The production of a magnetic field through the hydrodynamic motion of a plasma

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We show that when there are no external electric and magnetic fields, an electric current is generated via hydrodynamic motion of a plasma, and, hence, there occurs spontaneously a macroscopic magnetic field. The reason for the current generation is the difference in the electron and ion viscosities. We consider the spontaneous occurrence of a magnetic field for slow hydrodynamic motions of the plasma, and also when a beam of fast multiply charged ions are slowed down in the plasma, when the effect is greatly enhanced.

The average velocities of electrons and ions in a plasma may become different during hydrodynamic motion because of their different viscosities. This means that hydrodynamic motion can, when there are no external electric and magnetic fields, lead to the generation of an electric current in the plasma and, hence, to the spontaneous appearance of a magnetic field. The present paper is devoted to a study of that effect.

In Sec. 1 we consider relatively slow plasma motions. We show that the process of magnetic field generation soon acquires a non-linear character as the viscosity starts to depend on the magnitude of the magnetic field. Thanks to the non-linearity the growth of the field is slowed down and it becomes frozen into the plasma. The process considered here can therefore serve as one mechanism for the generation of a "seed field" in the problem of the turbulent hydro-magnetic dynamo.[1,2]

The effect is enhanced when the velocity of the plasma motion increases. In Sec. 2 we consider the generation of a magnetic field when a beam of multiply charged fast ions are slowed down in the plasma. In that limiting case one needs already a kinetic discussion to solve the problem.

1. HYDRODYNAMIC MOTION OF A FULLY IONIZED PLASMA

We consider a fully ionized plasma. Let the plasma motion satisfy the relations

$$L \gg l_{e,i}, \quad \Delta t \nu_{e,i} \gg 1, \tag{1}$$

for characteristic spatial and temporal scales L and Δt, where l_e, l_i, and ν_e, ν_i are the mean free paths and the collision frequencies of the electrons and ions. The motion of both the electronic and the ionic components of the plasma is then described by the hydrodynamics equations[3]

$$\frac{\partial N_e}{\partial t} + \operatorname{div}(N_e \mathbf{V}_e) = 0, \quad \frac{\partial N_i}{\partial t} + \operatorname{div}(N_i \mathbf{V}_i) = 0; \tag{2}$$

$$mN_e \left[\frac{\partial \mathbf{V}_e}{\partial t} + (\mathbf{V}_e \cdot \nabla) \mathbf{V}_e \right] = -\operatorname{grad} p_e - \operatorname{div} \hat{\pi}_e - eN_e \left(\mathbf{E} + \frac{1}{c}[\mathbf{V}_e \times \mathbf{H}] \right) + \mathbf{R}, \tag{3}$$

$$MN_i \left[\frac{\partial \mathbf{V}_i}{\partial t} + (\mathbf{V}_i \cdot \nabla) \mathbf{V}_i \right] = -\operatorname{grad} p_i - \operatorname{div} \hat{\pi}_i + eN_i \left(\mathbf{E} + \frac{1}{c}[\mathbf{V}_i \times \mathbf{H}] \right) - \mathbf{R}.$$

Here

$$p_e = N_e T_e, \quad p_i = N_i T_i, \tag{4}$$

are the electron and ion pressures. The electron and ion temperatures T_e and T_i are taken to be constant for the sake of simplicity. We assume to begin with also that there is no external magnetic field ($H = 0$). The viscous stress tensors for the electrons $\hat{\pi}_e$ and for the ions $\hat{\pi}_i$ then have the form

$$\pi_{\alpha\beta e,i} = -\eta_{0e,i} \left(\frac{\partial V_{\alpha e,i}}{\partial x_\beta} + \frac{\partial V_{\beta e,i}}{\partial x_\alpha} - \frac{2}{3} \delta_{\alpha\beta} \operatorname{div} \mathbf{V}_{e,i} \right); \tag{5}$$

$$\eta_{0e} = 0.73 N_e T_e / \nu_e, \quad \eta_{0i} = 0.96 N_i T_i / \nu_i, \tag{6}$$

$$\nu_e = \frac{4\sqrt{2\pi}}{3} \frac{e^4 N \ln \Lambda}{m^{1/2} T_e^{3/2}}, \quad \nu_i = \frac{4\sqrt{\pi}}{3} \frac{e^4 N \ln \Lambda}{M^{1/2} T_i^{3/2}}.$$

Here η_{0e}, η_{0i} and ν_e, ν_i are the viscosity coefficients and the collision frequencies of the electrons and ions and $\ln \Lambda$ is the Coulomb logarithm. The friction force is

$$\mathbf{R} = -\frac{mN\nu_e}{1.96} (\mathbf{V}_e - \mathbf{V}_i). \tag{7}$$

We simplify Eqs. (2) to (7). We use the fact that normally the characteristic scale L of the motion is large compared to the Debye radius and, hence, the plasma is quasi-neutral:

$$N_e \approx N_i = N.$$

The continuity equations (2) can then be rewritten in the form

$$\partial N / \partial t + \operatorname{div}(N \mathbf{V}) = 0,$$
$$\operatorname{div}(N \mathbf{V}) = \operatorname{div}(N \mathbf{V}_e) = \operatorname{div}(N \mathbf{V}_i). \tag{8}$$

The last equation determines the electric polarization field of a quasi-neutral plasma (see Refs. 4, 5).

