we have shown for comparison by a dotted curve the magnitude and the position of the principal maximum of $|W_q(\delta)|^2$. For $|q|\gg 1$ the asymptotic equation

$$V_q(\delta)\approx W_q(\delta) \tag{5.12}$$

must hold. From Figure 6, it can be seen that for large positive $q$ the equality (5.12) is attained quite rapidly while for negative $q$ the asymptotic transition from $V_q(\delta)$ to $W_q(\delta)$ occurs considerably more slowly. This fact is also illustrated by Fig. 4. From Fig. 4d, e it can be seen, for example, that for $q<0$ the transparency band for the plasma wave determined by the function $W_q(\delta)$ is situated at $\delta>0$. But if the plasma wave is determined by the function $V_q(\delta)$, then the transparency band for $q\sim -1$ is basically still situated at $\delta<0$. Thus, the maxima of $|W_q(\delta)|^2$ and $|V_q(\delta)|^2$ are situated respectively on opposite sides of the origin, while for $q>0$ the positions of the maxima of $|W_q(\delta)|^2$ and $|V_q(\delta)|^2$ approximately coincide (Fig. 4a, b).

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1)The inequality (4.6) may be not valid only for certain functions $\epsilon^T(k)$ devoid of physical meaning. As regards condition (4.7), it is much more rigid; for example, it is not valid if $\epsilon^T(\mathbf{k}) \sim k_x^{-2}$ (cf., the following section), but, in any case, it is always satisifed if the function $\epsilon^T(\mathbf{k})$ exhibits a power dependence with a positive exponent.

2)The dependence of $\epsilon^T$ on collisions indicated above occurs in a weakly ionized plasma, and in this case $\nu_{eff}=\nu_{in}$.


1. V. L. Ginzburg, Rasprostranenie elektromagnitnykh voln v plasme (Propagation of Electromagnetic Waves in a Plasma), Nauka, 1967.
2. V. B. Gil'denburg, Zh. Eksp. Teor. Fiz. **45**, 1978 (1963) [Sov. Phys. JETP **18**, 1359 (1964)].
3. A. D. Piliya, Zh. Tekh. Fiz. **36**, 818 (1966) [Sov. Phys. Tech. Phys. **11**, 609 (1966)].
4. A. D. Piliya and V. I. Federov, Zh. Eksp. Teor. Fiz. **57**, 1198 (1969) [Sov. Phys. JETP **30**, 653 (1970)].
5. V. E. Golant and A. D. Piliya, Usp. Fiz. Nauk **104**, 413 (1971) [Sov. Phys. Usp. **14**, 413 (1972)].
6. Élektrodinamika plasmy (Electrodynamics of Plasma), ed. by A. I. Akhiezer, Nauka, 1974.
7. Yu. A. Kravtsov, Zh. Eksp. Teor. Fiz. **55**, 1470 (1968) [Sov. Phys. JETP **28**, 769 (1969)].
8. L. Tonks, Phys. Rev. **37**, 1458 (1931); **38**, 1219 (1931).
9. A. Dattner, Ericsson Technics **13**, 309 (1957); **19**, 3 (1963).


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# Kinetic theory of the nonlinear wave interaction in a semi-bounded plasma

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We develop a kinetic theory of the nonlinear wave interaction in a semi-bounded plasma for the specular-reflection model. We obtain the nonlinear equation for the field; we use this to study the surface-wave resonant interaction that leads to decay and explosive instabilities.

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## 1. INTRODUCTION

It is well known that the electrodynamic properties of a spatially uniform plasma are described by giving the linear and non-linear electrical susceptibilities. The electromagnetic field is then given by the solution of the non-linear equations for a given distribution of the charges and currents in the plasma.[1] If the plasma is spatially bounded the electromagnetic field depends also on the conditions given on the surfaces bounding the plasma. A distinguishing property of a spatially bounded plasma is that together with bulk oscillations there exist in it also surface waves which propagate along the boundary surface and which are damped deep in the plasma.

The structure of the surface waves depends in an essential way on the shape of the surface and the nature of the boundary conditions. It is clear that the boundary conditions themselves must be determined by the nature of the interactions of the particles in a plasma with a bounding surface. The surface waves are described in the simplest way in the case of the so-called specular reflection model, when one assumes that all charged particles incident on the surface are specularly reflected from it.[2-5] A number of authors[6-9] (see also

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Refs. 10, 11) have studied in detail the properties of various surface waves for the simplest case of a semi-bounded plasma. The first indication of the existence of surface Langmuir waves in a semi-bounded plasma is in Refs. 12, 13. The excitation of surface Langmuir waves in a semi-bounded plasma during the motion of a charged particle along the boundary surface has been considered.[14] Romanov[6] has developed a kinetic theory of high-frequency Langmuir and low-frequency ion-acoustic surface waves in a semi-bounded plasma.

It is clear that if the intensity of the surface waves is sufficiently large, effects connected with non-linear wave interactions turn out to be important (the theory of the non-linear wave interaction in an unbounded plasma is expounded in a number of monographs[15-17]). A number of authors[18-20] have studied the non-linear wave interaction in a semi-bounded plasma in the hydrodynamic approximation; in particular, in Ref. 18 the decay instability of surface waves was studied. In Refs. 21, 22 the non-linear wave interaction in a semi-bounded plasma was considered in the kinetic approximation where it leads to echo surface oscillations.

In the present paper we develop a kinetic theory of the non-linear wave interaction in a semi-bounded plasma for the specular reflection model. We obtain the general non-linear equation for the electric field and use that to study various non-linear wave interaction effects. In particular, we consider the resonance three-wave interaction of surface waves which leads to decay and explosive instabilities.

## 2. NON-LINEAR EQUATION FOR A POTENTIAL FIELD IN A SEMI-BOUNDED PLASMA

It is convenient for the consideration of surface and bulk eigenoscillations and their non-linear interaction in a semi-bounded plasma, as in the case of an unbounded plasma, to start from the non-linear equation for the field which one can obtain using the kinetic equations for the particle distribution functions and the equation for the self-consistent field. We shall assume that the plasma fills the half-space $z>0$ and that it is spatially uniform and in a stationary state. We shall assume the half-space $z<0$ to be filled by a dielectric characterized by a dielectric constant $\epsilon_0$. We restrict the considerations to the electrostatic interaction between charged particles (in that case the self-consistent field is a potential one).

The kinetic equations for the electron and ion distribution functions, and also the equation for the self-consistent field in the $z>0$ region can be written in the form

$$\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial \mathbf{r}}+\frac{e}{m}\mathbf{E}\frac{\partial}{\partial \mathbf{v}}(f_0+f)=0, \tag{1}$$

$$\operatorname{div}\mathbf{E}=4\pi\left(\sum e\int d\mathbf{v}\,f+\rho^0\right), \tag{2}$$

where $f$ is the deviation of the electron or ion distribution function from the unperturbed distribution function $f_0$ (we must use for $f_0$ the Maxwell distribution function for an equilibrium plasma), $\mathbf{E}$ is the self-consistent electrical field, and $\rho^0$ the density of the external charges. The sum $\Sigma$ in (2) indicates summation over the electron and ion components. In the region of space $z<0$ the electrical field $\mathbf{E}$ satisfies the equation

$$\operatorname{div}\mathbf{E}=0. \tag{3}$$

(We assume that there are no external charges in the region of space outside the plasma.)

