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Magnetic properties of disordered media

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It is shown that in spatially disordered magnetic systems three macroscopically different types of magnetically disordered exchange structures are possible. Besides a disordered ferromagnet and a completely random structure of the spin-glass type, a structure corresponding to a disordered antiferromagnet with three mutually perpendicular antiferromagnetic moments is possible. Nonlinear equations are found that describe the dynamical properties of spin glasses and disordered ferromagnets and antiferromagnets with allowance for the external magnetic field and relativistic interactions. The spin-wave spectrum and magnetic-resonance frequencies are calculated.

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This paper is devoted to a macroscopic analysis of the magnetic properties of spatially disordered media, i.e., systems in which the spatial distribution of the magnetic atoms is, on the average, homogeneous and isotropic. We are concerned here with amorphous substances containing magnetic atoms or with weak solutions of magnetic atoms in nonmagnetic crystals. We shall assume that the appearance of one magnetic structure or another in the substances under consideration is due principally to exchange forces considerably greater than the relativistic interactions.

A spatially disordered system can be completely ordered magnetically. The only such case is complete ferromagnetic ordering of the spins of the magnetic atoms. Any other magnetic order is obviously incompatible with spatial disorder. The macroscopic properties of such a ferromagnet do not differ from the properties of ordinary crystalline ferromagnets and are described by the Landau-Lifshitz equation.[1]

In recent times, disordered systems with magnetic structures of a different type have been widely investigated (see the reviews[2,3]). These, firstly, are the so-called spin glasses (see Ref. 2), in which not only the positions but also the directions of the spins of the different atoms are randomly distributed. In addition, there exist systems (see Ref. 3) possessing a finite spontaneous magnetization whose value at zero temperature, however, differs substantially from the nominal value. The state of such a disordered ferromagnet is analogous to the state of a spin glass in an external magnetic field. There is partial ferromagnetic order superimposed on a fairly random (in general) distribution of spin orientations.

Below, on the basis of the symmetry arguments developed in the work of Marchenko and the author,[4] all the theoretically possible macroscopically different types of such partial order will be found. It turns out that, apart from the disordered ferromagnet, there exists only one other possible structure—a disordered antiferromagnet characterized by three mutually perpendicular antiferromagnetic moments.

The dynamical properties of disordered magnetic media can be described macroscopically in a manner analogous to the way in which ordinary amorphous solids are described by the theory of elasticity. In this case the analog of the spatial-displacement vector that appears in elasticity theory is a rotation of all the spins through the same angle. The exchange energy does not change under such a rotation. The change of energy is determined, therefore, by the time and space derivatives of the rotation angles, which are analogous to the velocity of the medium and the deformation. The description of the spin dynamics by means of rotation angles has been used in the study of the magnetic properties of the superfluid phases of He^3 [5,6] and of crystalline[7] and disordered[8] magnets. There is, however, an essential difference between elasticity theory and magnetic dynamics. Unlike spatial displacements, different rotations do not, generally speaking, commute. Therefore, the equations of magnetic dynamics are nonlinear even for small velocities and deformations. Below we shall derive the dynamical equations for all three types of disordered magnetic structures, i.e., for spin glasses and disordered ferromagnets and antiferromagnets. For our purposes it will be convenient to use the Lagrangian formalism applied by Maki[6] to investigate the magnetic properties of liquid He^3-*B* and by Dzyaloshinskiĭ and Kukharenko[9] for crystalline magnets. The spin-glass case is the simplest. The point is that, by the virtue of the complete randomness of the corresponding structure, the spin glass does not have any macroscopic characteristics that change under "deformation." Therefore, the form of the equations is determined by the geometry (see Ref. 10) of the rotation group. These equations, if we linearize them and neglect the magnetic field and relativisitic interactions, become equivalent to the equations of Halperin and Saslow.[6] The equations for the disordered antiferromagnet are highly analogous to the equations[6] describing the spin dynamics of He^3-B.

1. SYMMETRY OF DISORDERED MAGNETS

The symmetry of any macroscopically homogeneous magnetic medium with exchange forces is determined by specifying the exchange class (see Ref. 4), i.e., the point group of transformations consisting of all the combinations of spatial rotations and reflections, rotations in spin space, and time reversal under which the medium under consideration is invariant. If in the exchange class we formally identify all the spin rotations and time reversal with a single transformation, we obtain a certain point group G defining the purely spatial symmetry of the medium. This is the symmetry group of all the spin scalars characterizing the given medium. In the case of crystalline magnets the group G is one of the familiar 32 crystal classes. In the case we are considering, namely, that of macroscopically isotropic disordered media, the group G is the direct product $\mathrm{SO}(3)\times \mathrm{P}$ of the group of three-dimensional spatial rotations and the spatial inversion P.