We introduce instead of $\mathbf{V}_e$ and $\mathbf{V}_i$ new variables—the hydrodynamic velocity $\mathbf{V}$ and the velocity difference $\mathbf{U}$:

$$\mathbf{V} = \frac{M\mathbf{V}_i + m\mathbf{V}_e}{M + m}, \quad \mathbf{U} = \mathbf{V}_e - \mathbf{V}_i;$$

$$\mathbf{V}_i = \mathbf{V} - \frac{m}{M+m}\mathbf{U}, \quad \mathbf{V}_e = \mathbf{V} + \frac{M}{M+m}\mathbf{U}. \tag{9}$$

Adding and subtracting the two Eqs. (3) from one another we get (for $H = 0$)

$$(M+m)N\left(\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\nabla)\mathbf{V}\right)=-\mathrm{grad}\,(p_e+p_i)$$
$$-\mathrm{div}\,(\hat{\pi}_e+\hat{\pi}_i)-\frac{mM}{M+m}N(\mathbf{U}\nabla)\mathbf{U}, \qquad (10)$$

$$-\mathbf{R}+eN\mathbf{E}=-\mathrm{grad}\left(\frac{Mp_e-mp_i}{M+m}\right)-\mathrm{div}\left(\frac{M\hat{\pi}_e-m\hat{\pi}_i}{M+m}\right)$$
$$-N\frac{mM}{M+m}\left[\frac{\partial \mathbf{U}}{\partial t}+(\mathbf{U}\nabla)\mathbf{V}+(\mathbf{V}\nabla)\mathbf{U}+\frac{M-m}{M+m}(\mathbf{U}\nabla)\mathbf{U}\right]. \qquad (11)$$

We can neglect the last term in (11) because of condition (1) when compared to R. Moreover, we use (5), (9) to transform the viscous friction term in (11):

$$\mathrm{div}\left(\frac{M\pi_e-m\pi_i}{M+m}\right)=-\frac{M\eta_{0e}}{M+m}\left(\nabla^2\mathbf{V}_e+\frac{1}{3}\mathrm{grad\,div}\,\mathbf{V}_e\right)$$
$$+\frac{m\eta_{0i}}{M+m}\left(\nabla^2\mathbf{V}_i+\frac{1}{3}\mathrm{grad\,div}\,\mathbf{V}_i\right)=\frac{-M\eta_{0e}+m\eta_{0i}}{M+m}\left(\nabla^2\mathbf{V}+\frac{1}{3}\mathrm{grad\,div}\,\mathbf{V}\right)$$
$$-\frac{M^2\eta_{0e}+m^2\eta_{0i}}{(M+m)^2}\left(\nabla^2\mathbf{U}+\frac{1}{3}\mathrm{grad\,div}\,\mathbf{U}\right). \qquad (12)$$

The last term in (12) can also be neglected in comparison with R by virtue of (1). Finally we have instead of (11)

$$-\mathbf{R}+eN\mathbf{E}=-\mathrm{grad}\left(\frac{Mp_e-mp_i}{M+m}\right)+\frac{M\eta_{0e}-m\eta_{0i}}{M+m}\left(\nabla^2\mathbf{V}+\frac{1}{3}\mathrm{grad\,div}\,\mathbf{V}\right). \qquad (13)$$

Equations (13), (10), and (8) form a complete set decribing the hydrodynamic motion of a quasi-neutral plasma. It differs from the set traditionally used (see, for instance, Ref. 6) in taking into account the effect of the viscosity on the difference in electron and ion velocities (the last term in Eq. (13)). Below we shall show that just this generally weak process guarantees the generation of an electric current in the case of hydrodynamic motion of a plasma when there are no external electric and magnetic fields. Indeed, it follows from (8) that

$$\mathrm{div}\,\mathbf{j}=-e\,\mathrm{div}(N\mathbf{U})=0. \qquad (14)$$

We consider the incompressible flow

$$\mathrm{div}\,\mathbf{V}=0. \qquad (15)$$

Determining $N\mathbf{U}$ from (7), (13), and (15) and substituting it into (14) we are led to the equation

$$\nabla^2\varphi=\frac{MT_e-mT_i}{e(M+m)}\nabla^2\ln\frac{N}{N_0}, \qquad (16)$$

describing the electric polarization field of the plasma ($\mathbf{E}=-\nabla\varphi$). It follows from (16) that

$$\varphi=\frac{MT_e-mT_i}{e(M+m)}\ln\frac{N}{N_0}\approx\frac{T_e}{e}\ln\frac{N}{N_0}. \qquad (17)$$

Substituting now (17) into (13) we eliminate the polarization field E. We then find from (13) that

$$-\mathbf{R}=\frac{M\eta_{0e}-m\eta_{0i}}{M+m}\nabla^2\mathbf{V}.$$

This yields the velocity of the electron motion relative to that of the ions U, i.e., the electron current in the plasma $\mathbf{j}=eN\mathbf{U}$:

$$\mathbf{j}=g\nabla^2\mathbf{V}, \quad -g=1.43\,eNT_e/m v_e^2. \qquad (18)$$

By virtue of the condition (15) the current (18) has zero divergence—it does not lead to the appearance of electric charges or a polarization of the plasma.

The physical nature of the current (18) if quite lucid: it is connected with the difference in the viscosity coefficients for the electrons and the ions in the plasma. Thanks to the difference in the viscosities there appears a difference in the average velocity of the electron and ion motions and, hence, there appears an electric current. It is then important that the transfer of the total momentum is mainly caused by the ion viscosity (10), (6) so that the average velocity of the ion motion $\mathbf{V}_i$ is close to the velocity of the hydrodynamic motion of the plasma V given by (9). On the other hand, the average velocity of the electrons differs appreciably from it. This difference is caused by the electron viscosity. The electron viscosity also determines therefore the electric current (18) arising in the plasma.

