Behavior of the field of an electromagnetic wave in an inhomogeneous plasma near the point at which ϵ_{xx} vanishes

S. M. Dickman

Institute for Physical Problems, Academy of Sciences of the USSR
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The problem of the behavior of the field of an electromagnetic wave in an inhomogeneous anisotropic medium near the singularity $\epsilon_{xx}=0$ is solved generally in the case when the characteristic wavelength is much smaller than the size of the inhomogeneity. The solution is valid for any means of exciting the wave, and also does not depend on the type of resonance under consideration, which is determined by the specific nature of spatial dispersion or by collisions. Cases are considered of applying the theory developed in the present paper for investigating different plasma resonances arising in an inhomogeneous plasma near a singularity.

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1. INTRODUCTION. INITIAL EQUATIONS

The problem of the behavior of the field of a wave near a singularity in a one-dimensionally inhomogeneous (along the x axis) medium has been studied by many authors.[1-5] A rigorous mathematical solution within the framework of a theory linear in the field has been obtained by Pilya and Fedorov[4] by means of investigating a differential equation of the fourth order with a small coefficient of the highest order derivative. In their discussion a magnetoactive plasma was considered and it was assumed that near the singularity the wave was propagated at a small angle with respect to the x axis:

$$k_x \gg k_\perp \tag{1.1}$$

($\mathbf{k}_\perp$ lies in the y, z plane), while the "thermal" addition to the dielectric permittivity tensor $\epsilon^0_{ij}(x)$ for a cold plasma arising as a result of spatial dispersion has the form

$$\varepsilon_{xx}^T(\mathbf{k}) = \beta k_x^2. \tag{1.2}$$

However, it is well known that conditions (1.1)–(1.2) by no means exhaust all possible cases of the behvior of the wave near the singularity at $x=0$ (for convenience we place the singularity at the origin, so that $\epsilon^0_{xx}(0)=0$). In particular, (1.1) may not hold if the incident wave is formed outside the plasma with the aid of a special delay system as a result of which the component $\mathbf{k}_\perp$ of the propagation vector $\mathbf{k}$ directed along the surface of the plasma turns out to be much bigger than ω/c, and may attain the characteristic values of k_x near $x=0$. Moreover we note that (1.1) does not hold if the source of the wave is situated inside the plasma itself as is the case when the oscillations are excited by an electron beam.

In its turn the dependence of the permittivity tensor on $\mathbf{k}$ is determined by specific properties of the plasma and by the nature of the observed resonance at $x=0$, and even under conditions of weak spatial dispersion this dependence is not always described by formula (1.2). For example, in a nonisothermal isotropic plasma with $T_e \gg T_i$ the relation (1.2) does not hold if the singularity at $x=0$ is associated with the excitation of ionic Langmuir waves.[6] In exactly the same manner (1.2) does not apply to a plasma whose electrons move with respect to the ions, and also for a plasma with beams.

All these specific cases in which conditions (1.1), (1.2) do not hold indicate that it would be useful to solve the problem formulated above in a more general form than was done in Ref. 3, 4. In the present paper a method of solution is presented which does not depend on the specific form of the permittivity and on the relation between the components k_x and $\mathbf{k}_\perp$. It is essential only that the plasma (we shall speak of a plasma, although in principle the main results of the paper are valid for an arbitrary medium) is assumed to be weakly inhomogeneous, i.e., specifically that near the singularity the following inequalities hold

$$r_D \ll a, \tag{1.3}$$

$$|k_x a| \gg 1. \tag{1.4}$$

Here r_D is the Debye radius, a is a characteristic length of the inhomogeneity over which the parameters of the plasma are altered. We assume that the wave frequency is given, i.e., the time dependence is of the form $e^{-i\omega t}$. The material equation can be written in the form

$$D_i = \int \hat{\varepsilon}_{ij}(x, \mathbf{r}-\mathbf{r}') E_j(\mathbf{r}')\, d\mathbf{r}', \quad i,j = x, y, z.$$

If $k \gg \omega/c$, then the wave is an electrostatic one $E_i(\mathbf{r}) = -\partial\varphi(\mathbf{r})/\partial x_i$, where the potential satisfies the Maxwell equation

$$\frac{\partial}{\partial x_i}\left[\int \hat{\varepsilon}_{ij}(x, \mathbf{r}-\mathbf{r}') \frac{\partial \varphi(\mathbf{r})}{\partial x_j'}\, d\mathbf{r}'\right] = 0. \tag{1.5}$$

2. FORMAL SOLUTION OF EQ. (1.5)

First of all we note that in the zero order approximation with respect to $(k_x a)^{-1}$ and $r_D a^{-1}$ the kernel $\epsilon_{ij}(x_0, \mathbf{r}-\mathbf{r}')$ is known. For a given value of x_0 this is simply the kernel for the permittivity of a homogeneous plasma with the parameters of the initial inhomogeneous plasma in the plane $x = x_0$. We represent $\hat{\epsilon}_{ij}$ in the form of a sum of two terms:

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$$\hat{\varepsilon}_{ij}(x, \mathbf{r}-\mathbf{r}') = \varepsilon_{ij}^0(x)\,\delta(\mathbf{r}-\mathbf{r}') + \varepsilon_{ij}^T(x, \mathbf{r}-\mathbf{r}'),$$

where $\epsilon_{ij}^0(x)$ is the permittivity of a cold collisionless plasma

$$\varepsilon_{ij}^0(x)\,\delta(\mathbf{r}-\mathbf{r}') = \lim \hat{\varepsilon}_{ij}(x, \mathbf{r}-\mathbf{r}'), \quad T\to 0, \quad \nu_{eff}\to 0,$$

while the quantity ϵ_{ij}^T is determined by collisions and by spatial dispersion. We investigate the solution of equation (1.5) for $|x| \ll a$. With this in mind we expand the cold part of the permittivity in terms of x/a restricting ourselves to first order terms:

$$\varepsilon_{ij}^0 = a_{ij} - x b_{ij}. \tag{2.1}$$

Here $a_{ij} = \epsilon_{ij}^0(0)$, $b_{ij} = -\partial\epsilon_{ij}^0/\partial x\,|_{x=0}$. According to the condition