The macroscopic magnetic properties of a medium are uniquely determined by specifying the multipole moments $\mathbf{M}_{lm}=\{M^{\alpha}_{lm}\}$ (see Ref. 4), which transform as a vector (α is the spin-vector index) under spin rotations and according to one of the irreducible representations (l and m are, respectively, the indices specifying the irreducible representation and the label of the function transforming according to the given representation) under the action of transformations of the group G. In the case of crystals G is a finite group and the multipole moments have meaning for every irreducible representation. The point is that, as is clear from Ref. 4, in the general case multipole moments correspond to each irreducible representation that can be realized by scalar functions of the coordinates. For finite groups every representation can be realized in such a way. But for the continuous group $\mathrm{SO}(3)\times \mathrm{P}$ scalar functions of the coordinates realize only those representations for which the parity is equal to $(-1)^l$ (in the present case, l and m must be understood to be the orbital angular momentum and its projection; $l=0, 1, \ldots$; $m=-l, \ldots, l$).

It was shown in Ref. 4 that in the equilibrium state not more than three moments can be nonzero and that the different moments are perpendicular to each other, those belonging to the same multi-dimensional representation having the same length. It is clear that for the group $\mathrm{SO}(3)\times \mathrm{P}$ there are just three possibilities: 1) all the moments are equal to zero, 2) just one moment $\mathbf{M}$, corresponding to the identical representation $l=0$, is nonzero, and 3) three moments $\mathbf{M}_i$ (in place of $m=-1, 0, 1$ for $l=1$ it is convenient to introduce the spatial vector index $i=x, y, z$), equal in magnitude and perpendicular to each other, are nonzero. As applied to the magnetically disordered media discussed above, the first possibility corresponds to the spin glass, the second to the disordered ferromagnet, and the third to the disordered antiferromagnet.

We shall discuss the last possibility in more detail. Inasmuch as the spin is a pseudovector and changes sign under the time-reversal operation T, a disordered antiferromagnet with moments M^{α}_i is not invariant under P and T separately but is invariant under the product PT. If such a structure is realized in a dielectric, the magnetoelectric effect (the appearance of magnetization proportional to the external electric field) should occur, and is observed, e.g., in the crystalline antiferromagnet Cr_2O_3.[11,12]

When relativistic effects are taken into account a particular orientation of the triad of mutually perpendicular spin vectors $\mathbf{M}_i$ with respect to the coordinate space should be realized. Of all the orientations, that which, in the appropriate normalization, is characterized by moments $M^{\alpha}_i=\delta^{\alpha}_i$ is special. A medium with such moments is obviously isotropic even when the relativistic interactions are taken into account. Its point symmetry group (magnetic class) is the group $\mathrm{SO}(3)\times \mathbf{PT}$. Any other orientation can be obtained from the one discussed above by a rotation of all the spins through a certain angle $\theta (0<\theta<\pi)$ about an axis defined by a unit vector n. As a result the moments acquire the following values:

$$M_i^{\alpha}=\cos\theta\,\delta_i^{\alpha}+(1-\cos\theta)\,n_i n_{\alpha}+\sin\theta\, e_{i\alpha\beta}n_{\beta},$$

where e_{ikl} is the antisymmetric unit tensor.

The anisotropy energy U_{an}, which depends on the angle θ and is defined by averaging the Hamiltonian of the relativistic interactions, is, in the present case, a linear combination of the two invariants $M^{\alpha}_i M^{i}_{\alpha}$ and $(M^{\alpha}_{\alpha})^2$. Substituting the expression for the moments in terms of θ, we obtain

$$U_{\mathrm{an}}=(A\varphi^2+B\varphi^4)/(1+\varphi^2)^2, \tag{1}$$

where $\varphi=\tan(\theta/2)$, and A and B are the anisotropy constants, proportional to the square of the ratio of the electron velocity to the velocity of light. If $A>0$, the energy is a minimum at $\varphi=0 (\theta=0)$ or $\varphi=\infty$ $(\theta=\pi)$, depending the sign of B. If $A<0$ and $B>A/2$, the minimum is reached at the finite value $\varphi=[|A|/(|A|+2B)]^{1/2}$, while in the case $A<0$ and $B<A/2$ it is again found at $\varphi=\infty$.