We must supplement the kinetic Eq. (1) with boundary conditions which are imposed on the distribution-function deviations $f$ at the boundary surface $z=0$. If we assume that the particles are specularly reflected from the boundary surface, these conditions can be written in the form

$$f(x,y,z=0;\ v_x,v_y,v_z)=f(x,y,z=0;\ v_x,v_y,-v_z). \tag{4}$$

The electrical field at the boundary surface $z=0$ must satisfy the usual boundary conditions which are reduced to the requirement that the tangential components of the electrical field and the normal components of the electrical induction are continuous.

To find the solution of the set of Eqs. (1), (2) given in the region of space $z>0$ we use the following formal method. We continue into the region of space $z<0$ the components $\mathbf{E}_\perp$ of the electrical field so that they are even functions and the component $E_z$ so that it is an odd function (we denote the electrical field continued in this way by $\mathbf{E}^+$) and we assume that the kinetic equations determine the distribution functions in the whole of space (we denote these distribution functions by $f^+$):

$$\frac{\partial f^+}{\partial t}+\mathbf{v}\frac{\partial f^+}{\partial \mathbf{r}}+\frac{e}{m}\mathbf{E}^+\frac{\partial}{\partial \mathbf{v}}(f_0+f^+)=0. \tag{5}$$

As the differential operator occurring in these equations for the given continuation of the electrical field is invariant under the substitution $z, v_z\to -z, -v_z$, the solutions of the equations will also be invariant under that substitution

$$f(x,y,z;\ v_x,v_y,v_z)=f(x,y,-z;\ v_x,v_y,-v_z). \tag{6}$$

(We assume that the unperturbed distributions $f_0$ are even functions of $v_z$.) It is clear that in the region of space $z>0$ the distribution functions $f$ and $f^+$ must be the same as at $z=0$ the boundary conditions (4) follow directly from Eqs. (6).

The electrical field $\mathbf{E}^+$ satisfies the equation

$$\operatorname{div}\mathbf{E}^+=4\pi\left(e\int d\mathbf{v}\,f^+ +\rho^0\right)+2E_z^+(x,y,z=0)\,\delta(z), \tag{7}$$

which differs from (2) by taking into account an additional surface charge which guarantees the jump in the normal component of the field at the boundary surface. (The density of external charges $\rho^0$ is assumed to be continued into the region $z<0$ in even fashion.) The electrical field determined in the region of space $z<0$ by Eq. (3) is similarly continued into the region of space $z>0$. This field which we shall denote by $\mathbf{E}^-$ satisfies the equation

$$\operatorname{div}\mathbf{E}^- = -2E_z^-(x, y, z=0)\delta(z). \qquad (8)$$

It is clear that the solution of Eq. (7) in the region $z>0$ describes the electrical field in the plasma ($\mathbf{E}^+(\mathbf{r}) = \mathbf{E}(\mathbf{r})$) and the solution of Eq. (8) describes in the region $z<0$ the field outside the plasma ($\mathbf{E}^-(\mathbf{r}) = \mathbf{E}(\mathbf{r})$). At the surface of the plasma the following boundary conditions must be satisfied:

$$\mathbf{E}_\perp^+(x, y, z=0) = \mathbf{E}_\perp^-(x, y, z=0), \qquad (9)$$

$$E_z^+(x, y, z=0) = \varepsilon_0 E_z^-(x, y, z=0). \qquad (10)$$

It is convenient to use spatial Fourier transforms to write Eq. (7) in the form

$$ikE_\mathbf{k}^+ = 4\pi\left(e\int d\mathbf{v}\, f_\mathbf{k}^+ + \rho_\mathbf{k}^0\right) + 2E_{z\mathbf{k}_\perp}^+(0), \qquad (11)$$

and then, because of the longitudinal nature of the field

$$E_{z\mathbf{k}_\perp}^+(0) = \frac{1}{2\pi}\int dk_z \frac{k_z}{k} E_\mathbf{k}^+. \qquad (12)$$

Similarly we write Eq. (8) in the form

$$ikE_\mathbf{k}^- = -2E_{z\mathbf{k}_\perp}^-(0). \qquad (13)$$

The quantity $E_{z\mathbf{k}_\perp}^+(0)$ which occurs in Eq. (11) can be found from the boundary conditions and we can rewrite Eq. (11) in the form

$$ikE_\mathbf{k}^+ = 4\pi\left(e\int d\mathbf{v}\, f_\mathbf{k}^+ + \rho_\mathbf{k}^0\right) - 2i\varepsilon_0 E_{\perp\mathbf{k}_\perp}^+(0), \qquad (14)$$

where

$$E_{\perp\mathbf{k}_\perp}^+(0) = \frac{k_\perp}{2\pi}\int dk_z \frac{E_\mathbf{k}^+}{k}. \qquad (15)$$

For the sake of simplicity we shall in what follows drop the + index of **E**, i.e., we shall denote $\mathbf{E}^+$ by **E**.

Applying a space-time Fourier transformation to the kinetic Eq. (5) and solving the equation obtained by the method of successive approximations, we write the deviation of the distribution function $f_{\mathbf{k}\omega}^+$ as a power series in the field strength $E_{\mathbf{k}\omega}$. Substituting this expansion into (14) we get the following non-linear equation which completely determines the electrical field in the region of space occupied by the plasma:

$$\varepsilon(\omega, \mathbf{k})E_{\mathbf{k}\omega} + \sum_{\substack{\omega_1+\omega_2=\omega,\\ \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}} \varkappa^{(2)}(\omega_1, \mathbf{k}_1; \omega_2, \mathbf{k}_2)E_{\mathbf{k}_1\omega_1}E_{\mathbf{k}_2\omega_2}$$
$$+ \sum_{\substack{\omega_1+\omega_2+\omega_3=\omega,\\ \mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}}} \varkappa^{(3)}(\omega_1, \mathbf{k}_1; \omega_2, \mathbf{k}_2; \omega_3, \mathbf{k}_3)E_{\mathbf{k}_1\omega_1}E_{\mathbf{k}_2\omega_2}E_{\mathbf{k}_3\omega_3}$$
$$+ \ldots + 2\varepsilon_0\frac{k_\perp}{k}\sum_{k_z'}\frac{E_{\mathbf{k}'\omega}}{k'} = -\frac{4\pi i}{\omega}\rho_{\mathbf{k}\omega}^0, \qquad (16)$$

where $\epsilon(\omega, \mathbf{k})$ is the permittivity, and $\varkappa^{(2)}(\omega_1, \mathbf{k}_1; \omega_2, \mathbf{k}_2)$ and $\varkappa^{(3)}(\omega_1, \mathbf{k}_1; \omega_2, \mathbf{k}_2; \omega_3, \mathbf{k}_3)$ are the non-linear susceptibilities of the unbounded plasma.

The electrical field in the region outside the plasma is, according to (13) and (15), directly expressed in terms of the solution of Eq. (16):

$$E_{\mathbf{k}\omega}^- = 2\frac{k_\perp}{k}\sum_{k_z'}\frac{E_{\mathbf{k}'\omega}}{k'}. \qquad (17)$$

The complete solution of the problem of finding the field of a plasma half-space reduces thus to solving the non-linear Eq. (16).