$$a_{xx} = 0, \tag{2.2}$$

we can without loss of generality also assume that

$$b_{xx} = a^{-1} > 0. \tag{2.3}$$

In a cold collisionless plasma the solution of equation (1.5) becomes infinite in the plane $x=0$, and therefore condition (1.4) near this plane is certainly satisfied. This condition is also satisfied if the dispersion and the collisions are not significant, and this enables us to consider ϵ_{ij}^T in equation (1.5) in the zero order approximation in terms of $(k_x a)^{-1}, r_D a^{-1}, x/a$. At the same time we substitute ϵ_{ij}^0 into (1.5) in the form of (2.1). After this we apply to (1.5) the Fourier transformation:

$$i b_{ij} k_i k_j \frac{\partial \bar{\varphi}(\mathbf{k})}{\partial k_x} + i b_{ix} k_i \bar{\varphi}(\mathbf{k}) - a_{ij} k_i k_j \bar{\varphi}(\mathbf{k}) - \varepsilon_{ij}^T(\mathbf{k}) k_i k_j \bar{\varphi}(\mathbf{k}) = 0; \tag{2.4}$$

here $\bar{\varphi}(\mathbf{k})$ is the Fourier component of the potential

$$\bar{\varphi}(\mathbf{k}) = \int \varphi(\mathbf{r})\, e^{-i\mathbf{k}\mathbf{r}} d\mathbf{r},$$

$\epsilon_{ij}^T(\mathbf{k})$ is the thermal and collision part of the permittivity of the corresponding homogeneous plasma:

$$\varepsilon_{ij}^T(\mathbf{k}) = \int [\lim_{a\to\infty} \hat{\varepsilon}_{ij}(0, \mathbf{r}-\mathbf{r}')] \exp(-i\mathbf{k}\mathbf{r} + i\mathbf{k}\mathbf{r}')\, d\mathbf{r}.$$

In (2.4) we have also utilized the fact that the potential $\varphi(\mathbf{r})$ according to our assumption falls off outside the region $x \ll a$, so that in carrying out the integration one can assume that $\varphi(\mathbf{r})|_{x=\pm\infty} = 0$. The condition that the dispersion and the collisions are not significant evidently now takes on the form

$$|\varepsilon_{ij}^T(\mathbf{k})| \ll 1. \tag{2.5}$$

The solution of (2.4) is an asymptotic formula for the Fourier component of the potential for large $|k_x|$ (cf., (1.4)). Depending on the sign of k_x we obtain

$$\bar{\varphi}_\pm(\mathbf{k}) = \frac{\varphi_\pm^0(\mathbf{k}_\perp)}{(a b_{ij} k_i k_j)^{1/2}} Q(\mathbf{k}) \exp\left[R(\mathbf{k}) - i \int_{\tilde{k}}^{k_x} \varepsilon^T(\mathbf{k}')\, k'^2 (b_{ij} k_i' k_j')^{-1} dk_x'\right], \tag{2.6}$$

where

$$\varepsilon^T(\mathbf{k}) = k^{-2} \varepsilon_{ij}^T(\mathbf{k}) k_i k_j$$

is the thermal and collision part of the longitudinal permittivity; $\mathbf{k}'$ is a vector which has the components k_x', k_y', k_z';

$$Q(\mathbf{k}) = \exp[2^{-1}\Delta^{-1/2}(b_{x\alpha} - b_{\alpha x}) k_\alpha L],$$
$$R(\mathbf{k}) = -i(2b_{xx})^{-1}(a_{\alpha x} + a_{x\alpha}) k_\alpha \ln(a^2 b_{ij} k_i k_j)$$
$$+ i(2b_{xx})^{-1} k_\alpha k_\beta [(a_{\beta x} + a_{x\beta})(b_{\alpha x} + b_{x\alpha}) - 2b_{xx} a_{\alpha\beta}] \Delta^{-1/2} L,$$
$$\Delta = [(b_{\alpha x} + b_{x\alpha}) k_\alpha]^2 - 4 b_{\alpha\beta} k_\alpha k_\beta b_{xx},$$
$$L = \ln\{[(b_{xi} + b_{ix}) k_i - \Delta^{1/2}][(b_{xi} + b_{ix}) k_i + \Delta^{1/2}]^{-1}\}.$$

The Greek subscripts take on the values y and z, $\varphi_\pm^0(\mathbf{k}_\perp)$ are arbitrary functions of the given vector $\mathbf{k}_\perp$, $\tilde{k}$ is an arbitary constant of integration which must be chosen so that the integral in (2.6) should converge. In addition to (2.5) in order that the solution (2.6) be valid we also need the condition

$$|k_x x \varepsilon^T(\mathbf{k})| \ll 1, \tag{2.7}$$

which arises as a result of neglecting quantities of the order of $|x\hat{\epsilon}_{ij}^T|$ in the initial equation (1.5). If the principal contribution to the integral

$$\varphi(x, \mathbf{k}_\perp) = (2\pi)^{-1} \int \bar{\varphi}(\mathbf{k}) \exp(i k_x x)\, dk_x$$

is made by the values of k_x for which conditions (1.4), (2.5), (2.7), are satisfied then, neglecting the region $|k_x| \lesssim a^{-1}$, we obtain

$$\varphi(x, \mathbf{k}_\perp) = (2\pi)^{-1} \int_{-\infty}^{0} \bar{\varphi}_-(\mathbf{k}) \exp(i k_x x)\, dk_x + (2\pi)^{-1} \int_{0}^{\infty} \bar{\varphi}_+(\mathbf{k}) \exp(i k_x x)\, dk_x. \tag{2.8}$$

