supersonic law.[18] However, damping inhibits also the development of this instability mode, as it tightens the collapse regime; in any case the time of its development is not less than the collapse time itself.

When solving the problem of plasma heating in the presence of collapse condition (22) plays an important role. It just determines the total number of accelerated particles and the function $v_0(t)$. Hence it follows that just the sonic model of the collapse gives a more correct estimate for these quantities, since a collapsing soliton, according to that condition, is damped in a time of the order of collapse in such a way that it is impossible to neglect the effect of damping on the dynamics. However, even before strong resonance damping is switched on, the soliton can collapse according to the sonic law if we take into account the non-linear dissipation effect considered in Ref. 7 and the present paper.

We shall use the sonic model of collapse, but for a comparison we give the results obtained in the framework of the supersonic model; it turns out that the basic qualitative results of that part of the paper are independent of the model of the collapse.

It was shown in Ref. 8. that when strong damping of a collapsing soliton [in the heating problem this corresponds to collapse up to a scale $\delta\sim v_0(t)/\omega_{pe}$] is switched on, the dissipation of the Langmuir waves is not accompanied by the dissipation of their sonic envelopes—the density well in which the plasmons were trapped is preserved and evolves further as a free sonic perturbation. Ref. 5 also drew attention to the formation of sound when collapsing cavitons are damped. The characteristic wavenumber of such sound corresponds to the final size of the caviton:

$$k_s\sim k_f(t)=\omega_{pe}/v_0(t),$$

meaning that the beats of the Langmuir and sound waves have a phase velocity close to $v_0(t)$, i.e., induced scattering of Langmuir waves into sound waves[19] by electrons from the resonance "tail" is possible. This process was considered in Ref. 7 in the supersonic collapse approximation. We consider the problem in the framework of the sonic model, and in contrast to Ref. 7 we take into account the possibility that the collapsing soliton itself may be damped, and we establish the limits for the efficiency of the process and describe the non-linear heating of the electron "tail".

Since sound waves in k-space are concentrated in a drop in scales of the order of unity we shall, as a rule, use not the spectral density $W^s_{\mathbf{k}}$, but the total energy density of the sound W^s. We note that the non-linear conversion process includes not only the re-emission of Langmuir waves into ion-sound waves, but also pair production and absorption of l- and s-waves. Using the standard methods of weak-turbulence theory[19] we can confirm that these processes practically balance one another for sound which allows us to take only the production of sound by damped solitons into account:

$$(dW^s/dt)_+=N\mathscr{E}_f(k_f r_D)^2$$

and the linear damping $\gamma_s=-k_f c_s(m/M)^{1/2}$. Here $\mathscr{E}_f$ is the energy with which a collapsing soliton begins to get Landau-damped (owing to the same process of non-linear conversion, $\mathscr{E}_f$ may be smaller than $\mathscr{E}_i$). Using also (21) we get

$$W^s=\frac{Q_e}{\omega_{pi}}\varepsilon(k_f r_D)\left(\frac{M}{m}\right)^{1/2},\quad \varepsilon=\frac{\mathscr{E}_f}{\mathscr{E}_i}\leq 1. \tag{25}$$

The non-linear damping rate of Langmuir waves is (see, e.g., Ref. 7)

$$\gamma_{ls}\approx\frac{1}{9}\frac{W^s}{nT}\frac{\gamma(k_f)}{(k_f r_D)^4}.$$

From (22) and (24) it follows that $\gamma(k_f)=-k_f c_s$. The final expression for the non-linear damping rate takes the form

$$\gamma_{ls}\approx-\frac{1}{9}\frac{Q_e}{nT}\varepsilon\left(\frac{M}{m}\right)^{1/2}\frac{1}{(k_f r_D)^2}. \tag{26}$$

One can easily write down an equation for the quantity ϵ:

$$\varepsilon=\mathscr{E}_f/\mathscr{E}_i=\exp(\gamma_{ls}/k_i c_s),$$

where we have introduced the initial collapse scale $\delta_i=k_i^{-1}$ and taken into account that the collapse time $\tau(\delta_i)\approx(k_i c_s)^{-1}$.