One can use the non-linear Eq. (16) to study a number of effects in a semi-bounded plasma: the three-wave decay interaction; the four-wave interaction leading to the appearance of a shift in the eigenfrequencies and the non-linear Landau damping; non-resonance echo effects connected with the undamped nature of the oscillations of the distribution function, induced scattering of waves by particles, and so on. Even if one can restrict oneself in the analysis of the decay interaction of waves to the hydrodynamic approximation (although in that case there remains an arbitrariness in the determination of the coefficients) one needs the kinetic approach for a description of the other effects listed here.

## 3. DISPERSION EQUATIONS FOR BULK AND SURFACE WAVES (LINEAR APPROXIMATION)

Neglecting the non-linear terms in (16) and putting the density of the external charges equal to zero, we get the basic equation of the linear approximation which describes the eigenoscillations in a semi-bounded plasma:

$$\varepsilon(\omega, \mathbf{k})E_{\mathbf{k}\omega} + 2\frac{\varepsilon_0}{k}E_{\mathbf{k}_\perp\omega} = 0. \qquad (18)$$

Here and henceforth we introduce, to simplify the notation, the notation

$$E_{\mathbf{k}_\perp\omega} \equiv E_{\perp\mathbf{k}_\perp\omega}(0) = \frac{k_\perp}{2\pi}\int dk_z \frac{E_{\mathbf{k}\omega}}{k}, \qquad (19)$$

i.e., we understand by the quantity $E_{\mathbf{k}_\perp\omega}$ the value of the tangential component of the field strength at the surface of the plasma.

One easily finds from (18) the following equation for the quantity $E_{\mathbf{k}_\perp\omega}$:

$$\zeta(\omega, \mathbf{k}_\perp)E_{\mathbf{k}_\perp\omega} = 0, \qquad (20)$$

where

$$\zeta(\omega, \mathbf{k}_\perp) \equiv 1 + \frac{\varepsilon_0 k_\perp}{\pi}\int dk_z \frac{1}{k^2\varepsilon(\omega, \mathbf{k})}. \qquad (21)$$

Two kinds of eigenoscillations—bulk and surface oscillations—are according to (18) possible in a semi-bounded plasma.

The dispersion equation for the bulk eigenwaves ($E_{\mathbf{k}\omega} \neq 0, E_{\mathbf{k}_\perp\omega} = 0$) is determined by the same condition as in the case of an unbounded plasma,

$$\varepsilon(\omega, \mathbf{k}) = 0. \qquad (22)$$

We shall denote the eigenfrequencies which are the solutions of Eq. (22) for fixed value of $\mathbf{k}$ by $\omega_{\mathbf{k}}$ and we write the field of the eigenoscillations in the form

$$E_{\mathbf{k}\omega}=\pi E_{\mathbf{k}}\{\exp(-i\Phi_{\mathbf{k}})\delta(\omega-\omega_{\mathbf{k}})+\exp(i\Phi_{\mathbf{k}})\delta(\omega+\omega_{\mathbf{k}})\}, \tag{23}$$

where $E_{\mathbf{k}}$ and $\Phi_{\mathbf{k}}$ are the initial amplitude and phase. Using the boundary conditions (9) and (10) one can easily show that

$$E_{\mathbf{k}_\perp\omega}=-\frac{i}{\varepsilon_0}\frac{1}{2\pi}\int dk_z\frac{k_z E_{\mathbf{k}\omega}}{k},$$

and as $E_{\mathbf{k}}$, $\Phi_{\mathbf{k}}$, and $\omega_{\mathbf{k}}$ are even functions of $k_z$ the condition $E_{\mathbf{k}_\perp\omega}=0$ is satisfied for the bulk oscillations.

The dispersion equation for the surface eigenwaves ($E_{\mathbf{k}_\perp\omega}\neq 0$) is given by the condition

$$\zeta(\omega,\mathbf{k}_\perp)=0. \tag{24}$$

We denote the eigenfrequencies of the surface oscillations by $\omega_{\mathbf{k}_\perp}$ and we write the field of the surface oscillations in the form

$$E_{\mathbf{k}_\perp\omega}=\pi E_{\mathbf{k}_\perp}\{\exp(-i\Phi_{\mathbf{k}_\perp})\delta(\omega-\omega_{\mathbf{k}_\perp})+\exp(i\Phi_{\mathbf{k}_\perp})\delta(\omega+\omega_{\mathbf{k}_\perp})\}. \tag{25}$$

Since $\epsilon(\omega,\mathbf{k})\neq 0$ in the case of the surface oscillations, we easily get from (18) the total spatial field component for the surface oscillations

$$E_{\mathbf{k}\omega}=-\frac{2\varepsilon_0}{k}\frac{E_{\mathbf{k}_\perp\omega}}{\varepsilon(\omega,\mathbf{k})}. \tag{26}$$

One can show that the field of the surface oscillations decreases exponentially when one goes away from the boundary surface.

## 4. SURFACE WAVES

We consider various types of surface waves in a semi-bounded plasma, which are given by the dispersion Eq. (24). In the high-frequency region the dispersion of the surface waves is determined, just as for the bulk waves, by the electron component of the plasma. In the long-wavelength limit $a^2k_\perp^2\ll 1$ ($a$ is the Debye radius) the eigenfrequency and the damping coefficient of the high-frequency surface waves are described by the formulae

$$\omega_{\mathbf{k}_\perp}=\frac{\Omega}{(1+\varepsilon_0)^{1/2}}\left(1+\frac{(3\varepsilon_0)^{1/2}}{2}a|\mathbf{k}_\perp|+\dots\right),$$
$$\gamma_{\mathbf{k}_\perp}=(2/3\pi)^{1/2}|\mathbf{k}_\perp|s,\quad s^2=3T/m. \tag{27}$$

In the low-frequency region the dispersion of the surface waves depends in an essential way on both the electrons and the ions. We assume that the electron temperature is appreciably higher than that of the ions $T_e\gg T_i$ (strongly non-isothermal plasma) and we consider the frequency range satisfying the condition $s\gg\omega/k_\perp\gg s_i$. In that case we can use for the dielectric permittivity the approximate expression

$$\varepsilon(\omega,\mathbf{k})=1+1/a^2k^2-\Omega_i^2/\omega^2,$$

and using this we find easily

$$\zeta(\omega,\mathbf{k}_\perp)=1+\frac{\varepsilon_0}{\varepsilon_i}\left(1+\frac{1}{a^2k_\perp^2\varepsilon_i}\right)^{-1/2},\quad \varepsilon_i=1-\frac{\Omega_i^2}{\omega^2}. \tag{28}$$

Putting after this the quantity $\zeta(\omega,\mathbf{k}_\perp)$ equal to zero we get for the eigenfrequencies and damping coefficients of the surface waves in the long- and short-wavelength limits, respectively: when $a^2k_\perp^2\ll 1$

$$\omega_{\mathbf{k}_\perp}=k_\perp v_s\left(1-\frac{1}{2}a^2k_\perp^2\right), \tag{29a}$$

$$\gamma=\left(\frac{\pi}{8}\right)^{1/2}\left(\frac{m_e}{m_i}\right)^{1/2}\Omega_i a k_\perp\left[1+\left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2}\exp\left(-\frac{T_e}{2T_i}\right)\right]$$

and when $a^2k_\perp^2\gg 1$

$$\omega_{\mathbf{k}_\perp}=\frac{\Omega_i}{(1+\varepsilon_0)^{1/2}}\left(1-\frac{1}{2a^2k_\perp^2(1+\varepsilon_0)}\right),\quad \gamma\approx\left(\frac{2}{\pi}\right)^{1/2}k_\perp s_i. \tag{29b}$$