3. INVESTIGATION OF THE OBTAINED SOLUTION

As is expected for $\epsilon^T = 0$ (i.e., in the absence of dispersion and collisions) the integrals in (2.8) diverge at the point $x=0$. We investigate the convergence of these integrals for $\epsilon^T \neq 0$. According to (2.3) the first integral in (2.8) converges if $\mathrm{Im}\,\epsilon^T(\mathbf{k}) \geq 0$ for $k_x \to -\infty$, and the second one converges if $\mathrm{Im}\,\epsilon^T(\mathbf{k}) \leq 0$ for $k_x \to +\infty$. This conclusion follows directly from the Hermitian nature of the tensors a_{ij} and b_{ij}, and also from the fact that for sufficiently large values of $|k_x|$ the inequalities $b_{ij} k_j k_j > 0$, $\Delta > 0$ hold as the result of which L is real while $R(\mathbf{k})$ and $\ln Q(\mathbf{k})$ are purely imaginary. The quantity $\mathrm{Im}\,\epsilon^T(\mathbf{k})$ always contains a positive term associated with collisions. If this term is sufficiently large then collisions play the determining role in the question of the convergence of the integrals (2.8). In this case in order for the solution to have a meaning we must set the constant of integration $\varphi_+^0(\mathbf{k}_\perp)$ indentically equal to zero, as a result of which we obtain

$$\varphi(x, \mathbf{k}_\perp) = \varphi_-^0(\mathbf{k}_\perp)(2\pi)^{-1} \int_{-\infty}^{0} (a b_{ij} k_i k_j)^{-1/2} Q(\mathbf{k}) \exp\left[i k_x x + R(\mathbf{k}) - i \int_{\tilde{k}}^{k_x} \mathscr{E}(k_x', \mathbf{k}_\perp)\, dk_x'\right] dk_x, \tag{3.1}$$

where

$$\mathscr{E}(k_x, \mathbf{k}_\perp) = (b_{ij} k_i k_j)^{-1} \varepsilon^T(\mathbf{k}) k^2.$$

But, on the other hand, our solution is valid only if conditions (2.5), (2.7) are satisfied. If collisions are rare, then it might happen that they must be taken into account in $\epsilon^T(\mathbf{k})$ only for values of k_x lying outside the limits allowed by the inequalities (2.5), (2.7). In this case the question of convergence of the integrals (2.8) must be solved without taking collisions into account. The limits of integration $\pm\infty$ must be interpreted as values of k_x for which any one of conditions (2.5), (2.7) ceases to hold. It is well known that for a medium in thermodynamic equilibrium one must always have

$$\mathrm{Im}\,\varepsilon^T(\mathbf{k})>0.$$

In this case the plasma is stable, the electromagnetic waves are absorbed (in the absence of collisions the absorption takes place according to the Landau mechanism), and generation of waves, i.e., transfer of energy from the plasma to the electromagnetic field, is impossible. In such a plasma the solution as before is determined by (3.1) which represents a superposition of plane longitudinal waves traveling in the direction of negative x (from right to left). This result has a completely understandable physical meaning, it is related to the fact that longitudinal waves cannot exist outside a plasma, they originate in the neighborhood of $x=0$ and are propagated in the region of transparency until they are absorbed. By assumption the plasma occupies the half-space on the right and therefore the region of absorption is situated to the left of the transparency zone. In a stable plasma there will be no waves propagated since otherwise this would mean that instead of absorption generation of waves is taking place.

The situation is altered if one considers an unstable plasma in which the inequality $\mathrm{Im}\epsilon^T<0$ is possible. For example, let the plasma contain an electron beam propagating along the x axis from left to right with a velocity u greater than the thermal velocity of the plasma particles. If damping of the oscillations by particles belonging to the plasma is negligibly small, then the following inequalities hold[6]

$$\mathrm{Im}\,\varepsilon^T>0 \quad \text{for} \quad k_x<\omega/u,$$
$$\mathrm{Im}\,\varepsilon^T<0 \quad \text{for} \quad k_x>\omega/u.$$

Therefore if the values of $k_x>\omega/u$ satisfy conditions (2.5), (2.7) then both integrals in (2.8) are convergent. In this case there are no reasons to regard the constant of integration φ_+^0 to be identically equal to zero, and the solution is determined by the general formula (2.8) in which the second integral describes longitudinal waves propagated from left to right. These waves are excited in the plasma by the resonance particles of the beam, i.e., a transfer of energy occurs from the medium to the electromagnetic field. In future we shall not consider unstable plasmas and shall dwell in greater detail on the investigation of the solution (3.1).

4. ASYMPTOTIC FORMULAS

A rigorous mathematical investigation can be carried out only if one introduces additional specific assumptions into the problem posed above. For example, in Ref. 4 the solution was obtained in the form of an integral obtained from (3.1) if the relations (1.1), (1.2) are satisfied. The corresponding asymptotic expressions for the solution were also found in the same paper. Nevertheless it is possible to obtain certain natural estimates directly from the general expression (3.1). We shall consider k_x to be a complex variable. The contour of integration in (3.1) can be deformed in such a manner that the free end would recede to infinity in the region in which $\mathrm{Re}[ik_x x+\ln\overline{\varphi}_-(\mathbf{k})]<0$. In obtaining asymptotic estimates an important role can be played by saddle points in which the derivative of the integrand vanishes. Differentiating the rapidly varying exponential in (3.1) we obtain the eikonal equation:

$$k^2\varepsilon(x,\mathbf{k})=a_{ij}k_ik_j-xb_{ij}k_ik_j+\varepsilon^T(\mathbf{k})k^2=0; \tag{4.1}$$

$\epsilon(x,\mathbf{k})$ is the longitudinal permittivity of the plasma, (4.1) is a transcendental equation with respect to k_x which, generally speaking, has infinitely many solutions. Among the solutions there are three "cold" ones which remain finite for $\epsilon^T=0$. Of these three solutions only one is essential for us: $k_x=k_0(x,\mathbf{k}_\perp)$, which for $x=0$ produces in virtue of (2.2) a wave propagating parallel to the x axis ($k_0(0,\mathbf{k}_\perp)=\pm\infty$). The "hot" solutions of (4.1) which become infinite for $\epsilon^T=0$ are associated with the presence of plasma waves. We assume that a significant contribution to the integral is made by not more than one plasma wave and we denote the corresponding root of (4.1) by $k_T(x,\mathbf{k}_\perp)$. For an arbitrary relationship between the small quantity xa^{-1} and $\epsilon^T(\mathbf{k})$ the separation of the solutions into "hot" and "cold" ones is of an arbitrary nature and therefore for the sake of definiteness we assume that the plasma wave corresponds to the root of greater absolute value, i.e., $|k_0|<|k_T|$. If the contour of integration can be made to pass through the point k_T in the direction of "steepest descent," and also if

$$\left|k_T^2\frac{d^2\ln\varphi_-(\mathbf{k})}{dk_x^2}\bigg|_{k_x=k_T}\right|\gg1, \tag{4.2}$$

$$\left|(k_T-k_0)^2\frac{d^2\ln\varphi_-(\mathbf{k})}{dk_x^2}\bigg|_{k_x=k_T}\right|\gg1, \tag{4.3}$$

then the potential for the plasma wave can be easily calculated. Indeed, let AB be the direction "steepest descent" near the saddle point k_T (cf., Fig. 1). We assume that the contour of integration in (3.1) can be represented in the form of the dashed line AB with $|k_B|\gtrsim|k_T|$, $|k_T-k_B|\gtrsim|k_T|$. In virtue of the inequalities (4.2), (4.3) the integral over the segment AB can be evaluated by the saddle-point method. From this we obtain

$$\varphi_T(x,\mathbf{k}_\perp)=\varphi_-^0(\mathbf{k}_\perp)\left[2\pi a(k^2\partial\varepsilon/\partial k_x)|_{k_x=k_T}\right]^{-1/2}$$
$$\times(Q(\mathbf{k})|_{k_x=k_T})\exp\left(\Psi(\mathbf{k}_\perp,x)\right). \tag{4.4}$$