The mechanism for the appearance of a current is made clear in Fig. 1. The solid line shows the profile of the hydrodynamic velocity of the motion of the plasma in the y-direction or the profile of the ion velocity which is close to it. The electrons are more mobile than the ions, they possess a larger kinematic viscosity, and tend to flatten out a velocity gradient. Their velocity profile is shown by the dashed curve. The current appearing in the plasma is proportional to $\mathbf{V}_i-\mathbf{V}_e$ and it changes sign in the inflection points of the hydrodynamic velocity profiles.

The magnetic field produced by the current (18) equals

$$\mathbf{H}=-\frac{4\pi g}{c}\mathrm{rot}\,\mathbf{V}+\mathrm{grad}\,\Psi, \qquad (19)$$

where Ψ is an arbitrary harmonic function ($\nabla^2\Psi=0$). It is determined by the conditions at the boundaries.

As an example we consider a one-dimensional plasma flow: let the hydrodynamic velocity V in the chosen frame of reference be in the direction of the y-axis and let it change in magnitude along the x-axis, i.e.,

$$\mathbf{V}=(0,\,V_y,\,0), \quad V_y=V_y(x). \qquad (20)$$

The incompressibility condition (15) is then always satisfied. The electric viscosity current j is directed along the y-axis, i.e., it is parallel to V. It is equal to

$$\mathbf{j}=(0,\,j_y,\,0), \quad j_y=g\partial^2 V_y/\partial x^2. \qquad (21)$$

In particular, for Poiseuille flow[7] the current has a constant magnitude:

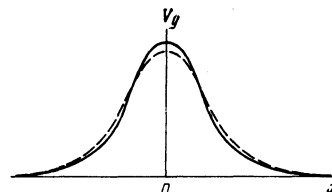FIG. 1.

$$j_y - j_0 = \frac{g}{\eta}\left(\frac{dp}{dy}\right)_0 = -1.05\frac{e(T_e/T_i)^{3/2}}{(mM)^{1/2}v_e}\left(\frac{dp}{dy}\right)_0. \qquad (22)$$

Here $\eta \approx \eta_{0i}$ is the viscosity coefficient, p the total pressure (4), (10), $(dp/dy)_0$ the constant pressure gradient. The magnetic field H produced by the current (21), (22) is in the direction of the z-axis. In magnitude it equals (cf. (19))

$$H_z = H = -\frac{4\pi g}{c}\frac{\partial V_y}{\partial x} + \text{const}. \qquad (23)$$

In particular, for Poiseuille flow in the region $-a \leqslant x \leqslant a$, where a is the boundary of the plasma, we have

$$H = H_z = -\frac{4\pi a j_0}{c}\begin{cases} -1 & \text{when } x \leqslant -a \\ x/a & \text{when } -a \leqslant x \leqslant a. \\ 1 & \text{when } x \geqslant a \end{cases}$$

It is here important that at the boundary $x = \pm a$ the gradient $|\partial V_y/\partial x| \neq 0$. If, however, the hydrodynamic motion occupies a region unbounded in x inside the plasma, the constant vanishes in Eq. (23). A magnetic field arises in that case only in the region where the plasma moves

$$H_z = -\frac{4\pi g}{c}\frac{\partial V_y}{\partial x}.$$

A completely analogous situation occurs also for an axially symmetric flow of the plasma.

The magnetic field arising in the plasma can turn out to have an appreciable effect on the motion of the electrons and the ions. First, the viscous stress tensor $\hat{\pi}_{e,i}$ and the friction force R can change. Of course, in that case the condition for the excitation of the dielectric viscosity current (18) is also changed which in turn affects the magnetic field itself. For sufficiently large values of H the process of generating a magnetic field thus becomes non-linear. The linear approximation considered above is valid under the restriction

$$\omega_H \ll v_e, \quad \omega_H = eH/mc, \qquad (24)$$

when the effect of the magnitude field on the friction force and on the tensor $\hat{\pi}_{e,i}$ is unimportant. Using (19), (18) we rewrite condition (24) in the form

$$|\text{rot } V| \ll m^2c^2v_e^3/18e^2NT_e. \qquad (25)$$

The restriction (25) is very strong and is often violated. It is therefore of interest to study the generation of a magnetic field in the non-linear case when condition (25) is not satisfied and when there is an appreciable magnetic field in the plasma. When there is a magnetic field present in the plasma Eqs. (2), (8) retain their form. After the transformation (9) Eqs. (3) can be written in a form analogous to (10), (13):

$$(M + m)\left(\frac{\partial V}{\partial t} + (V\nabla)V\right)$$
$$= -\text{grad}(p_e + p_i) - \frac{eN}{c}[U \times H] - \text{div}(\hat{\pi}_e + \hat{\pi}_i) - N\frac{Mm}{M+m}(U\nabla)U,$$
$$-R + \frac{eN}{c}\frac{M-m}{M+m}[U \times H] + eN\left(E + \frac{1}{c}[V \times H]\right)$$
$$= -\text{grad}\left(\frac{Mp_e - mp_i}{M+m}\right) - \text{div}\left(\frac{M\hat{\pi}_e - m\hat{\pi}_i}{M+m}\right). \qquad (26)$$