Thus, three types of states of a disordered antiferromagnet are possible. Besides the isotropic structure ($\varphi=0$) discussed above, structures with an exact spatial symmetry, of the type of that of nematic liquid crystals, are also possible. The unit rotation vector $\mathbf{n}$ plays the role of the director, the states $\mathbf{n}$ and $-\mathbf{n}$ being different for finite φ and identical for $\varphi=\infty$. The nonisotropic antiferromagnetic phases are highly analogous to the spin system of liquid He^3-B. The principal qualitative difference is that in our case the moments M_i^α describe long-range order arising against a background of a system of randomly oriented spins.

2. THE SPIN GLASS

We shall consider the equilibrium state of a spin glass. We carry out an arbitrary spin rotation, the same for all spatial points. If we neglect the relativistic interactions, then, in the absence of an external magnetic field, we obtain as a result another equilibrium state with energy equal to the energy of the original state. Let the parameters $\varphi^\alpha(\alpha=1, 2, 3)$ realize a certain parametrization of the group of spin rotations. The state obtained by means of any rotation φ^α, where φ^α does not depend on the coordinates, is an equilibrium state. Now let the parameters $\varphi^\alpha(x_i)$ be slowly varying functions of the coordinates. In this case the state of the system at the point x_i can be obtained from the initial state at the same point by the rotation $\varphi^\alpha(x_i)$. The energy of the state $\varphi^\alpha(x_i)$ exceeds that of the equilibrium state by a quantity U that is quadratic in the spatial derivatives of $\varphi^\alpha(x_i)$. To calculate U we shall consider the rotations $\varphi^\alpha(x_i)$ and $\varphi^\alpha(x_i+dx_i)=\varphi^\alpha(x_i)+d\varphi^\alpha$ corresponding to two close points. Let

$$\delta\theta=2\lambda_\alpha(\varphi)\,d\varphi^\alpha$$

(the factor 2 is introduced for convenience in the following) be the infinitesimal rotation angle defining the rotation such that successive application of the rotations φ^α and $\delta\theta$ gives $\varphi^\alpha+d\varphi^\alpha$. It is clear that in a spin glass the only quantity on which U can depend is

$$\left(\frac{\delta\theta}{dx_i}\right)^2=4\lambda_\alpha(\varphi)\lambda_\beta(\varphi)\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_i}.$$

By definition, the quantities $\lambda_\alpha(\varphi)\cdot\lambda_\beta(\varphi)$ are the metric tensor $g_{\alpha\beta}(\varphi)$ specifying (see Ref. 10) the metric of the Riemannian space corresponding to the group of three-dimensional rotations. We have

$$U=\int dV\frac{a}{2}g_{\alpha\beta}(\varphi)\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x},\qquad(2)$$

where $a>0$ is a certain constant.

The state $\varphi^\alpha(x_i)$ is not an equilibrium state. Together with the small spatial derivatives $\partial\varphi^\alpha/\partial x_i$ small time derivatives of φ^α appear. The kinetic energy K of the system is quadratic in $\dot\varphi^\alpha$. In analogy with (2), we obtain

$$K=\int dV\frac{a}{2c^2}g_{\alpha\beta}(\varphi)\dot\varphi^\alpha\dot\varphi^\beta,\qquad(3)$$

where c is a constant which, as will be seen below, is equal to the spin-wave velocity.

The Lagrangian of the spin glass, determining its macroscopic spin dynamics in the absence of an external magnetic field and relativistic interactions, is thus equal to

$$L=K-U=\int dV\frac{a}{2}g_{\alpha\beta}(\varphi)\left(\frac{1}{c^2}\dot\varphi^\alpha\dot\varphi^\beta-\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_i}\right).\qquad(4)$$

The corresponding Lagrange equations have the form

$$\ddot\varphi^\alpha-c^2\Delta\varphi^\alpha+\Gamma_{\beta\gamma}{}^\alpha\left(\dot\varphi^\beta\dot\varphi^\gamma-c^2\frac{\partial\varphi^\beta}{\partial x_i}\frac{\partial\varphi^\gamma}{\partial x_i}\right)=0.\qquad(5)$$

Here $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols corresponding to the metric $g_{\alpha\beta}$.