Here $\varphi_T(x,\mathbf{k}_\perp)$ is the potential for the plasma wave

$$\Psi(\mathbf{k}_\perp,x)=-\frac{i\pi}{4}+ik_Tx+R(\mathbf{k})|_{k_x=k_T}-i\int\limits_{\bar{k}}^{k_T}\mathscr{E}(k_x,\mathbf{k}_\perp)\,dk_x',$$

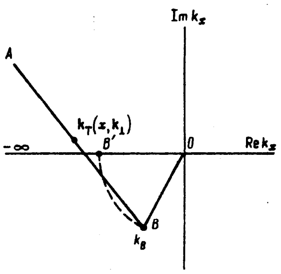

FIG. 1.

and it is assumed that

$$-\frac{3}{2}\pi<\arg\left(\frac{k^2}{b_{ij}k_ik_j}\frac{\partial\varepsilon}{\partial k_x}\right)\bigg|_{k_x=k_T}<\frac{\pi}{2}\ .$$

It can be easily verified that

$$d\Psi/dx=ik_T(x,\ \mathbf{k}_\perp),$$

so that the expression (4.4) can be represented in the form of the formula

$$\varphi_T(x,\mathbf{k}_\perp)=\varphi_-^0(\mathbf{k}_\perp)\left[2\pi ak^2\frac{\partial\varepsilon}{\partial k_x}\bigg|_{k_x=k_T}\right]^{-1/2}(Q(\mathbf{k})|_{k_x=k_T})$$
$$\times\exp\left[i\int_{\tilde{x}}^{x}k_T(x',\mathbf{k}_\perp)\,dx'-\frac{i\pi}{4}+i\tilde{k}x+R(\mathbf{k})\bigg|_{k_x=\tilde{k}}\right],\qquad(4.5)$$

in which $\tilde{x}$ is found from the equation $[\epsilon(\tilde{x},\mathbf{k})|_{k_x=\tilde{k}}]=0$. A necessary condition to obtain this estimate for the integral over AB is the requirement that $\varphi_T(x,\mathbf{k}_\perp)$ should be regular for $x\to\pm\infty$. Formulas (4.4), (4.5) must satisfy this condition automatically.

We note the following important circumstance. The function $\varphi_T(x,k_\perp)$ represents a geometrical-optics approximation obtained by us from formula (3.1) which, in its turn, is valid if the conditions (2.5), (2.7) are satisfied. Of course, in actual fact in order for (4.5) to be valid it is sufficient to have conditions (4.2), (4.3) satisfied, and the propagation vector must be determined from the eikonal equation $\epsilon_{ij}(x,\mathbf{k})k_ik_j=0$, in which the expansion in terms of x/a is not utilized. Formula (4.5) can be obtained directly from equation (1.5) by the method of the WKB approximation. If conditions (4.2), (4.3) are not satisfied, then conditions (2.5), (2.7) will certainly hold and the solution should be sought in the form (3.1).

In addition to the segment AB it is necessary to take into account the contribution which is given by integration over BO. It may be easily seen that from conditions (4.2), (4.3) with (4.1) taken into account the inequality

$$|x|^{-1}\ll|k_T(x,\ \mathbf{k}_\perp)|\qquad(4.6)$$

follows as a rule. If for $|k_x|\sim|x|^{-1}$ the condition[1)]

$$\left|\int_{k}^{k_x}[\mathscr{E}(k_x',\ \mathbf{k}_\perp)-\mathscr{E}(0,\ \mathbf{k}_\perp)]\,dk_x'\right|\ll1,\qquad(4.7)$$

is also satisfied, then in the integral (3.1) over BO one need not take into account the dependence of $\mathscr{E}(k_x',\mathbf{k}_\perp)$ on k_x', since the integral converges for values $|k_x|\sim|x|^{-1}$ long before this dependence becomes essential. As a result of integrating over the contour $BB'O$ and of neglecting the small integral over the arc BB' compared with the integral over BO' we obtain for the latter:

$$\varphi_0(x,\mathbf{k}_\perp)=\varphi_-^0(\mathbf{k}_\perp)(2\pi)^{-1}\int_{-\infty}^{0}(b_{ij}k_ik_ja)^{-1/2}$$
$$\times Q(\mathbf{k})\exp[R(\mathbf{k})+ik_x(x-\mathscr{E}(0,\mathbf{k}_\perp))]\,dk_x.\qquad(4.8)$$

The potential $\phi_0(x,\mathbf{k}_\perp)$ describes the "cold" waves in the plasma, including the characteristic longitudinal wave with $k_x=k_0(x,\mathbf{k}_\perp)$. The field is obtained by the sum of expressions (4.5) and (4.8):

$$\varphi(x,\ \mathbf{k}_\perp)=\varphi_T(x,\ \mathbf{k}_\perp)+\varphi_0(x,\ \mathbf{k}_\perp).\qquad(4.9)$$

Differentiating (4.9) with respect to x one can obtain the component E_x of the electric field and it turns out that for a real k_T the expression $|\partial\varphi_T/\partial x|\gg|\partial\varphi_0/\partial x|$ holds, and therefore E_x is determined basically by the plasma wave. But if $|\mathrm{Im}k_Tx|\gg1$, then the potential (4.5) is exponentially small and this corresponds to a strong damping of the plasma wave, and in this case $E_x\approx-\partial\varphi_0/\partial x$.