The high-frequency electron and low-frequency ion-sound and ion surface waves considered here are characterized by positive energies. It is well known that in an unbounded non-equilibrium plasma there are possible not only waves with positive energy, but also waves with negative energy.[23] In a non-equilibrium semi-bounded plasma surface waves with negative energy are also possible. As an example we consider a plasma through which a compensated low-density ion beam ($n_0'\ll n_0$) passes with velocity $\mathbf{u}$ (the vector $\mathbf{u}$ is parallel to the boundary surface). Neglecting the thermal motion of the ions we can write the permittivity of the plasma in the form

$$\varepsilon(\omega,\mathbf{k})=1+\frac{1}{a^2k^2}-\frac{\Omega_i^2}{\omega^2}\left[1+\eta\frac{\omega^2}{(\omega-\mathbf{k}_\perp\mathbf{u})^2}\right],\quad \eta=\frac{n_0'}{n_0}\ll 1. \tag{30}$$

(This expression is applicable in the frequency range $\omega\ll k_\perp s$.) Substituting (30) into the general Eq. (21) and integrating over $k_z$ we get Eq. (28) for the quantity $\zeta(\omega,\mathbf{k}_\perp)$ in which we must understand by $\epsilon_i$

$$\varepsilon_i=1-\frac{\Omega_i^2}{\omega^2}\left[1+\eta\frac{\omega^2}{(\omega-\mathbf{k}_\perp\mathbf{u})^2}\right].$$

After that putting $\zeta(\omega,\mathbf{k}_\perp)$ equal to zero and restricting the discussion to the long-wavelength limit $a^2k_\perp^2\ll 1$ we can write the dispersion equation in the form

$$1+\frac{1}{a^2k_\perp^2}+a^2k_\perp^2\varepsilon_0^2=\frac{\Omega_i^2}{\omega^2}\left[1+\eta\frac{\omega^2}{(\omega-\mathbf{k}_\perp\mathbf{u})^2}\right]. \tag{31}$$

Assuming the beam density to be sufficiently small ($\eta\ll 1$) we can easily find the roots of that equation which correspond to the eigenfrequencies of the surface waves:

$$\omega_{\mathbf{k}_\perp}^{(1)}=k_\perp v_s/(1+a^2k_\perp^2)^{1/2},$$
$$\omega_{\mathbf{k}_\perp}^{(2,3)}=\mathbf{k}_\perp\mathbf{u}\mp\eta^{1/2}\Omega_i[1+1/a^2k_\perp^2-\Omega_i^2/(\mathbf{k}_\perp\mathbf{u})^2]^{-1/2}. \tag{32}$$

Waves corresponding to the eigenfrequencies $\omega_{\mathbf{k}_\perp}^{(1)}$ and $\omega_{\mathbf{k}_\perp}^{(3)}$ are characterized by positive energies, while the wave corresponding to the eigenfrequency $\omega_{\mathbf{k}_\perp}^{(2)}$ is characterized by a negative energy.

## 5. NON-LINEAR INTERACTION OF SURFACE WAVES

The non-linear interaction of bulk and surface waves in a semi-bounded plasma is described by the general non-linear Eq. (16). Assuming there to be no external charges we rewrite this equation in the form

$$\varepsilon(\omega,\mathbf{k})E_{\mathbf{k}\omega}+\sum_{\substack{\omega_1+\omega_2=\omega,\\ \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}}\varkappa^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)E_{\mathbf{k}_1\omega_1}E_{\mathbf{k}_2\omega_2}$$

$$+\sum_{\substack{\omega_1+\omega_2+\omega_3=\omega,\\ \mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}}}\varkappa^{(3)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2;\omega_3,\mathbf{k}_3)E_{\mathbf{k}_1\omega_1}E_{\mathbf{k}_2\omega_2}E_{\mathbf{k}_3\omega_3}+\ldots+\frac{2\varepsilon_0}{k}E_{\mathbf{k}_\perp\omega}=0. \tag{33}$$

Using the method of multiple time consecutive approximations we can obtain from this equation a hierarchy of equations which determines the time-dependence of the amplitudes caused by the non-linear resonance interaction of the waves.

The simplest example of a non-linear resonance wave interaction is the three-wave resonance which occurs in the case when the frequencies of the interacting waves satisfy the condition

$$\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}=\omega_{\mathbf{k}}. \tag{34}$$

It is clear that three-wave resonance is possible in a semi-bounded plasma in the case of the interaction of three bulk waves, two bulk and one surface wave (two cases are possible: as the result of the interaction of two bulk waves a surface wave is formed, and the interaction of a bulk wave with a surface wave leads to the formation of a bulk wave), two surface waves with a bulk wave, and of three surface waves. In the case when there are no three-wave resonances the four-wave resonance interaction which occurs when the following resonance condition between frequencies holds,

$$\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}+\omega_{\mathbf{k}_3}=\omega_{\mathbf{k}} \tag{35}$$

turns out to be the most important one.

We restrict ourselves to a more detailed discussion of the resonance interaction of surface waves. We multiply Eq. (33) by the quantity $k_\perp/k\epsilon(\omega,\mathbf{k})$ and integrate over $k_z$ and then, using the surface nature of the interacting waves, we use Eq. (26) to express the fields $E_{\mathbf{k}_1\omega_1}$, $E_{\mathbf{k}_2\omega_2}$, ... in terms of the surface components WE$_{\mathbf{k}_1\perp\omega_1}$, $E_{\mathbf{k}_2\perp\omega_2}$, ... As a result the basic equation describing the interaction of surface waves in a semi-bounded plasma can be written in the form

$$\zeta(\omega,\mathbf{k}_\perp)E_{\mathbf{k}_\perp\omega}+\sum_{\substack{\omega_1+\omega_2=\omega,\\ \mathbf{k}_{1\perp}+\mathbf{k}_{2\perp}=\mathbf{k}_\perp}}\tilde{\varkappa}^{(2)}(\omega_1,\mathbf{k}_{1\perp};\omega_2,\mathbf{k}_{2\perp})E_{\mathbf{k}_{1\perp}\omega_1}E_{\mathbf{k}_{2\perp}\omega_2}$$

$$+\sum_{\substack{\omega_1+\omega_2+\omega_3=\omega,\\ \mathbf{k}_{1\perp}+\mathbf{k}_{2\perp}+\mathbf{k}_{3\perp}=\mathbf{k}_\perp}}\tilde{\varkappa}^{(3)}(\omega_1,\mathbf{k}_{1\perp};\omega_2,\mathbf{k}_{2\perp};\omega_3,\mathbf{k}_{3\perp})E_{\mathbf{k}_{1\perp}\omega_1}E_{\mathbf{k}_{2\perp}\omega_2}E_{\mathbf{k}_{3\perp}\omega_3}+\ldots=0, \tag{36}$$

where $\tilde{\varkappa}^{(2)}$ and $\tilde{\varkappa}^{(3)}$ are the non-linear surface susceptibilities of the plasma, determined by the equations

$$\tilde{\varkappa}^{(2)}(\omega_1,\mathbf{k}_{1\perp};\omega_2,\mathbf{k}_{2\perp})=\frac{\varepsilon_0^2k_\perp}{\pi^2}\int dk_{1z}\int dk_{2z}\frac{\varkappa^{(2)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2)}{k_1k_2k\varepsilon(\omega_1,\mathbf{k}_1)\varepsilon(\omega_2,\mathbf{k}_2)\varepsilon(\omega,\mathbf{k})} \tag{37}$$

$$\varkappa^{(3)}(\omega_1,\mathbf{k}_{1\perp};\omega_2,\mathbf{k}_{2\perp};\omega_3,\mathbf{k}_{3\perp})$$

$$=-\frac{\varepsilon_0^3k_\perp}{\pi^3}\int dk_{1z}\int dk_{2z}\int dk_{3z}\frac{\varkappa^{(3)}(\omega_1,\mathbf{k}_1;\omega_2,\mathbf{k}_2;\omega_3,\mathbf{k}_3)}{k_1k_2k_3k\varepsilon(\omega_1,\mathbf{k}_1)\varepsilon(\omega_2,\mathbf{k}_2)\varepsilon(\omega_3,\mathbf{k}_3)\varepsilon(\omega,\mathbf{k})}. \tag{38}$$