The viscous stress tensors of the electrons $(\hat{\pi}_e)$ and of the ions $(\hat{\pi}_i)$ are given in Ref. 3 for the case when there is a magnetic field present. Here we write down only the tensor $\hat{\pi}_e$ for the case when

$$\omega_H \gg v_e. \qquad (27)$$

Taking the z-axis parallel to H we have

$$(\text{div }\hat{\pi}_e)_x = -\eta_{1e}\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2}\right) - \eta_{2e}\frac{\partial^2 V_x}{\partial z^2}$$
$$-\eta_{3e}\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2}\right) - \eta_{4e}\frac{\partial^2 V_y}{\partial z^2} + (\eta_{0e} - \eta_{2e})\frac{\partial^2 V_z}{\partial x\partial z} - \eta_{4e}\frac{\partial^2 V_z}{\partial y\partial z},$$
$$(\text{div }\hat{\pi}_e)_y = -\eta_{1e}\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2}\right) - \eta_{2e}\frac{\partial^2 V_y}{\partial z^2}$$
$$+\eta_{3e}\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2}\right) + \eta_{4e}\frac{\partial^2 V_x}{\partial z^2} + (\eta_{0e} - \eta_{2e})\frac{\partial^2 V_z}{\partial y\partial z} + \eta_{4e}\frac{\partial^2 V_z}{\partial x\partial z},$$
$$(\text{div }\hat{\pi}_e)_z = -(2\eta_{0e} - \eta_{2e})\frac{\partial^2 V_z}{\partial z^2} - \eta_{2e}\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2}\right) - \eta_{4e}\left(\frac{\partial^2 V_y}{\partial z\partial x} - \frac{\partial^2 V_x}{\partial z\partial y}\right). \qquad (28)$$

The coefficient η_{0e} is defined here as before by Eq. (6) and the remaining coefficients have the form

$$\eta_{1e} = 0.51\frac{NT_e v_e}{\omega_H^2}, \quad \eta_{2e} = 2.04\frac{NT_e v_e}{\omega_H^2},$$
$$\eta_{3e} = -\frac{NT_e}{2\omega_H}, \quad \eta_{4e} = -\frac{NT_e}{\omega_H}. \qquad (29)$$

The friction force is

$$R = -mNv_e(U_\parallel/1.96 + U_\perp), \qquad (30)$$

where $U = V_e - V_i$ and the components $U_\parallel$ and $U_\perp$ are taken along and at right angles to the magnetic field.

We consider the simple case of one-dimensional flow in the plasma. As before, we take the y-axis in the direction of the velocity V, with $V_y = V_y(x)$ as in (20). Under the conditions (25) there arises then a magnetic field directed along the z-axis. We assume that it retains its direction also under conditions of strong anisotropy (27) (we shall show this in what follows). We can then write Eqs. (26), (28), and (30) in the form

$$mNv_eU_x + mN\omega_HU_y = \eta_{3e}\partial^2 V_y/\partial x^2,$$
$$mNv_eU_y - mN\omega_HU_x = \eta_{1e}\partial^2 V_y/\partial x^2, \qquad (31)$$
$$mNv_eU_z = 0.$$

We have here eliminated the polarization field $E = -\nabla\varphi$, which is determined by the relation (see (17))

$$E = -\frac{1}{c}[V \times H] - \frac{T_e}{e}\nabla\ln\frac{N}{N_0}.$$

It follows from (29) and (31) that

$$U_y = -\frac{T_e(1-4.08\nu_e{}^2/\omega_H{}^2)}{2m(\omega_H{}^2+\nu_e{}^2)}\frac{\partial^2 V_y}{\partial x^2}, \quad U_z = -\frac{2.54T_e\nu_e}{m(\omega_H{}^2+\nu_e{}^2)\omega_H}\frac{\partial^2 V_y}{\partial x^2},$$

$$U_x = 0. \tag{32}$$

However, the divergence of the current $\mathbf{j}_0 = -eN\mathbf{U}$ is in this case not zero. There arises then an additional polarization field $\mathbf{E}_1$ such that the total current $\mathbf{j} = \mathbf{j}_0 + \hat{\sigma}\mathbf{E}_1$ has zero divergence, (14):

$$\text{div }\mathbf{j} = \text{div }(\mathbf{j}_0 + \sigma\mathbf{E}_1) = 0. \tag{33}$$

We find from (33) that

$$\mathbf{E}_1 = (E_{1x}, 0, 0), \quad E_{1x} = -\frac{\partial\varphi_1}{\partial x} = -\frac{2.54eT_e\nu_e}{m\omega_H{}^3\sigma_1}\frac{\partial^2 V_y}{\partial x^2}, \tag{34}$$

where σ_1 is the conductivity of the plasma across the magnetic field:

$$\sigma_1 = e^2 N\nu_e/m\omega_H{}^2. \tag{35}$$

The total current caused by the simultaneous action of the viscosity $\mathbf{j}_0$ and the additional polarization field $\mathbf{E}_1$ is equal to

$$\mathbf{j} = \mathbf{j}_0 + \hat{\sigma}\mathbf{E}_1 = (-eNU_x + \sigma_1 E_{1x}, -eNU_y + \sigma_2 E_{1x}, 0).$$

Here σ_2 is the Hall conductivity:

$$\sigma_2 = e^2 N/m\omega_H.$$

It follows from (32), (34), and (35) that $j_x = 0$. Hence, the current arising in the plasma is, as in case (21), directed along the y-axis which is parallel to the hydrodynamic velocity $\mathbf{V}$. As to magnitude, it equals (cf. (21), (18))

$$j_y = g\frac{\partial^2 V_y}{\partial x^2}, \quad -g = \frac{2.04eT_eN}{m\omega_H{}^2}.$$

Such a current generates a magnetic field directed along the z-axis, and this was already used above.