For a comparison with the results of Ref. 4 we consider separately formulas (4.5) and (4.8) in the case when

$$k_\perp\ll|k_T|.$$

In (4.8) the dependence of $\mathscr{E}(0,\mathbf{k}_\perp)$ on $k_\perp$ is then not essential. Neglecting collisions we evaluate the integral (4.8) for values of $|x|\ll k_\perp^{-1}$:

$$\varphi_0=\varphi_-^0(2\pi)^{-1}\int_{-\infty}^{0}(-ak_x)^{-1-i\sigma}\exp(ik_xx)\,d(k_xa).\qquad(4.10)$$

Here $\sigma=b_{xx}^{-1}(a_{\alpha x}+a_{x\alpha})k_\alpha$. For $\sigma=0$ the integral (4.10) diverges at the origin. But this divergence is fictitious. It can be removed by replacing the upper limit of integration by the quantity

$$\beta\sim-\max(k_\perp,\ a^{-1}).$$

We obtain finally

$$\varphi_0=\varphi_-^0(2\pi)^{-1}[\Gamma(-i\sigma)(a^{-1}x)^{i\sigma}e^{-\sigma\pi/2}-i\sigma^{-1}|\beta a|^{-i\sigma}].\qquad(4.11)$$

As is expected φ_0 for $\sigma=0$ depends on x logarithmically. The strongest singularity at the origin occurs in the component E_x^0 of the electric field:

$$E_x^0=\varphi_-^0(2\pi a)^{-1}\Gamma(1-i\sigma)(xa^{-1})^{i\sigma-1}e^{-\sigma\pi/2}.\qquad(4.12)$$

We calculate the power dissipated in the plasma. The field is determined by the sum (4.9), but the principal contribution to the integral

$$P=-\omega(8\pi)^{-1}\,\mathrm{Im}\int_{-\infty}^{+\infty}(\mathbf{ED}^*)\,dx$$

is given by the component E_x^0 (4.11). Going around the singularity at $x=0$ on the lower side we obtain

$$P=2^{-5}\pi^{-2}a^{-1}\omega|\varphi_-^0|^2.\qquad(4.13)$$

Here P differs from zero because the permittivity tensor for the cold plasma (2.1) is not Hermitian for complex x. The flux of energy transported by the wave for values of x such that $|\mathrm{Im}k_T| \ll |\mathrm{Re}k_T|$, is determined basically by the solution (4.5) and therefore can be calculated by means of the Poynting vector

$$S_x = -\omega(16\pi)^{-1}\left(k^2 \frac{\partial \varepsilon}{\partial k_x}\Big|_{k_x=k_T}\right)|\varphi_T|^2$$

(cf., Ref. 7). From this we obtain the result $S_x + P = 0$ which follows from the law of conservation of energy. It can be easily seen that (4.11)–(4.13) differ from the corresponding formulas in Ref. 4, 5 only by constant factors.

5. SOME SPECIAL CASES

First of all we consider an isotropic plasma without a magnetic field:[8, 9]

$$a_{ij} = 0, \quad b_{ij} = a^{-1}\delta_{ij},$$
$$\varepsilon^T(\mathbf{k}) = -3r_{D_e}^2(k_x^2 + k_\perp^2) \ll 1. \tag{5.1}$$

The singularity at $x=0$ in a cold plasma is associated with the excitation of electron Langmuir oscillations. Collisions are not taken into account: $\nu_{eff}/\omega \ll \epsilon^T$. Then (3.1) assumes the form ($\tilde{k}=0$)

$$\varphi(x, \mathbf{k}_\perp) = \varphi_-^0(\mathbf{k}_\perp)(2\pi)^{-1}\int_{-\infty}^{\infty}(k_x^2+k_\perp^2)^{-1/2} \times \exp(ik_x x' + ir_{D_e}^2 k_x^3 a)\,dk_x. \tag{5.2}$$

Here we have set $x' = x + 3r_{D_e}^2 k_\perp^2 a$. From this one can easily find formula (4.5) for the potential of the Langmuir plasma wave (cf., Ref. 4, 5):

$$\varphi_T(x, \mathbf{k}_\perp) = \varphi_-^0(\mathbf{k}_\perp)(-4\pi x r_{D_e}^{-1})^{-1/2}(-x'/3a)^{-1/4} \times \exp[-i\pi/4 + 2i(-x'/3)^{3/2} a^{-1/2} r_{D_e}^{-1}]. \tag{5.3}$$

The potential (4.8) is expressed in terms of special functions:

$$\varphi_0(x, \mathbf{k}_\perp) = \varphi_-^0(\mathbf{k}_\perp)(2\pi)^{-1}\int_{-\infty}^{0}(k_x^2+k_\perp^2)^{-1/2}\exp(ik_x x')\,dk_x = {}^1/_4\varphi_-^0(\mathbf{k}_\perp) \times [i(\mathbf{L}_0(k_\perp x') - I_0(x'k_\perp)) + 2\pi^{-1}K_0(x'k_\perp)]. \tag{5.4}$$

Formulas (5.3), (5.4) are valid if the inequality

$$|x'|^3 \gg r_{D_e}^2 a$$

is satisfied. The transparency zone for the plasma wave (5.3) is situated at $x'<0$. For positive values of $x'>0$ the plasma wave is exponentially damped and the potential (4.9) is determined only by the one cold wave (5.4). If $|x'|^3 \lesssim r_{D_e}^2 a$, then it is necessary to utilize the general expression (5.2). Figure 2 shows the dependence of the square of the absolute value of the component E_x of the electric field on x for different values of $k_\perp$. The dimensionless function

$$U(\xi, \varkappa) = \int_0^\infty (p^2+\varkappa^2)^{-1/2} p \exp(-ip\xi - ip^3)\,dp$$

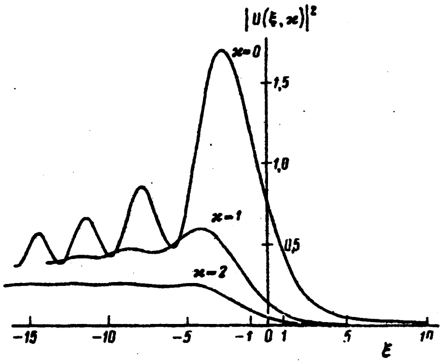

FIG. 2. The dependence of the function U on the coordinate $\xi = x r_{D_e}^{-2/3} a^{-1/3}$ (the parameter is $\kappa = k_\perp r_{D_e}^{2/3} a^{1/3}$).

is related to E_x by the expression

$$E_x(x, k_\perp) = i\varphi_-^0(\mathbf{k}_\perp)(2\pi)^{-1} r_{D_e}^{-2/3} a^{-1/3} U(x' r_{D_e}^{-2/3} a^{-1/3}, k_\perp r_{D_e}^{2/3} a^{1/3}).$$

Figure 3 shows graphs of the real and the imaginary parts of $U(\xi, \kappa)$.