We apply to the non-linear Eq. (36) the method of multiple time consecutive approximations. In first approximation the fields of the surface waves are, as before, given by Eq. (25), but if the non-linear interaction of the waves is taken into account the amplitudes $E_{\mathbf{k}_\perp}$ and phases $\Phi_{\mathbf{k}_\perp}$ must be assumed to be slowly varying functions of the time. The equations determining the time-dependence of the amplitudes $E_{\mathbf{k}_\perp}$ and the phases $\Phi_{\mathbf{k}_\perp}$ can be found from the conditions that the secular parts of the higher approximations of Eq. (36) must vanish. Under the conditions of three-wave resonance for surface waves:

$$\omega_{\mathbf{k}_{1\perp}}+\omega_{\mathbf{k}_{2\perp}}=\omega_{\mathbf{k}_\perp} \tag{39}$$

the equation determining the time-dependence of the amplitude of the linear approximation of the surface waves has the form

$$\frac{\partial}{\partial t}E_{\mathbf{k}_\perp}\exp(-i\Phi_{\mathbf{k}_\perp})=\frac{i}{2}\left(\frac{\partial\zeta(\omega_{\mathbf{k}_\perp},\mathbf{k}_\perp)}{\partial\omega_{\mathbf{k}_\perp}}\right)^{-1}$$

$$\times\sum_{\mathbf{k}_\perp=\mathbf{k}_{1\perp}+\mathbf{k}_{2\perp}}\tilde{\varkappa}^{(2)}(\omega_{\mathbf{k}_{1\perp}},\mathbf{k}_{1\perp};\omega_{\mathbf{k}_{2\perp}},\mathbf{k}_{2\perp})E_{\mathbf{k}_{1\perp}}E_{\mathbf{k}_{2\perp}}\exp[-i(\Phi_{\mathbf{k}_{1\perp}}+\Phi_{\mathbf{k}_{2\perp}})]. \tag{40}$$

This equation describes three-wave decay processes, i.e., processes in which two waves fuse to form one wave and processes in which one wave decays into two others. The decay interaction appears also when we neglect the thermal motion of the particles, i.e., it can be described also on the basis of the hydrodynamical discussion.

If condition (39) is not satisfied, the correction to the field is in second approximation expressed in terms of the field in the first approximation as follows:

$$E^{(2)}_{\mathbf{k}_\perp\omega}=-\frac{1}{\zeta(\omega,\mathbf{k}_\perp)}\sum_{\substack{\omega=\omega_1+\omega_2,\\ \mathbf{k}_\perp=\mathbf{k}_{1\perp}+\mathbf{k}_{2\perp}}}\tilde{\varkappa}^{(2)}(\omega_1,\mathbf{k}_{1\perp};\omega_2,\mathbf{k}_{2\perp})E_{\mathbf{k}_{1\perp}\omega_1}E_{\mathbf{k}_{2\perp}\omega_2}. \tag{41}$$

The time-dependence of the amplitude and phase can in this case be found from the condition for removing the secularity in the equations of the third approximation. Resonance interaction occurs if the following relation holds between the frequencies

$$\omega_{\mathbf{k}_{1\perp}}+\omega_{\mathbf{k}_{2\perp}}+\omega_{\mathbf{k}_{3\perp}}=\omega_{\mathbf{k}_\perp}. \tag{42}$$

In the four-wave resonance case the equation determining the time-dependence of the field of the surface wave has the form

$$\frac{\partial}{\partial t}E_{\mathbf{k}_\perp}\exp(-i\Phi_{\mathbf{k}_\perp})=-\frac{i}{4}\left(\frac{\partial\zeta(\omega_{\mathbf{k}_\perp},\mathbf{k}_\perp)}{\partial\omega_{\mathbf{k}_\perp}}\right)^{-1}$$

$$\times\sum_{\mathbf{k}_\perp=\mathbf{k}_{1\perp}+\mathbf{k}_{2\perp}+\mathbf{k}_{3\perp}}{}'\left\{\frac{2}{\zeta(\omega_{\mathbf{k}_{2\perp}}+\omega_{\mathbf{k}_{3\perp}},\mathbf{k}_{2\perp}+\mathbf{k}_{3\perp})}\right.$$

$$\times\tilde{\varkappa}^{(2)}(\omega_{\mathbf{k}_{1\perp}},\mathbf{k}_{1\perp};\omega_{\mathbf{k}_{2\perp}}+\omega_{\mathbf{k}_{3\perp}},\mathbf{k}_{2\perp}+\mathbf{k}_{3\perp})\varkappa^{(2)}(\omega_{\mathbf{k}_{2\perp}},\mathbf{k}_{2\perp};\omega_{\mathbf{k}_{3\perp}},\mathbf{k}_{3\perp})$$

$$\left.-\tilde{\varkappa}^{(3)}(\omega_{\mathbf{k}_{1\perp}},\mathbf{k}_{1\perp};\omega_{\mathbf{k}_{2\perp}},\mathbf{k}_{2\perp};\omega_{\mathbf{k}_{3\perp}},\mathbf{k}_{3\perp})\right\}E_{\mathbf{k}_{1\perp}}E_{\mathbf{k}_{2\perp}}E_{\mathbf{k}_{3\perp}}$$

$$\times\exp[-i(\Phi_{\mathbf{k}_{1\perp}}+\Phi_{\mathbf{k}_{2\perp}}+\Phi_{\mathbf{k}_{3\perp}})]. \tag{43}$$

The prime on the summation sign on the right-hand side of (43) indicates the necessity to take all possible wave combinations which are in accordance with condition (42) for various signs in front of the frequencies into account.

Equation (43) describes induced wave decay processes and the processes of induced scattering of waves by particles which lead to the appearance of a non-linear shift of the eigenfrequencies and non-linear Landau damping. These effects, like the linear damping, can only be described in the framework of the kinetic theory.