We now introduce a concrete parametrization of $\{\varphi^\alpha\}=\boldsymbol{\phi}$ that is convenient for the following. We put $\boldsymbol{\phi}=\mathbf{n}\varphi$, where $\mathbf{n}$ is the unit vector along the rotation axis and $\varphi=\tan(\theta/2)$; θ is the angle of rotation about $\mathbf{n}(0<\theta<\pi)$. Any spin vector $\boldsymbol{\sigma}$ is transformed under the action of a spin rotation $\boldsymbol{\phi}$ in accordance with the formula

$$\boldsymbol{\sigma}'=\boldsymbol{\sigma}+\frac{2}{1+\varphi^2}\{[\boldsymbol{\varphi}[\boldsymbol{\varphi}\times\boldsymbol{\sigma}]]+[\boldsymbol{\varphi}\times\boldsymbol{\sigma}]\}.\qquad(6)$$

The advantages of this parametrization are displayed in the simple law determining the product of two rotations. If we perform first the rotation $\boldsymbol{\phi}_1$ and then $\boldsymbol{\phi}_2$, as a result we obtain a rotation corresponding to the parameter

$$\boldsymbol{\varphi}_{21}=\frac{\boldsymbol{\varphi}_2+\boldsymbol{\varphi}_1+[\boldsymbol{\varphi}_2\times\boldsymbol{\varphi}_1]}{1-\boldsymbol{\varphi}_2\boldsymbol{\varphi}_1}.\qquad(7)$$

The "length" element dl of the group space is easily determined with the aid of formula (7). We have

$$dl^2\equiv g_{\alpha\beta}d\varphi^\alpha d\varphi^\beta=(\delta\theta/2)^2=(\delta\boldsymbol{\omega})^2,$$

where $\delta\omega$ is obtained from (7) by the substitution $\boldsymbol{\phi}_1=-\boldsymbol{\phi}$, $\boldsymbol{\phi}_2=\boldsymbol{\phi}+d\boldsymbol{\phi}$:

$$\delta\boldsymbol{\omega}=\frac{d\boldsymbol{\varphi}+[\boldsymbol{\varphi}\times d\boldsymbol{\varphi}]}{1+\varphi^2}.\qquad(8)$$

As a result we find

$$g_{\alpha\beta}=\frac{\delta_{\alpha\beta}}{1+\varphi^2}-\frac{\varphi^\alpha\varphi^\beta}{(1+\varphi^2)^2}.\qquad(9)$$

In the chosen parametrization the equations of motion (5) acquire the following form:

$$\ddot{\boldsymbol{\varphi}}-c^2\Delta\boldsymbol{\varphi}=\frac{\dot{\boldsymbol{\varphi}}(\varphi^2)^{\boldsymbol{\cdot}}-c^2\boldsymbol{\varphi}_i(\varphi^2)_i}{1+\varphi^2},\qquad(10)$$

where

$$(\varphi^2)^{\boldsymbol{\cdot}}=\frac{\partial\varphi^2}{\partial t},\quad \boldsymbol{\varphi}_i=\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\quad (\varphi^2)_i=\frac{\partial\varphi^2}{\partial x_i}.$$

The metric (9) is the metric of a sphere in four-dimensional Euclidian space. Indeed, writing dl^2 in the variables $\theta/2, \vartheta$ and Φ, where ϑ and Φ are the polar angle and azimuth of the direction $\mathbf{n}$, we obtain

$$dl^2=(d\theta/2)^2+\sin^2(\theta/2)(d\vartheta^2+\sin^2\vartheta\, d\Phi^2),$$

which coincides (see Ref. 13) with the metric of a three-dimensional sphere of unit radius. It is clear, therefore, that the equations of motion are invariant under the six-parameter group O(4) of four-dimensional rotations. Infinitesimal transformations of the coordinates ϕ, conserving the metric, are defined by the formulas

$$\delta\varphi = \frac{\varphi + \epsilon_1 - [\varphi \times \epsilon_1]}{1 - \varphi\epsilon_1} - \varphi \approx \epsilon_1 + \varphi(\varphi\epsilon_1) + [\epsilon_1 \times \varphi], \tag{11}$$

$$\delta\varphi = [\epsilon_2 \times \varphi].$$

The first of these transformations is an infinitesimal spin rotation through the angle $\delta\theta = 2\epsilon_1$.