We now consider the case when the singularity at $x=0$ is associated with the excitation in an isotropic plasma of ionic Langmuir waves

$$T_e \gg T_i, \quad r_{D_e}^{-1} \ll k_x \ll r_{D_i}^{-1}.$$

If at the same time we also have $(\nu_{eff}/\omega)^{1/3} \gg (k_x r_{D_e})^{-1} \gg k_x r_{D_i}$, then the thermal motion of the ions need not be taken into account and the function $\epsilon^T(\mathbf{k})$ takes on the form[2)]

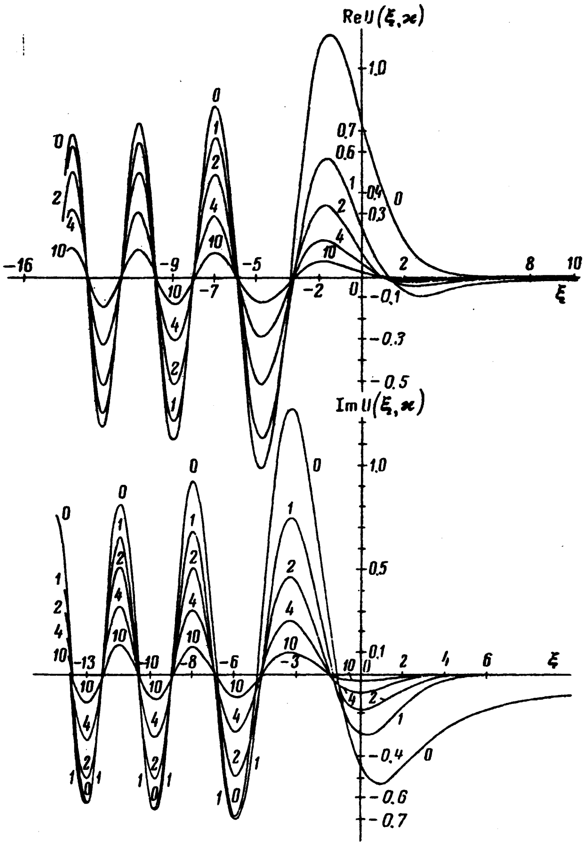

FIG. 3.

$$\varepsilon^T(\mathbf{k})=(r_{D_e}k)^{-2}+i\nu_{eff}/\omega, \quad |\varepsilon^T(\mathbf{k})|\ll 1. \tag{5.5}$$

For the sake of simplicity we assume that condition (1.1) is satisfied, then using formula (3.1) we obtain setting $\tilde{k}=-\infty$,

$$\varphi(x)=-\varphi_-^0(2\pi)^{-1}\int_{-\infty}^{h}\exp(ik_x x+iar_{D_e}^{-2}k_x^{-1}$$

$$+k_x a\nu_{eff}/\omega)k_x^{-1}\,dk_x, \quad h\sim\max(k_\perp, r_{D_e}^{-1}).$$

The plasma wave in the geometrical-optics approximation is determined as before by formulas (4.4), (4.5), but formulas (4.8), (4.10) are now no longer valid, since the inequality (4.7) for the function (5.5) does not hold. Nevertheless one can easily obtain asymptotic estimates by noting that the field E_x in the plasma is expressed in terms of a Bessel function:

$$E_x=i\varphi_-^0(2\pi)^{-1}\int_0^\infty\exp(-ipx-iar_{D_e}^{-2}p^{-1}-p\nu_{eff}a/\omega)\,dp$$

$$=i\varphi_-^0(\pi r_{D_e})^{-1}(a/x'')^{1/2}K_1[2r_{D_e}^{-1}i(ax'')^{1/2}].$$

Here

$$x''=x-ia\nu_{eff}/\omega, \quad -\pi<\arg x''<0.$$

From this we obtain for $|x''|\ll r_{D_e}^2/a$ in accordance with (4.4), (4.5)

$$E_x=\tfrac{1}{2}\varphi_-^0 a^{1/4}(\pi r_{D_e})^{-1/2}x''^{-3/4}\exp[\tfrac{1}{4}i\pi-2ir_{D_e}^{-1}(ax'')^{1/2}]. \tag{5.6}$$

While for $|x''|\ll r_{D_e}^2/a$ we obtain $E_x=\varphi_-^0/2\pi x''$. We note that the transparency zone in which the plasma wave (5.6) is propagated is situated at $x>0$, i.e., to the right of the singularity.

We now turn to the case when the spatial dispersion is not associated with the thermal motion of the particles but is determined by the relative macroscopic motion of electrons and ions. We assume that the electrons move with respect to the ions along the x axis with velocity u, then

$$\varepsilon=1-\frac{\omega_{pe}^2(x)}{(\omega-k_x u)^2}-\frac{\omega_{pi}^2(x)}{\omega^2}.$$

If $|k_x u|\ll\omega$, then for $\omega_{pe}(x)\approx\omega$ the weak spatial dispersion approximation is valid:

$$\varepsilon=-x/a+\varepsilon^T(k_x), \quad \varepsilon^T(k_x)=-2k_x u/\omega\ll 1. \tag{5.7}$$

From this in accordance with (3.1) we obtain [it is assumed that the relations (1.1) and (5.1) are valid, and moreover $\tilde{k}=0$]:

$$\varphi=-\varphi_-^0(2\pi)^{-1}\int_{-\infty}^{h}\exp(ik_x x+ik_x^2 ua\omega^{-1})k_x^{-1}\,dk_x,$$

$$h\sim-\max(k_\perp, a^{-1}). \tag{5.8}$$

The electric field is expressed in terms of the probability integral $\Phi(z)$:

$$E_x=i\varphi_-^0(2\pi)^{-1}\int_0^\infty\exp(-ipx+ip^2au\omega^{-1})\,dp$$

$$=\tfrac{1}{4}i\varphi_-^0\omega^{1/2}(\pi a|u|)^{-1/2}\exp(\tfrac{1}{4}i\pi\,\mathrm{sign}\,u$$

$$-i\omega x^2/4au)[1-\Phi(\tfrac{1}{2}e^{i\theta}|x|\omega^{1/2}a^{-1/2}|u|^{-1/2})]. \tag{5.9}$$