The magnitude of H_z is now determined by the nonlinear equation

$$-\frac{\partial H_z}{\partial x} = \frac{4\pi}{c}j_y = -\frac{4\pi \cdot 2.04eT_eN}{mc\omega^2{}_H}\frac{\partial^2 V_y}{\partial x^2},$$

$$\omega_H = eH_z/mc.$$

Its solution with the boundary conditions $\partial V_y/\partial x \to 0$, $H_z \to 0$ when $x \to \pm\infty$ has the form

$$H_z = \left(\frac{24.5\pi T_eNmc}{e}\frac{\partial V_y}{\partial x}\right)^{1/2}. \tag{36}$$

It is convenient to introduce the dimensionless parameter

$$G = \frac{24.5\pi e^2 NT_e}{m^2 c^2\nu_e{}^2}\frac{\partial V_y}{\partial x}. \tag{37}$$

We can then write the solutions of (18), (23), and (36) in the form

$$\omega_H/\nu_e = G/4.28, \quad G \lesssim 1; \quad \omega_H/\nu_e = G^{1/2}, \quad G \gg 1. \tag{38}$$

From this it is clear that for large values of G the magnetic field continues to grow with increasing G although less energetically than in the linear case. We note that replacing $\partial V_y/\partial x$ in (37) by $|\text{curl }\mathbf{V}|$ we can use Eq. (38) for an estimate of the magnitude of the magnetic field generated in the plasma in the general case.

Under real cosmic conditions, for the interstellar ionized gas or, for instance, for the solar chromosphere and corona, the parameter G is always a large quantity. For instance, for the interstellar gas, putting $T_e \sim 1$ eV, $N \sim 1$ cm^{-3}, $V \sim 10^5$ cm/s, $L \sim 10^{20}$ cm, we have $G \sim 10^2$. It then follows from (38) that the effect considered here leads under cosmic conditions to the generation of a magnetic field guaranteeing the satisfying of the condition for the freezing-in of field lines. This effect can thus serve as one mechanism for the generation of a seed field for the turbulent hydromagnetic dynamo.[1,2]

We note also that although we considered above only a completely ionized plasma, the effect of the generation of an electric current and a magnetic field due to viscosity occurs also for any degree of ionization. In particular, in a weakly ionized plasma the electric viscosity current is as before described by Eq. (18):

$$\mathbf{j} = g_n\nabla^2\mathbf{V}_n,$$

where $\mathbf{V}_n$ is the velocity of the neutral gas and satisfies the incompressibility condition (div $\mathbf{V}_n = 0$) and the coefficient

$$-g_n = \frac{eN}{N_n{}^2}\left(\frac{k_e}{\sigma_e{}^2} - \frac{k_i}{\sigma_i{}^2}\right).$$

Here σ_e and σ_i are the transport cross-sections for collisions of electrons and ions with neutral molecules, k_e and k_i are numerical coefficients which depend on the nature of the interaction between the particles. The difference with the case of a completely ionized plasma consists in that now the velocity of the hydrodynamic motion is determined by the neutrals while the electrons and ions make contributions of the same order to the electric current: $\mathbf{j}_{e,i} \sim eNl_{e,i}^2\nabla^2\mathbf{V}_n$.

2. SLOWING DOWN OF A BEAM OF ACCELERATED MULTIPLY CHARGED IONS

Above we considered the case of a relatively slow hydrodynamic motion of the plasma. When the velocity of the motion increases the generation of the electric current and of the magnetic field is enhanced. It is enhanced also when the interaction between the particles increases, in particular, when the ionic charge Z increases. In this section we consider another limiting case when the motion is fast: $V \gg v_{Ti}$ and the ion charge $Z \gg 1$. As an example of such a motion we can mention the deceleration of a beam of fast multiply charged ions in a dense plasma. We assume the ion beam to be compensated by electrons so that, as before, there is no a priori given current or magnetic field at all in the system.[1]

We consider a completely ionized plasma filling the half-space $x > 0$ with a compensated beam of Z-ions of

mass M_z streaming into it with a velocity V_0 in the z-direction. The density of the electrons and ions in the plasma N is much larger than the density of the Z-ions: $N \gg N_z$. The slowing-down process of the Z-ions in the plasma is therefore independent of the interaction between them, i.e., it has a linear character. Moreover, taking into account that we consider here only fast multiply charged ions with a velocity of the same order as the thermal electron velocity we can to a first approximation neglect the motion of the main ions in the plasma. The fast multiply charged ions lose their energy mainly through interactions with the electrons in the plasma. In that case their scattering is small so that to a first approximation we may assume that the ions do not change the direction of their motion. The deceleration of the Z-ions, i.e., the change in their velocity V_z and their density N_z is then, as the beam penetrates into the plasma, described by the equation[9,10]

$$\frac{dV_z}{dx} = -\left[\frac{M_z}{m}G(y)+\frac{M_z}{M}\right]\nu_z, \quad N_z = \frac{N_{z0}V_0}{V_z},$$

$$y=\frac{V_z}{v_{Te}}, \quad G(y)=\Phi(y)-\frac{2ye^{-y^2}}{\pi^{1/2}}, \quad \Phi(y)=\frac{2}{\pi^{1/2}}\int_0^y e^{-t^2}dt, \quad (39)$$

$$\nu_z=\frac{4\pi e^4 N Z^2 \ln \Lambda}{M_z^2 V_z^3}, \quad V_z|_{x=0}=V_0, \quad v_{Te}=\left(\frac{2T_e}{m}\right)^{1/2}.$$