The variation of the action W considered as a function of the coordinates ϕ[14,9] for arbitrary $\delta\phi$ is equal to

$$\delta W = \frac{a}{c^2} \int dV\, g_{\alpha\beta} \dot{\varphi}^\beta \delta\varphi^\alpha.$$

Substituting the formulas (11) into this we find the quantities conserved by virtue of the 0(4) symmetry:

$$\mathbf{S} = \frac{\delta W}{2\varepsilon_1} = \frac{a}{2c^2} \int dV \frac{\varphi + [\varphi \times \dot{\varphi}]}{1 + \varphi^2},$$

$$\mathbf{N} = \frac{\delta W}{\varepsilon_2} = \frac{a}{c^2} \int dV \frac{[\varphi \times \dot{\varphi}]}{1 + \varphi^2}. \qquad (12$$

The quantity **S** is the mechanical spin moment, and this immediately permits us to write down the expression for the magnetic moment **M** per unit volume in the absence of an external magnetic field:

$$\mathbf{M} = \frac{a\gamma}{2c^2} \frac{\dot{\varphi} + [\varphi \times \dot{\varphi}]}{1 + \varphi^2} +$$

here, γ is the gyromagnetic ratio.

If there is an external magnetic field the additional term $\mathbf{M} \cdot \mathbf{H}$ appears in the expression for the Lagrangian. Calculating **S** and $\mathbf{M} = \gamma\mathbf{S}$ for the new Lagrangian in the same way as above, we obtain

$$\mathbf{M} = \frac{a\gamma}{2c^2} \frac{\dot{\varphi} + [\varphi \times \dot{\varphi}]}{1 + \varphi^2} + \frac{a\gamma^2}{4c^2} \mathbf{H}.$$

From this it follows that the equality $a\gamma^2/4c^2 = \chi$ should be fulfilled, where χ is the magnetic susceptibility.

It remains for us to write in the Lagrangian the additional terms describing the relativistic anisotropy energy U_{an}. The quantity U_{an} does not contain derivatives of the rotation angles and should be expressed in terms of the orthogonal spin-rotation matrix $O_{\alpha\beta}$, which, according to (6), is equal to

$$O_{\alpha\beta} = \delta_{\alpha\beta} + \frac{2}{1 + \varphi^2} \{\varphi^\alpha \varphi^\beta - \varphi^2 \delta_{\alpha\beta} + e_{\alpha\gamma\beta} \varphi^\gamma\}.$$

Inasmuch as U_{an} is obtained from the Hamiltonian of the spin-orbit and spin-spin interactions, which contains terms linear and quadratic in the spins, U_{an} can contain only combinations of $O_{\alpha\beta}$ that are linear or quadratic in $O_{\alpha\beta}$ and, in addition, are invariant under rotations. In all there are two such combinations:

$$O_{\alpha\alpha} = 3 - \frac{4\varphi^2}{1 + \varphi^2}, \quad O_{\alpha\beta} O_{\beta\alpha} = 3 - \frac{16\varphi^2}{(1 + \varphi^2)^2}.$$

It is clear that the anisotropy energy of the spin glass can be written in the form

$$U_{\mathrm{an}} = \frac{\alpha\varphi^2 + \beta\varphi^4}{(1 + \varphi^2)^2}, \tag{13}$$

where α and β are certain positive anisotropy constants. It is essential to note that these constants are proportional to the fourth power of the ratio of the electron velocity to the velocity of light, i.e., they are considerably smaller than the anisotropy constants A and B of the disordered antiferromagnet, appearing in formula (1). In fact, to calculate the anisotropy energy we should replace the electron spin operators σ_1^α, σ_2^β, ... by $O_{\alpha\gamma}\sigma_1^\gamma$, $O_{\beta\delta}\sigma_2^\delta$, ... in the Hamiltonian of the relativistic interactions and average over the equilibrium state. If we neglect the relativistic interactions, then

$$\langle\sigma_1^\gamma\rangle = \langle\sigma_2^\delta\rangle = 0, \quad \langle\sigma_1^\gamma \sigma_2^\delta\rangle \propto \delta_{\gamma\delta},$$

and, by virtue of the orthogonality of the rotation matrix, $O_{\alpha\gamma}O_{\beta\gamma} = \delta_{\alpha\beta}$. In order to obtain the dependence of the energy on the angle of spin rotation it is necessary to take the relativistic interactions into account once again. Thus, the constants α and β are proportional to the square of the relativistic interactions.