Finally we get

$$\frac{\ln\varepsilon}{\varepsilon}=-\frac{1}{9}\frac{Q_e}{nT\omega_{pi}}\left(\frac{M}{m}\right)^{1/2}\frac{1}{k_f^2 k_i r_D^3}. \tag{27}$$

We further consider the energy balance. Using the fact that the energy density of the Langmuir condensate is

$$W^l\approx(k_i r_D)^2 nT,$$

we can obtain the following relation between the power dissipated by collapsing solitons and the total pumping power:

$$Q=Q_c-\gamma_{ls}W^l=Q_c\left[1+\frac{\varepsilon}{9}\left(\frac{M}{m}\right)^{1/2}\left(\frac{k_i}{k_f}\right)^2\right]. \tag{28}$$

We first of all consider a regime when the whole of the pumping energy is dissipated by the collapsing solitons:

$$q=\frac{1}{9}\left(\frac{M}{m}\right)^{1/2}\left(\frac{k_i}{k_f}\right)^2\varepsilon<1. \tag{29}$$

In that case non-linear conversion may be appreciable, if $\epsilon\ll 1$. Using Eqs. (27),(29) we bring this condition to the form

$$|\ln\varepsilon|=q\frac{Q}{nT\omega_{pi}(k_i r_D)^3}>1. \tag{30}$$

Substituting into (21) $Q_c = Q$ we have

$$Q = N\mathscr{E}_i \sim NnTr_D^2 k_i^{-1}. \tag{31}$$

The maximum number of solitons is determined by the condition of "dense packing"

$$N/k_i c_s \leqslant k_i^3. \tag{32}$$

It follows from (31) and (32) that

$$Q/nT\omega_{pi}(k_i r_D)^3 \leqslant 1, \tag{33}$$

that is, for weak pumping the quantity k_i may be independent (for instance, be determined by the boundary of the differential-transfer region), but when the power Q increases the modulational wavenumber starts to change accordingly. It is thus impossible to satisfy condition (30). If

$$k_i/k_f < 3(m/M)^{1/4},$$

non-linear conversion dissipates the power directly from the condensate:

$$Q_1 = \frac{Q}{9}\left(\frac{M}{m}\right)^{1/2}\left(\frac{k_i}{k_f}\right)^2$$

and from the collapsing solitons:

$$Q_2 = \frac{Q}{9}\left(\frac{M}{m}\right)^{1/2}\left(\frac{k_i}{k_f}\right)^2 \frac{Q}{nT\omega_{pi}(k_i r_D)^3},$$

but the main dissipation channel is Landau damping.

Let now the main dissipative process be the non-linear conversion directly from the Langmuir condensate. We have from (28)

$$Q_e = \frac{9}{\varepsilon}\left(\frac{m}{M}\right)^{1/2}\left(\frac{k_f}{k_i}\right)^2 Q. \tag{34}$$

Substituting (34) into (27) gives

$$|\ln \varepsilon| = Q/nT\omega_{pi}(k_i r_D)^3. \tag{35}$$

We emphasize that in that case $Q \gg Q_c$ so that the right-hand side of (35) can also be much larger than unity. The condition under which the predominantly non-linear dissipation is realized is the opposite of condition (29 and can be written in the form

$$\frac{k_i}{k_f} > 3\left(\frac{m}{M}\right)^{1/4} \exp\frac{Q}{2nT\omega_{pi}(k_i r_D)^3}. \tag{36}$$

Even for weak pumping when the exponential on the right-hand side of (36) can be neglected, the inertial range turns out to be rather narrow. One can obtain the same result qualitatively also in the framework of the supersonic model of the collapse. Of course, it can be used only when $\epsilon \approx 1$. One can show that also in the supersonic case Eq. (35) remains valid. Using the conditions for the realization of the supersonic regime the calculations yield

$$\frac{k_i}{k_f} > \frac{3^{4/3}}{(k_i r_D)^{1/3}}\left(\frac{m}{M}\right)^{1/2} \text{ when } \frac{Q}{nT\omega_{pi}} < (k_i r_D)^3 < \left(\frac{m}{M}\right)^{3/2},$$

$$\frac{k_i}{k_f} > \frac{3^{4/3}}{(k_i r_D)^{1/3}}\left(\frac{m}{M}\right)^{1/3}\left(\frac{Q}{nT\omega_{pi}}\right)^{1/9} \tag{36'}$$

$$\text{when } (k_i r_D)^3 > \frac{Q}{nT\omega_{pi}} > \left(\frac{m}{M}\right)^{3/2}.$$

As in the time-dependent problem of plasma heating during collapse the reaching of even relativistic velocities is connected with a relatively small total energy contribution, one can state that, starting with the quasi-linear regime, the heating process ultimately necessarily proceeds when $v_0(t)$ increases into the non-linear conversion regime. In this connection it is of interest to see how the "tail" parameters change in that case and, in particular, whether the number of accelerated particles remains constant.