The solution of Eq. (39) determining implicitly the function $y(x)$, i.e., the ion velocity $V_z(x)$ is

$$\Psi(y)=\Psi(y_0)-\frac{\zeta}{L}, \quad \Psi(y)=\int_0^y \frac{y^2 dy}{G(y)+m/M}, \quad \zeta=\int_0^x \frac{N(x)}{N_0}dx,$$

$$L=\frac{M_z l_e}{mZ^2}, \quad l_e=\frac{T_e^2}{\pi e^4 N_0 \ln \Lambda}, \quad y_0=V_0/v_{Te}. \quad (40)$$

Here l_e is the electron mean free path and L a characteristic deceleration length for a Z-ion. In particular, when $V_z > v_{Te}$, it follows from (40) that

$$V_z = V_0(1-4\zeta/L)^{1/4}.$$

It is clear from (40) that close to the point

$$\zeta_\kappa = L\Psi(y_0), \quad (41)$$

multiply charged ions are stopped (to be more precise, their velocity becomes equal to the thermal velocity of the ions in the plasma).

We now consider how multiply charged ions slowed down in the plasma excite an electric current. For the solution of this problem it is necessary to use kinetic theory. It is clear from (40) that the characteristic length L for the deceleration of Z-ions in the plasma is much longer than the electron mean free path l_e. To determine the perturbation of the electron distribution function we can thus restrict ourselves to a locally uniform problem, assuming that the velocity V_z and density N_z of the multiply charged ions are given at each point x by (39), (40). We write the kinetic equation for the electrons in a system of coordinates moving with a velocity equal to the average velocity of the main ions in the plasma. It has the form

$$S_{ee}+S_{ei}=-S_{ez}, \quad (42)$$

where S_{ee}, S_{ei}, and S_{ez} are the collision integrals for collisions of the electrons with one another, with the main ions in the plasma, and with the Z-ions.

The integral S_{ez} is the source for the perturbation of the electron distribution function.[2] Considering it, we bear in mind that there is only one preferred direction in velocity space—the direction of motion of the Z-ions—the x-axis. Hence, the electron distribution function depends only on the modulus v of the velocity and on the angle θ with the x-direction: $f=f(v,\theta)$. The multiply charged ion distribution function also depends solely on the modulus v_z of their velocity and the angle θ_z. Neglecting scattering of the Z-ions, it has the form

$$F(v_z,\theta_z)=\frac{N_z}{2\pi V_z^2}\delta(v_z-V_z)\delta(\cos\theta_z-1). \quad (43)$$

Starting from the Landau integral,[11] and using (43) and the fact that there is only one preferred direction in the problem,[5] we are led to the following expression for S_{ez}:

$$S_{ez}=\frac{1}{v^2}\frac{\partial}{\partial v}(v^2 j_v)+\frac{1}{v\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta j_\theta), \quad (44)$$

where

$$j_v=\frac{1}{2}\left[(A_1+A_3)f+A_2\frac{\partial f}{\partial v}+\frac{A_4}{v}\frac{\partial f}{\partial\theta}\right],$$

$$j_\theta=\frac{1}{2}\left[(B_1+B_3)f+B_2\frac{\partial f}{\partial v}+\frac{B_4}{v}\frac{\partial f}{\partial\theta}\right];$$

$$A_1=-\frac{3m}{M_z}\frac{v^2(v-V_z\cos\theta)\nu(u)}{u^2}(1-\cos^2\theta), \quad (44')$$

$$A_2=-V_z^2\nu(u)(1-\cos^2\theta),$$

$$A_3=\frac{m}{M_z}\nu(u)(V_z-v\cos\theta)\cos\theta,$$

$$A_4=V_z\nu(u)(v-V_z\cos\theta)\sin\theta;$$

$$B_1=\frac{2m}{M_z}V_z\nu(u)v^2\left[1-\frac{3V_z^2(1-\cos^2\theta)}{2u^2}\right]\sin\theta,$$

$$B_2=V_z\nu(u)(v-V_z\cos\theta)\sin\theta, \quad B_3=\frac{m}{M_z}\frac{u^2\nu(u)}{V_z}\sin\theta,$$

$$B_4=-\nu(u)(v-V_z\cos\theta)^2;$$

$$u=(V_z^2+v^2-2V_zv\cos\theta)^{1/2}, \quad \nu(u)=\frac{4\pi e^4 Z^2 N_z \ln\Lambda}{m^2 u^3}.$$