Here θ takes on the values $3\pi/4$, $\pi/4$, $-\pi/4$, $-3\pi/4$ depending on which of the relationships $x>0$, $u>0$; $x>0$, $u<0$; $x<0, u>0$; $x<0$, $u<0$ respectively hold. The asymptotic expressions can be sought either from formulas (4.4), (4.5), (4.8), (4.10), or directly from expression (5.9). As a result we obtain the electric field of the plasma wave:

$$E_x^T=\frac{1}{4}\frac{i\varphi_-^0\omega^{1/2}}{(a|u|\pi)^{1/2}}\exp\left[\frac{i\pi}{4}\,\mathrm{sign}\,u-\frac{i\omega x^2}{4au}\right][1+\mathrm{sign}(ux)]. \tag{5.10}$$

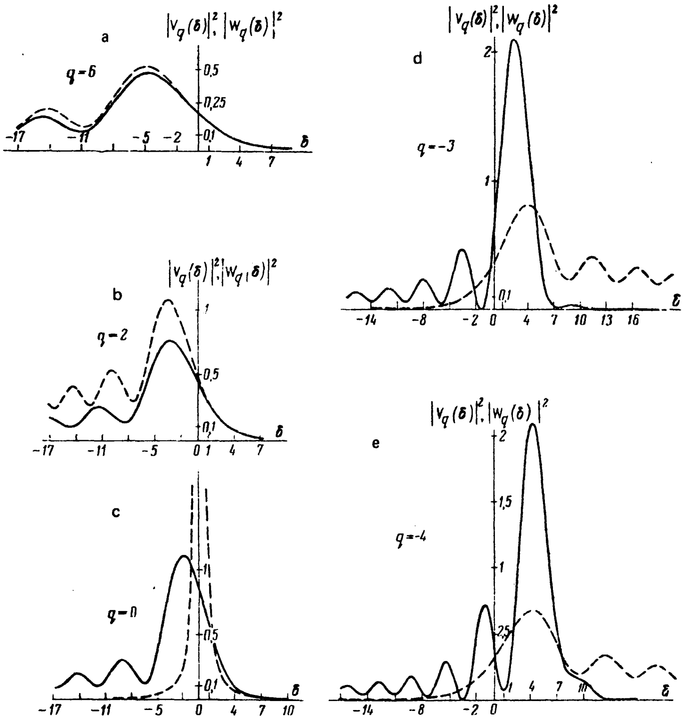

FIG. 4. The dependence on δ of $|V_q(\delta)|^2$ (solid curves) and of $|W_q(\delta)|^2$ (dotted curves) for different q.

The wave (5.10) is excited only if u and x have the same sign. This is associated with the fact that the saddle point k_T of the integrand of (5.8) lies on the negative semiaxis in the case when $ux>0$. But if $ux<0$, then $k_T>0$ and the neighborhood of k_T is not significant for the evaluation of the integral (5.8). The "cold" solution E_x^0 does not depend on the signs of u and x and is always determined by the formula:

$$E_x^0=\varphi_-^0(2\pi x)^{-1}. \qquad (5.10')$$

Expressions (5.10), and (5.10′) are valid when $x^2\omega \gg a|u|$. The field in the plasma is the sum of these expressions: $E_x=E_x^T+E_x^0$. If the opposite inequality $x^2\omega \ll a|u|$ holds then the field does not depend on x:

$$E_x=\frac{1}{4}i\varphi_-^0\left(\frac{\omega}{au\pi}\right)^{1/2}\exp\left(\frac{i\pi}{4}\operatorname{sign}u\right).$$

In conclusion we consider a magnetoactive plasma. We take the propagation vector $\mathbf{k}_\perp$ to be small, assuming that the inequalities (1.1) and $|\sigma|\ll 1$ are satisfied. Without taking collisions into account we consider the first two terms in the expansion of $\epsilon^T(k)$ in terms of k_x:

$$\varepsilon^T(k)=-\beta k_x^2-\gamma k_x^4. \qquad (5.11)$$

The field in the plasma is determined by the integral

$$E_x=i\varphi_-^0(2\pi)^{-1}\int_0^\infty \exp\left(-ipx-\frac{1}{3}i\beta p^3-\frac{1}{5}i\gamma p^5\right)dp.$$

For certain directions of the magnetic field the coefficient β vanishes (cf., the expression for β given in Ref. 5). In this case the behavior of $E_x(x)$ near the origin is determined by the magnitude of γ. In particular, γ determines the value of the pricipal maximum of the amplitude of the field $|E_x|^2$. Figure 4 shows the dependence of the square of the absolute value of the function

$$V_q(\delta)=\int_0^\infty \exp(-i\delta t-iqt^3-it^5)\,dt$$

on δ (cf. also Fig. 5). The field E_x is related to V_q by the expression

$$E_x(x,\ \beta,\ \gamma)=i(2\pi)^{-1}\varphi_-^0(5/\gamma)^{1/5}V_q[x(5/\gamma)^{1/5}],$$

in which $q=\beta(5/\gamma)^{3/5}/3$. The dotted curve shows the dependence on δ of the square of the absolute value of the function.

$$W_q(\delta)=\int_0^\infty \exp(-i\delta t-iqt^3)\,dt.$$

The field in the plasma is proportional to $W_q(\delta)$ if we do not take into account the second term in (5.11). Figure 6 shows the dependence on the parameter q of the magnitude of the principal maximum of $|V_q|^2$. The maximum is determined for different values of δ for a given q. As can be seen from the diagram this dependence is nonmonotonic for $q<0$ (i.e., for negative β). Moreover, the position of the principal maximum δ_m (the thin solid line) behaves in a discontinuous manner for certain negative values of q. Such a discontinuous behavior