## 6. THREE-WAVE SURFACE WAVE DECAYS

We consider the resonance interaction of three surface waves with frequencies $\omega_{\mathbf{k}_\perp}$, $\omega_{\mathbf{k}_{1\perp}}$, and $\omega_{\mathbf{k}_{2\perp}}$ and fixed values of the wavevectors $\mathbf{k}_\perp$, $\mathbf{k}_{1\perp}$, and $\mathbf{k}_{2\perp}$ for which the resonance conditions

$$\omega_{\mathbf{k}_{1\perp}}+\omega_{\mathbf{k}_{2\perp}}=\omega_{\mathbf{k}_\perp},\quad \mathbf{k}_{1\perp}+\mathbf{k}_{2\perp}=\mathbf{k}_\perp \tag{44}$$

are satisfied. Each of the interacting waves is characterized by an energy

$$W_{\mathbf{k}_\perp}=-\frac{\varepsilon_0}{8\pi}\frac{\omega_{\mathbf{k}_\perp}}{k_\perp}\frac{\partial\zeta(\omega_{\mathbf{k}_\perp},\mathbf{k}_\perp)}{\partial\omega_{\mathbf{k}_\perp}}|E_{\mathbf{k}_\perp}|^2. \tag{45}$$

The energy of the separate waves can be either positive or negative (the nature of the wave energy is determined by the sign of the derivative $\zeta'_{\mathbf{k}_\perp}\equiv\partial\zeta(\omega_{\mathbf{k}_\perp},\mathbf{k}_\perp)/\partial\omega_{\mathbf{k}_\perp}$). For the sake of convenience we introduce the amplitudes $A_{\mathbf{k}_\perp}$ and sign factors $s_{\mathbf{k}_\perp}$ defining them by means of the equations

$$A_{\mathbf{k}_\perp}=[\varepsilon_0|\zeta_{\mathbf{k}_\perp}'|/8\pi k_\perp]^{1/2}E_{\mathbf{k}_\perp}\exp(-i\Phi_{\mathbf{k}_\perp}),\quad s_{\mathbf{k}_\perp}=-\operatorname{sign}\zeta_{\mathbf{k}_\perp}'. \tag{46}$$

The expressions for the energy and momentum of the surface waves then take the form

$$W_{\mathbf{k}_\perp}=s_{\mathbf{k}_\perp}\omega_{\mathbf{k}_\perp}|A_{\mathbf{k}_\perp}|^2,$$
$$\mathbf{P}_{\mathbf{k}_\perp}=s_{\mathbf{k}_\perp}\mathbf{k}_\perp|A_{\mathbf{k}_\perp}|^2. \tag{47}$$

Using the definitions (46) we can write the basic Eq. (40) in the form of a Schrödinger equation in the interaction representation

$$i\frac{\partial A_{\mathbf{k}_\perp}}{\partial t}=s_{\mathbf{k}_\perp}V_{\mathbf{k}_\perp;\mathbf{k}_{1\perp};\mathbf{k}_{2\perp}}A_{\mathbf{k}_{1\perp}}A_{\mathbf{k}_{2\perp}}, \tag{48}$$

where $V_{\mathbf{k}_\perp;\mathbf{k}_{1\perp};\mathbf{k}_{2\perp}}$ is an interaction matrix element defined by means of the equation

$$V_{\mathbf{k}_\perp;\mathbf{k}_{1\perp},\mathbf{k}_{2\perp}}=-\left(\frac{2\pi k_{1\perp}k_{2\perp}}{\varepsilon_0 k_\perp}\right)^{1/2}\frac{\tilde{\varkappa}^{(2)}(\omega_{\mathbf{k}_{1\perp}},\mathbf{k}_{1\perp};\omega_{\mathbf{k}_{2\perp}},\mathbf{k}_{2\perp})}{|\zeta_{\mathbf{k}_\perp}'\zeta_{\mathbf{k}_{1\perp}}'\zeta_{\mathbf{k}_{2\perp}}'|^{1/2}}. \tag{49}$$

Using the symmetry properties of the non-linear plasma susceptibility $\tilde{\varkappa}^{(2)}(\omega_{\mathbf{k}_{1\perp}},\mathbf{k}_{1\perp};\omega_{\mathbf{k}_{2\perp}},\mathbf{k}_{2\perp})$ one can easily show that the time-dependence of the amplitudes of surface waves with frequencies $\omega_{\mathbf{k}_{1\perp}}$ and $\omega_{\mathbf{k}_{2\perp}}$ is described by the equations

$$i\frac{\partial A_{\mathbf{k}_{1\perp}}}{\partial t}=s_{\mathbf{k}_{1\perp}}V^*_{\mathbf{k}_\perp;\mathbf{k}_{1\perp},\mathbf{k}_{2\perp}}A_{\mathbf{k}_\perp}A^*_{\mathbf{k}_{2\perp}},$$
$$i\frac{\partial A_{\mathbf{k}_{2\perp}}}{\partial t}=s_{\mathbf{k}_{2\perp}}V^*_{\mathbf{k}_\perp;\mathbf{k}_{1\perp},\mathbf{k}_{2\perp}}A_{\mathbf{k}_\perp}A^*_{\mathbf{k}_{1\perp}}, \tag{50}$$

in which occurs the same interaction matrix element as in Eq. (48). The set of coupled Eqs. (48) and (50) completely describe the dynamics of three interacting surface waves. This set can be solved exactly.[24]

In the case of the interaction of three surface waves with energies of the same (positive or negative) sign, i.e., when the condition

$$s_{\mathbf{k}_{1\perp}}=s_{\mathbf{k}_{2\perp}}=s_{\mathbf{k}_\perp}, \tag{51}$$

is satisfied, a decay instability arises in the system. Let a wave of frequency $\omega_{\mathbf{k}_\perp}$ initially at time $t=0$ be characterized by a large amplitude $|A_{\mathbf{k}_\perp}|^2\gg|A_{\mathbf{k}_{1\perp}}|^2$ and $|A_{\mathbf{k}_\perp}|^2\gg|A_{\mathbf{k}_{2\perp}}|^2$. As a result of the resonance interaction the amplitude $A_{\mathbf{k}_\perp}$ in the first stage of the temporal evolution will change insignificantly, while the amplitudes $A_{\mathbf{k}_{1\perp}}$ and $A_{\mathbf{k}_{2\perp}}$ will exponentially increase with time. The growth rate of the waves with frequencies $\omega_{\mathbf{k}_{1\perp}}$ and $\omega_{\mathbf{k}_{2\perp}}$ is determined by the intensity of the wave of frequency $\omega_{\mathbf{k}_\perp}$:

$$|\gamma|=|V_{\mathbf{k}_\perp,\mathbf{k}_{1\perp},\mathbf{k}_{2\perp}}A_{\mathbf{k}_\perp}|. \tag{52}$$

The reciprocal of (52) determines the decay time. As an example of a decay interaction of surface waves in a semi-bounded plasma we consider the decay of a Langmuir surface wave into a Langmuir and an ion-sound surface wave, the dispersion of which is given by Eqs. (27) and (29) ($ak_\perp\ll 1$). The frequencies and wavevectors of the interacting waves satisfy the decay conditions which take the form

$$\frac{\Omega}{(1+\varepsilon_0)^{1/2}}+\frac{1}{2}\left(\frac{3}{1+\varepsilon_0}\right)^{1/2}\Omega a|\mathbf{k}_\perp|=\frac{\Omega}{(1+\varepsilon_0)^{1/2}}$$
$$+\frac{1}{2}\left(\frac{3}{1+\varepsilon_0}\right)^{1/2}\Omega a|\mathbf{k}_{1\perp}|+|\mathbf{k}_{2\perp}|v_s, \tag{53}$$

$$k_{2\perp}^2=k_\perp^2+k_{1\perp}^2-2|\mathbf{k}_\perp||\mathbf{k}_{1\perp}|\cos\alpha, \tag{54}$$

where $\alpha$ is the angle between $\mathbf{k}_\perp$ and $\mathbf{k}_{1\perp}$. Putting $|\mathbf{k}_\perp|=|\mathbf{k}_{1\perp}|=k_0$ we have $k_{2\perp}=2k_0\sin\frac{1}{2}\alpha$ and the decay conditions are then satisfied, if

$$\sin\frac{\alpha}{2}>2\left(\frac{1+\varepsilon_0}{3}\right)^{1/2}\left(\frac{m}{M}\right)^{1/2}.$$