We now expand, as usual, the electron distribution function in a series in Legendre polynomials $P_k(\cos\theta)$:

$$f(v,\theta)=f_0(v)+f_1(v)\cos\theta+f_2(v)P_2(\cos\theta)+\ldots$$

For the integrals S_{ezk} which determine the perturbation of each harmonic $f_k(v)$ in (42):

$$S_{ezk}=\frac{2k+1}{2}\int_{-1}^1 S_{ez}P_k(\cos\theta)d\cos\theta,$$

we then get

$$S_{ez0}=-\frac{V_z^2}{3v^2}\frac{\partial}{\partial v}\left\{v^2\frac{\partial f_0}{\partial v}\begin{bmatrix}\nu_{ez}(v), & v>V_z\\ \nu_{ez}(V_z), & v<V_z\end{bmatrix}\right\} \quad (45)$$

$$S_{ez1}=-\frac{3V_z^2}{5v^2}\frac{\partial}{\partial v}\left\{v^2\frac{\partial f_0}{\partial v}\begin{bmatrix}\frac{V_z}{v}\nu_{ez}(v), & v>V_z\\ \frac{v}{V_z}\nu_{ez}(V_z), & v<V_z\end{bmatrix}\right\}$$

$$+V_z\frac{\partial f_0}{\partial v}\begin{bmatrix}\nu_{ez}(v)(1-3V_z^2/5v^2), & v>V_z\\ {}^2/_5\nu_{ez}(V_z), & v<V_z\end{bmatrix}$$

where

$$\nu_{ez}(v) = \frac{4\pi e^4 Z^2 N_z \ln \Lambda}{m^2 v^3}.$$

In deriving Eq. (45) we used the fact that when

$$Z^2 N_z/N \ll 1$$

the perturbations of the electron distribution function are small and the main role in the integral S_{ez} is thus played by the main symmetric function $f_0(v)$. We also used the fact that when $V_Z^2 \gg mv^2/M_Z$ the terms A_1, A_3 and B_1, B_3 in (44') are small.

Knowing the integral S_{ez1} we can easily determine from Eq. (42) the directed part $f_1(v)$ of the electron distribution function and, hence, the average velocity of motion of the electrons relative to the ions, i.e., the electron current. In particular,[5,6]

$$S_{e11} = \nu_e(v)f_1, \quad \nu_e(v) = \frac{4\pi e^4 N \ln \Lambda}{m^2 v^3}, \tag{46}$$

so that we find at once from (42), (45) the function f_1 and the current when we neglect the electron-electron collisions:

$$j_e = -\frac{4\pi e}{3} \int_0^\infty v^3 f_1 dv = -j_0 F(y), \quad j_0 = -Zj_z = -eZ^2 N_z V_z. \tag{47}$$

Here j_z is the current of the multiply charged ions and the function

$$F(y) = \left(1 + \frac{6}{5}y^2\right)(1 - \Phi(y)) + \frac{112}{5\pi^{1/2}y^3}\left[1 - \left(1 + y^2 + \frac{23}{56}y^4\right)e^{-y^2}\right], \quad y = V_z/v_{Te}.$$

It is shown by the dashed curve in Fig. 2.

When the electron-electron collisions are taken into account Eq. (42) for the function f_1 is an integro-differential equation. It is solved by the usual method.[10] As before the electron current is given by Eq. (47): only the form of the function $F(y)$ is changed. It is shown by the solid curve in Fig. 2. As $y \to 0$, $F(y) = 1 + O(y^2)$, when $y \gg 1$ we have $F = 4.80/y^3$. The maximum value $F_m \approx 1.18$ is reached for $y \approx 0.8$. We give an analytical expression for $F(y)$ which is obtained in second order in the expansion of the distribution function in Laguerre polynomials:
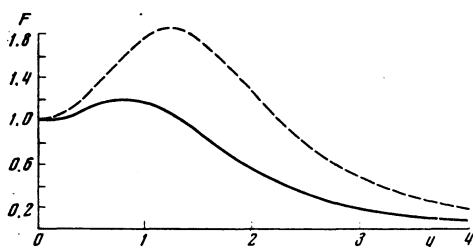
FIG. 2.

$$F(y) = 1.950F_0(y) - 0.5545F_1(y) - 0.06295F_2(y),$$
$$F_0 = {}^3/_4\pi^{1/2}\Phi(y) - {}^3/_2ye^{-y^2},$$
$$F_1 = -{}^3/_4\pi^{1/2}\Phi(y) + {}^3/_2y(1 + y^2)e^{-y^2},$$
$$F_2 = {}^{15}/_8y^3(1 - 2y^2)e^{-y^2}. \tag{48}$$

The accuracy of Eq. (48) is one to two per cent.

In equation (47) j_z is the multiply charged ion current density. It is clear from (47) that the electron current arising in the plasma is about Z times larger than the current of the Z-ions. Because of that we can neglect at $Z \gg 1$ the current of the multiply charged ions themselves in comparison with the electron current. When the Z-ions are slowed down there arises thus in the plasma an appreciable enhancement of the current.[3] This effect is analogous to the drag of multiply charged ions in a plasma by an electron stream (escape of multiply charged ions) considered in Refs. 9, 13.