One can write the non-linear diffusion tensor for our process as follows:

$$D_{\alpha\beta}(\mathbf{v}) = \frac{W^l}{\omega_{pe}}\int d\mathbf{k}\frac{W_{\mathbf{k}}^s}{kc_s}T(\mathbf{k},\mathbf{v})\delta(\omega_{pe}-\mathbf{kv})k_\alpha k_\beta. \tag{37}$$

In the same notation, the non-linear damping rate has the form

$$\gamma_{ls} = nm\int d\mathbf{k}\frac{W_{\mathbf{k}}^s}{kc_s}\int d\mathbf{v}\, T(\mathbf{k},\mathbf{v})\mathbf{k}\frac{\partial f}{\partial \mathbf{v}}\delta(\omega_{pe}-\mathbf{kv}). \tag{38}$$

We now use the fact that $W^l = Q/\gamma_{ls}$. Substituting (38) into (37) we get the diffusion equation in a very simple form:

$$\frac{\partial f}{\partial t} = \frac{1}{v^2}\frac{\partial}{\partial v}v^2 D(v)\frac{\partial f}{\partial v},$$

$$D(v) = \frac{\pi}{4}\frac{Q}{nT}\left(\frac{M}{m}\right)^{1/2}\frac{v_{Te}v_0^4(t)}{v^3}. \tag{39}$$

In the derivation of the diffusion coefficient (39) we used the fact that in the consistent problem the condition $\gamma = -k_f c_s$ must be satisfied and the self-similar solution obtained must conserve this quantity.

Strictly speaking, there still occurs in the diffusion coefficient the ratio

$$\int W_{\mathbf{k}}^s\left(\frac{k}{k_f}\right)^n dk \Big/ \int W_{\mathbf{k}}^s\left(\frac{k}{k_f}\right)^m dk, \quad \frac{n}{m} \sim 1, \tag{40}$$

which in the self-similar regime is a constant of the order of unity.

Introducing the scaling variables

$$f(v,t) = \tau^\alpha \Phi(\xi), \quad \tau = \frac{\pi}{4}\frac{Q}{nT}\left(\frac{M}{m}\right)^{1/2} t,$$

$$\xi = v/v_0, \quad v_0 \sim \tau^\beta$$

and using the fact that

$$Q \sim \frac{\partial}{\partial t}\int f v^4 d\mathbf{v} = \text{const},$$

we arrive at the following result:

$$\beta = 1, \quad \alpha = -4,$$

which corresponds exactly to the self-similar regime of resonance heating in sonic collapse.[16] Such self-similarity, indeed, retains the condition $\gamma = -k_f c_s$ which determines the total number of accelerated particles. The function $v_0(t)$ is determined by the conservation law for energy and, hence, also is the same as the result of Ref. 16. Only the asymptotic behavior of the solution for $v > v_0(t)$ changes because of the fast decrease of $D(v)$ with velocity:

$$\Phi(\xi) \propto \exp(-\xi^5) \quad \text{when} \quad \xi > 1. \tag{41}$$

Other characteristics of the solution are directly taken over from the quasi-linear problem:

$$\begin{aligned} f_\infty(v) &\approx c_s/v^2, \quad v \ll v_0(t), \\ v_0(t) &\approx v_{Te} \frac{Qt}{nT}\left(\frac{M}{m}\right)^{1/2}, \\ n_R(t) &\approx n\left(\frac{m}{M}\right)^{1/2} \frac{v_{Te}}{v_0(t)}. \end{aligned} \tag{42}$$

The transition from the quasi-linear to the non-linear heating regime is thus a smooth one without changes in the self-similarity law and retaining all main parameters of the electron "tail".

It is interesting to note that this result is also conserved when one goes over to the supersonic collapse model. In particular, the results obtained in the quasi-linear approximation are retained for the number of resonant particles and their characteristic velocities:

a) when $\frac{Q}{nT\omega_{pi}} < (k_i r_D)^3 < \left(\frac{m}{M}\right)^{3/2}$

$$\frac{n_R}{n} \approx \left(\frac{m}{M}\right)^{1/4} \left(\frac{v_{Te}}{v_0(t)}\right)^{4/3}, \quad v_0(t) \approx \frac{(Qt)^2}{m^2 n^2 v_{Te}^3 (m/M)^{1/2}}; \tag{43a}$$

b) when $(k_i r_D)^3 > \frac{Q}{nT\omega_{pi}} > \left(\frac{m}{M}\right)^{3/2}$

$$\frac{n_R}{n} \approx \left(\frac{m}{M}\right)^{1/2} \left(\frac{nT\omega_{pi}}{Q}\right)^{1/4} \left(\frac{v_{Tc}}{v_0(t)}\right)^{3/2}, \tag{43b}$$

$$v_0(t) = \left(\frac{Qt}{nT}\right)^2 \left(\frac{Q}{nT\omega_{pi}}\right)^{1/3} v_{Te} \left(\frac{M}{m}\right)^{1/2}.$$

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