Using (49) and (37) we find the growth rate of the surface oscillations ($E_{\mathbf{k}_\perp}=E_0$):

$$\gamma^2=\frac{1}{9}\left(\frac{eE_0}{m}\right)^2\frac{\varepsilon_0^5}{(1+\varepsilon_0)^{5/2}}\frac{a^2k_0^3v_s}{\Omega\Omega_i^2}\sin^3\frac{\alpha}{2}. \tag{55}$$

The expression (55) for the non-linear growth rate was obtained assuming that the linear damping rates of the interacting surface waves were small compared to the corresponding eigenfrequencies. It is clear that the growth of the Langmuir and ion-sound surface waves will occur, if the non-linear growth rate given by Eq. (55) turns out to be larger than the linear damping rates given by Eqs. (27) and (29), i.e., when the conditions

$$\gamma^2>\frac{4}{3\pi}k_0^2a^2\Omega^2,$$

$$\gamma^2>\frac{\pi}{2}\left(\frac{m_e}{m_i}\right)^2\left[1+\left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2}\exp\left(-\frac{T_e}{2T_i}\right)\right]^2k_0^2a^2\Omega^2\sin^2\frac{\alpha}{2}.$$

are satisfied. However, in that case it is necessary for the validity of the considerations that the non-linear growth rate does not exceed in magnitude the eigenfrequencies of the growing surface waves, i.e., that the conditions

$$\gamma^2<\Omega^2/(1+\varepsilon_0), \quad \gamma^2<4k_0^2v_s^2\sin^2(\alpha/2)$$

are satisfied.

In the case of the decay of a Langmuir bulk wave into Langmuir and ion-sound bulk waves the growth rate is given by the expression:[25]

$$\bar{\gamma}^2=\frac{1}{8}\left(\frac{eE_0}{m}\right)^2\frac{k_0^2v_s}{\Omega\Omega_i^2}\sin^2\frac{\alpha}{2}\cos^2\alpha. \quad (56)$$

We note that the growth rate of the decay instability of surface waves is appreciably larger than the growth rate of the decay instability of bulk waves ($\gamma/\bar{\gamma}\sim 1/ak_0 \gg 1$).

In the case of a resonance interaction of three surface waves with different signs of the energy, for instance, when the condition

$$s_{\mathbf{k}_{1\perp}}=s_{\mathbf{k}_{2\perp}}=-s_{\mathbf{k}_\perp} \quad (57)$$

holds, an explosive instability occurs in the system in which the amplitudes of the interacting waves turn to infinity at some finite value of the time $t_\infty$. The wave with negative energy gives energy to the waves with positive energy (or the waves with negative energy give energy to the wave with positive energy) and the amplitudes of the interacting waves increase without bounds, notwithstanding the conservation of the total energy of the system. By an appropriate choice of the initial conditions one can achieve that the amplitude of the (initially) strongest wave changes with time according to

$$A_{\mathbf{k}_\perp}(t)=\frac{A_{\mathbf{k}_\perp}(0)}{(1-t/t_\infty)}. \quad (58)$$

The explosive time $t_\infty$ will in that case be determined by the initial value of the amplitude and the matrix element of the non-linear interaction

$$t_\infty^{-1}=|V_{\mathbf{k}_\perp,\mathbf{k}_{1\perp},\mathbf{k}_{2\perp}}A_{\mathbf{k}}(0)|. \quad (59)$$

The explosive instability is possible in a semi-bounded plasma when a compensated beam of ions passes through it as the result of the resonance interaction of three surface waves, the dispersion of which is given by Eqs. (33). Putting $|\mathbf{k}_\perp|=|\mathbf{k}_{1\perp}|=k_0$ we find from the decay conditions $k_{2\perp}=-2\eta^{1/2}k_0v_s/u$.

The explosive time $t_\infty$ for the decay of a wave with negative energy $\omega^{(2)}_{\mathbf{k}_\perp}$ into waves with positive energies $\omega^{(3)}_{\mathbf{k}_{1\perp}}$ and $\omega^{(1)}_{\mathbf{k}_{2\perp}}$ is given by the expression

$$t_\infty^{-1}=8\varepsilon_0^{5/2}\frac{eE_0}{mu}\eta\left(\frac{M}{m}\right)^2\left(\frac{v_s}{u}\right)^{1/2}(k_0a)^3. \quad (60)$$

We note that in the case of the explosive instability caused by the interaction of bulk waves the explosion time equals

$$\bar{t}_\infty^{-1}=\frac{1}{8^{1/2}}\frac{eE_0}{mu}\eta^{1/2}\left(\frac{M}{m}\right)^2\left(\frac{v_s}{u}\right)^{5/2}. \quad (61)$$

The ratio of the explosive time for bulk waves to the explosion time for surface waves is a quantity of the order of

$$\bar{t}_\infty/t_\infty\sim\eta^{1/2}(k_0a)^3(u/v_s)^2.$$

In conclusion we note that the characteristics of the decay and explosive instabilities discussed by us (growth rate and explosion time) depend on the nature of the boundary conditions at the surface bounding the plasma. For the applicability of the specular reflection model it is necessary that all characteristic dimensions of the problem (wavelengths, Debye radii, and so on) are appreciably larger than the size of the inhomogeneity of the plasma density near the boundary. As we have in this paper studied three-wave interactions of just the long-wavelength surface oscillations the choice of such a model is qualitatively justified.[10,11]


1A. G. Sitenko, Fluktuatsii i nelineĭnoe vzaimodeĭstvie voln v plazme (Fluctuations and non-linear interactions of waves in a plasma) Naukova dumka, Kiev, 1977.

2L. D. Landau, Zh. Eksp. Teor. Fiz. **16**, 574 (1946) [Collected Papers of L. D. Landau, Pergamon Press, Oxford, 1965, p. 445].

3G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. London **A195**, 336 (1948).

4V. D. Shafranov, Zh. Eksp. Teor. Fiz. **34**, 1475 (1958) [Sov. Phys. JETP **7**, 1019 (1958)].

5V. P. Silin and E. P. Fetisov, Zh. Eksp. Teor. Fiz. **41**, 159 (1961) [Sov. Phys. JETP **14**, 115 (1962)].

6Yu. A. Romanov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **7**, 242 (1964); Fiz. Tverd Tela (Leningrad) **7**, 970 (1965) [Sov. Phys. Solid State **7**, 782 (1965)].

7K. N. Stepanov, Zh. Tekh. Fiz. **35**, 1349 (1965) [Sov. Phys. Tech. Phys. **10**, 1048 (1966)].

8A. N. Kondratenko, Nuclear Fusion **5**, 267 (1965).

9R. L. Guernsey, Phys. Fluids **12**, 1852 (1969).

10V. L. Ginzburg and A. A. Rukhadze, Volny v magnitoaktinvoĭ plazme (Waves in a magneto-active plasma) Nauka, 1975.