The electron current $\mathbf{j}_e$ of (47) has a non-vanishing divergence. This leads to the appearance of an electric polarization field in the plasma $\mathbf{E} = -\nabla\varphi$. The total current, determined by the action of the Z-ions ($\mathbf{j}_e$) and the electric polarization current ($\mathbf{j}_E$) has zero divergence, (14): $\mathrm{div}(\mathbf{j}_e + \mathbf{j}_E) = 0$. Hence follows the equation which determines as usual, (33), the potential of the polarization field:

$$\mathrm{div}(\hat{\sigma}\nabla\varphi) = \mathrm{div}\,\mathbf{j}_e. \tag{49}$$

Here $\hat{\sigma}$ is the conductivity tensor of the plasma and $\mathbf{j}_e$ the source current (47). If we can neglect the effect on the conductivity of the magnetic field which arises in the plasma (linear approximation (24)) and if the change in the electron temperature is unimportant, $\hat{\sigma}$ is a constant and Eq. (49) is identical with the Poisson equation. Its solution is found by expanding in spherical harmonics. In the case of an axially symmetric beam we have for the electric and magnetic fields

$$\varphi = \sum_{k=0}^\infty \varphi_k(r)P_k(\mu), \quad H_\varphi = \sum_{k=0}^\infty H_{\varphi k}(r)P_k(\mu),$$

where (see Ref. 14)

$$\varphi_k(r) = -\frac{4\pi}{2k+1}\left\{r^k \int_r^{R_0} t^{-k+1}\rho_k(t)dt + r^{-k-1}\int_0^r t^{k+2}\rho_k(t)dt\right\}, \tag{50}$$

$$H_{\varphi k} = -\frac{4\pi r}{k(k+1)c}\left(\frac{d\varphi_k}{dr} + j_{erk}\right),$$

$$\rho_k = \frac{2k+1}{8\pi}\int_{-1}^1 \left[\frac{\partial j_e}{\partial x} + \frac{\partial j_e^-}{\partial x}\right]_{r,\theta} P_k(\mu)d\mu. \tag{51}$$

Here r, θ, φ is a spherical frame with origin in the center of the ion beam at the boundary of the plasma $x = 0$ and the angle θ reckoned from the x-axis; $\mu = \cos\theta$; j_{er} is the radial component of the current j_e, (47); $\partial j_e^-/\partial x$ is a fictitious source in the region $x < 0$ (outside the plasma) arising through the condition that there should be no current flowing through the boundary of the plasma $x = 0$ ($j_e^-(-x) = -j_e(x)$). An important role is played in Eq. (51) by the discontinuities of the current j_e occurring at the plasma boundary $x = 0$ and in the point where the Z-ion stream stops $x = x_k$ (see (41)); in fact, the region of the discontinuity is spread out over a distance of the order of l_e.

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The total electron current forms according to (47), (50), and (51) a closed configuration. In the region occupied by the ion beam it is directed, like the ion current, in the $x > 0$ direction the current from the source (47) dominates here. Outside this region the electron current has the opposite direction and is caused by the electrostatic field. The current density steeply rises near the plasma boundary $x \approx 0$ and at the point where the beam stops $x \approx x_k$. The magnetic field of the total current has only the component H_φ, (50). It has an annular structure and encircles the ion beam. The maximum value is $H_{\varphi m} \sim R_0 Z j_{z0}/c$, where R_0 is the radius of the ion beam. Such a value of H_φ is, however, reached only when

$$l < R_0 < L. \qquad (52)$$

If, on the other hand, the beam is wide $R_0 > L$, the source current j_e is effectively cancelled by the polarization field current j_E so that the total current diminishes with increasing R_0. As $R_0/L \to \infty$ the total current $j = j_e + j_E \to 0$. This, of course, should be the case, as for a one-dimensional problem $j = (j_x, 0, 0)$, $j_x = j_x(x)$, the quasineutrality condition $\mathrm{div}\, j = 0$ is essentially equivalent to the condition $j_x = 0$. In that case the source current is completely compensated by the polarization current.

The lower bound in Eq. (52) is due to the condition of local uniformity of the problem: only in that case is Eq. (42) valid. The applicability of Eqs. (50), (51) is also restricted by the condition that the linear approximation (24) holds, which in our case can, when (52) is satisfied, be written in the form

$$eR_0 Z j_{z0}/mc^2 v_e < 1. \qquad (53)$$

When condition (53) is violated the growth of the magnetic field with increasing current of the multiply charged ions j_{z0} is weakened (see Sec. 1). We note also the limitations on the applicability of the theory following from the stability conditions of the plasma when it is perturbed by a multiply charged ion beam.[15]

We also emphasize that we considered above only an established, stationary state of the plasma. The buildup process is accompanied by the appearance of the electric induction field which prevents the appearance of an electric current. The characteristic time for the establishment of a stationary state (under conditions (24), (53)) is

$$\Delta t \sim (\pi R_0/\lambda)^2 v_e^{-1}.$$

Here $\lambda = 2\pi c/\omega_0$, ω_0 is the electron Langmuir frequency.

[1] The results of this section have been briefly reported earlier in Ref. 8

[2] We note that through the action of the Z-ions on the plasma there also arises an electric polarization field in the plasma. It also causes a perturbation in the electron distribution function and must be taken into account in the kinetic equation. However, in the linear approximation the perturbations of the distribution function caused by the multiply charged ions and by the electric polarization field are independent and can be considered separately. We have therefore to begin with considered only the perturbations caused by the Z-ions (42). The effect of the electric field will be taken into account in what follows.

[3] Using the integral S_{eZ0}, (45), one can easily determine the heating of the electrons in the plasma due to their interaction with the multiply charged ions:

$$d\varepsilon/dt = mV_z^2 \nu_{ez} N [\Phi(y) - 2\pi^{-1/2} y (1 - {}^2/_3 y^2) e^{-y^2}], \qquad y = V_z/v_{Te}.$$

Here $d\varepsilon/dt$ is the energy transferred on average to the electrons in the plasma per unit time and unit volume. The possibility to use multiply charged ions to heat a dense plasma is discussed in Ref. 12.

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