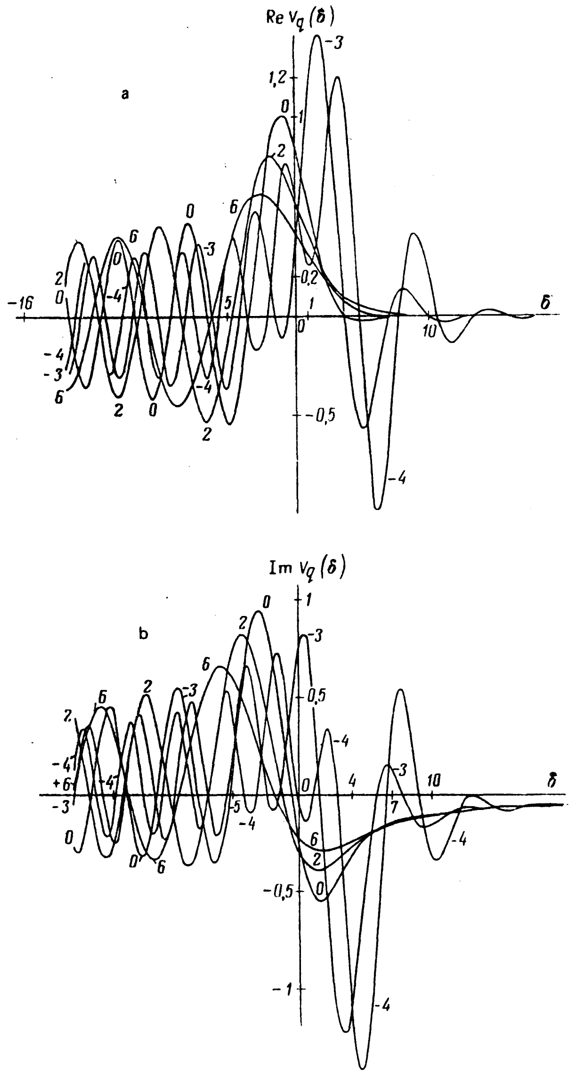

FIG. 5.

of δ_m is explained by the fact that for certain particular q the principal maximum undergoes a change: the main maximum decreases in value while one of the auxiliary maxima increases and becomes the main one. In Fig. 6

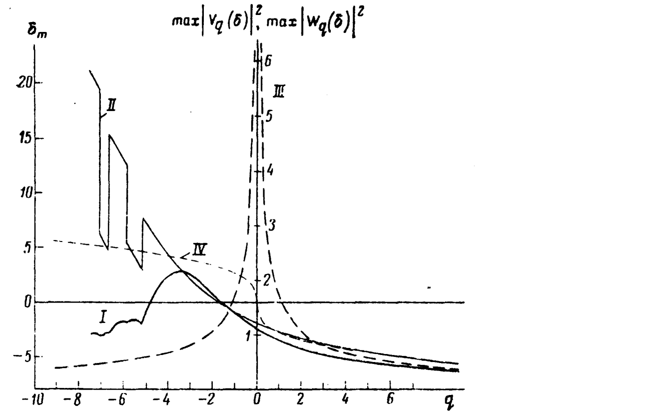

FIG. 6. The dependence of the value of the principal maximum of $|V_q(\delta)|^2$ on q-curve I, and the dependence of the position δ_m of the principal maximum on the parameter q-curve II; curves III and IV illustrate the corresponding dependences for the function $W_q(\delta)$.

we have shown for comparison by a dotted curve the magnitude and the position of the principal maximum of $|W_q(\delta)|^2$. For $|q|\gg 1$ the asymptotic equation

$$V_q(\delta)\approx W_q(\delta) \tag{5.12}$$

must hold. From Figure 6, it can be seen that for large positive q the equality (5.12) is attained quite rapidly while for negative q the asymptotic transition from $V_q(\delta)$ to $W_q(\delta)$ occurs considerably more slowly. This fact is also illustrated by Fig. 4. From Fig. 4d, e it can be seen, for example, that for $q<0$ the transparency band for the plasma wave determined by the function $W_q(\delta)$ is situated at $\delta>0$. But if the plasma wave is determined by the function $V_q(\delta)$, then the transparency band for $q\sim -1$ is basically still situated at $\delta<0$. Thus, the maxima of $|W_q(\delta)|^2$ and $|V_q(\delta)|^2$ are situated respectively on opposite sides of the origin, while for $q>0$ the positions of the maxima of $|W_q(\delta)|^2$ and $|V_q(\delta)|^2$ approximately coincide (Fig. 4a, b).

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1) The inequality (4.6) may be not valid only for certain functions $\epsilon^T(k)$ devoid of physical meaning. As regards condition (4.7), it is much more rigid; for example, it is not valid if $\epsilon^T(\mathbf{k}) \sim k_x^{-2}$ (cf., the following section), but, in any case, it is always satisifed if the function $\epsilon^T(\mathbf{k})$ exhibits a power dependence with a positive exponent.

2) The dependence of ϵ^T on collisions indicated above occurs in a weakly ionized plasma, and in this case $\nu_{eff}=\nu_{in}$.

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Kinetic theory of the nonlinear wave interaction in a semi-bounded plasma

A. G. Sitenko

Institute of Theoretical Physics, Academy of Sciences of the Ukrainian SSR

V. N. Pavlenko

Institute of Physics of the Academy of Sciences of the Ukrainian SSR
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We develop a kinetic theory of the nonlinear wave interaction in a semi-bounded plasma for the specular-reflection model. We obtain the nonlinear equation for the field; we use this to study the surface-wave resonant interaction that leads to decay and explosive instabilities.

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1. INTRODUCTION

It is well known that the electrodynamic properties of a spatially uniform plasma are described by giving the linear and non-linear electrical susceptibilities. The electromagnetic field is then given by the solution of the non-linear equations for a given distribution of the charges and currents in the plasma.[1] If the plasma is spatially bounded the electromagnetic field depends also on the conditions given on the surfaces bounding the plasma. A distinguishing property of a spatially bounded plasma is that together with bulk oscillations there exist in it also surface waves which propagate along the boundary surface and which are damped deep in the plasma.

The structure of the surface waves depends in an essential way on the shape of the surface and the nature of the boundary conditions. It is clear that the boundary conditions themselves must be determined by the nature of the interactions of the particles in a plasma with a bounding surface. The surface waves are described in the simplest way in the case of the so-called specular reflection model, when one assumes that all charged particles incident on the surface are specularly reflected from it.[2–5] A number of authors[6–9] (see also

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