11E. A. Aba-Asali, B. A. Alterkop, and A. A. Ruchadze, Plasma Phys. **17**, 218 (1975).

12W. O. Shuman, Z. Naturforsch. **5a**, 181 (1950).

13Ya. B. Fainberg, Proc. Symposium CERN **1**, 84 (1956).

14A. G. Sitenko and V. S. Tkalich, Zh. Tekh. Fiz. **29**, 1074 (1959) [Sov. Phys. Tech. Phys. **4**, 981 (1960)].

15B. B. Kadomtsev, Plasma Turbulence, Academic Press, New York, 1965.

16R. Z. Sagdeev and A. A. Galeev, Nonlinear Plasma Theory, Benjamin, New York, 1969.

17V. V. Pustovalov and V. P. Silin, Proc. Phys. Inst. Acad. Sc. USSR, Nauka, 1972.

18K. S. Karplyuk and V. N. Oraevskiĭ, Zh. Tekh. Fiz. **38**, 1214 (1968) [Sov. Phys. Tech. Phys. **13**, 1003 (1969)].

19K. S. Karplyuk, Ya. I. Kolesnichenko, and V. N. Oraevsky, Nuclear Fusion **10**, 3 (1970).

20A. N. Kondratenko and V. G. Shaptala, Ukr. Fiz. Zh. **14**, 1092 (1969).

21A. G. Sitenko, V. N. Pavlenko, and V. I. Zasenko, Ukr. Fiz. Zh. **20**, 324 (1975).

22A. G. Sitenko, V. N. Pavlenko, and V. I. Zasenko, Fiz. Plazmy **2**, 804 (1976) [Sov. J. Plasma Phys. **2**, 448 (1976)].

23B. B. Kadomtsev, A. B. Mikhaĭlovskiĭ, and A. V. Timofeev,

Zh. Eksp. Teor. Fiz. 47, 2266 (1964) [Sov. Phys. JETP 20, 1517 (1965)].
[24]B. Coppi, M. N. Rosenbluth, and R. N. Sudan, Ann. Phys. (N.Y.) 55, 207, 248 (1969).
[25]V. N. Oraevskiĭ and R. Z. Sagdeev, Zh. Tekh. Fiz. 32, 1291 (1962) [Sov. Phys. Tech. Phys. 7, 955 (1963)].

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# Ionization instabilities of an electromagnetic wave

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An analysis is made of instabilities of a homogeneous high-frequency discharge maintained by the field of a traveling plane electromagnetic wave in a cold gas of electronegative molecules; a low electron collision frequency is assumed. A general dispersion equation is obtained for arbitrary wave perturbations of the field and plasma density. The maximum increments are found, as well as the characteristic scales of the main types of instability, which are large-scale transverse and longitudinal modulation, backscattering, and small-scale stratification in the direction of the electric field vector.

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1. One of the important tasks in the theory of high-frequency discharges is the determination of the main types of instability of possible steady-state postbreakdown states.[1-3] We shall discuss instabilities of a homogeneous discharge maintained by the field of a traveling plane electromagnetic wave in a cold gas of electronegative molecules. The wave frequency $\omega$ is assumed to be high compared with the electron collision frequency $\nu(\omega\gg\nu)$ and the plasma is regarded as a nonabsorbing medium with a real permittivity $\epsilon=1-N/N_c>0$ ($N$ is the electron density and $N_c=m\omega^2/4\pi e^2$ is the critical value of this density). The adopted homogeneous absorption-free model makes it possible to reveal instabilities of real discharges due to perturbations of characteristic scale smaller than the wave attenuation length. A similar problem has been considered by Gurevich and Shvartsburg[2] for the opposite limiting case of $\nu\gg\omega$, $\mathrm{Re}\,\epsilon=1$ assuming a specific type of perturbation (long-wavelength modulation in the longitudinal direction).

We shall begin from the vector wave equation for the slowly varying (with time) complex amplitude of the electric field $\mathbf{E}(\mathbf{r},t)$

$$\Delta\mathbf{E}+\nabla\left(\frac{1}{\varepsilon}\mathbf{E}\,\nabla\varepsilon\right)+\frac{\omega^2}{c^2}\left(\varepsilon\mathbf{E}-\frac{2i}{\omega}\frac{\partial\mathbf{E}}{\partial t}\right)=0 \quad (1)$$

and from the rate equation for the electron density

$$\frac{\partial N}{\partial t}=D\,\Delta N+(\nu_i-\nu_a)N, \quad (2)$$

in which the frequency of ionization by electron impact $\nu_i$ is regarded as a given function of the field amplitude $[\nu_i=\nu_i(|E|)]$,[1)] whereas the diffusion coefficient $D$ and the frequency of capture by neutral molecules $\nu_a$, both depending much less strongly on $|E|$, are assumed to be constant.

We shall introduce dimensionless variables by the substitution:

$$\nu_a t\to t,\quad \frac{\omega}{c}\mathbf{r}\to\mathbf{r},\quad \frac{N}{N_c}\to N,\quad \frac{E}{E_c}\to E. \quad (3)$$

Here, $E_c$ is the amplitude (known as the breakdown value) corresponding to $\nu_i(|E|)=\nu_a$, i.e., the amplitude at which a homogeneous discharge is in equilibrium. In terms of the new variables, Eqs. (1) and (2) become

$$-2i\delta\frac{\partial\mathbf{E}}{\partial t}+\Delta\mathbf{E}+\nabla\left(\frac{1}{\varepsilon}\mathbf{E}\,\nabla\varepsilon\right)+\varepsilon\mathbf{E}=0, \quad (4)$$

$$-\frac{\partial N}{\partial t}+L^2\,\Delta N+f(|E|)N=0. \quad (5)$$

Here,

$$\delta=\frac{\nu_a}{\omega}\ll 1,\quad \varepsilon=1-N,\quad L=\frac{\omega}{c}\left(\frac{D}{\nu_a}\right)^{1/2}$$

is the dimensionless diffusion capture length; $f(|E|)=\nu_i/\nu_a-1$; for $|E|=1$, we have $f=0$ and $df/d|E|>0$.

Let us assume that, under steady-state conditions the field is a plane wave of unit (breakdown) amplitude $E=y_0\exp(-i\epsilon_0^{1/2}x)$, traveling along the $x$ axis in a homogeneous plasma with an arbitrary value of $N=N_0<1$ ($\epsilon=\epsilon_0=1-N_0>0$).[2)] We shall investigate the stability of this state in the presence of small perturbations. Assuming that

$$N=N_0+N_1(\mathbf{r},t),\quad E_y=(1+E_1(\mathbf{r},t))\exp(-i\varepsilon_0^{1/2}x)$$

and linearizing Eqs. (4) and (5), we obtain the following equations for the perturbations $E_1$ and $N_1$:

$$-2i\delta\frac{\partial E_1}{\partial t}+\Delta E_1-2i\varepsilon_0^{1/2}\frac{\partial E_1}{\partial x}-\frac{1}{\varepsilon_0}\frac{\partial^2 N_1}{\partial y^2}-N_1=0, \quad (6)$$

$$-\frac{\partial N_1}{\partial t}+L^2\Delta N_1+\frac{1}{2}\alpha N_0(E_1+E_1^*)=0 \quad (7)$$

(here $\alpha=df/d|E|$ for $|E|=1$), or, representing $E_1$ in the form $E_1=u_1+iv_1$, where $u_1$ and $v_1$ are real functions:

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