We now write down the complete expression for the Lagrangian of a spin glass with allowance for the external magnetic field and relativistic interactions:

$$L = \frac{2\chi}{\gamma^2} \left(\frac{\dot{\varphi}^2 - c^2 \varphi_i^2}{1 + \varphi^2} - \frac{(\varphi\dot{\varphi})^2 - c^2 (\varphi\varphi_i)^2}{(1 + \varphi^2)^2} \right) + \frac{2\chi}{\gamma} \dot{\varphi} \frac{\mathbf{H} + [\mathbf{H} \times \varphi]}{1 + \varphi^2} - \frac{\alpha\varphi^2 + \beta\varphi^4}{(1 + \varphi^2)^2}. \tag{14}$$

The linearized equations of motion corresponding to this Lagrangian have the form

$$\ddot{\varphi} - c^2 \Delta\varphi + \gamma[\mathbf{H} \times \dot{\varphi}] + \frac{\alpha\gamma^2}{2\chi} \varphi = 0,$$

whence we easily find the frequencies of the longitudinally polarized (with respect to the magnetic field) spin waves:

$$\omega_l^2 = \alpha\gamma^2/2\chi + c^2 k^2 \tag{15}$$

and of the transverse spin waves:

$$\omega_t^2 = \omega_l^2 + (\gamma H)^2/2 \pm (\gamma^4 H^4/4 + \gamma^2 H^2 \omega_l^2)^{1/2}. \tag{16}$$

For $H = \alpha = 0$ the formulas obtained go over into the results of the work of Halperin and Saslow.[8]

3. THE DISORDERED FERROMAGNET

A disordered ferromagnet in the equilibrium state is characterized by a constant spontaneous-magnetization vector $\mathbf{M}_0$. In the given case, however, unlike that of ordinary ferromagnets, it is convenient to write the equations of motion not for **M** but for the rotation angles $\phi(x_i, t)$. In the state characterized by the functions $\phi(x_i, t)$ the spontaneous moment $\mathbf{M}(x_i, t)$ is, by virtue of (6), equal to

$$\mathbf{M} = \mathbf{M}_0 + \frac{2}{1 + \varphi^2} \{\varphi(\varphi\mathbf{M}_0) - \varphi^2 \mathbf{M}_0 + [\varphi \times \mathbf{M}_0]\}. \tag{17}$$

To avoid an excess of accuracy, we begin by elucidating the relationship between the exact magnetic moment per unit volume, which, in this section, we shall denote by

$\mathfrak{M}$, and the spontaneous moment **M**. If we neglect the derivatives of $\boldsymbol{\phi}$, then $\mathfrak{M} = \mathbf{M}$. In the next approximation terms containing derivatives appear in the right-hand side of the latter equality. Inasmuch as **M** depends on the rotation angles $\boldsymbol{\phi}$ themselves and not on their derivative, it is clear that by a slight renormalization of the angles $\boldsymbol{\phi}$ it is always possible to make the directions of the moments $\mathfrak{M}$ and **M** coincide. Thus, by definition of the direction of the vector **M**, in the nonequilibrium case we have $\mathfrak{M} = \mathrm{const}\cdot\mathbf{M}$. In equilibrium the constant is equal to unity. For the constant in the nonequilibrium case an expansion in the small quantities

$$\dot{\omega}=\delta\omega/dt, \quad \omega_i=\delta\omega/dx_i$$

should be written ($\delta\omega$ is defined by formula (8)). In the absence of an external magnetic field, if we take into account the symmetry under time reversal, this expansion can be written in the form

$$\mathfrak{M}=\mathbf{M}+b\mathbf{M}(\mathbf{M}\dot{\omega}),$$

where b is a constant.

For this moment $\mathfrak{M}$ to be obtained, the Lagrangian in the absence of a magnetic field should be equal to

$$\frac{b}{\gamma}(\mathbf{M}\dot{\omega})^2+\frac{2}{\gamma}\mathbf{M}\dot{\omega}$$

plus terms not containing time derivatives. There are two such terms: ω_i^2 and $(M\omega_i)^2$, and, in addition, the anisotropy energy. The latter, as in the spin glass, is determined by formula (13), since the components of the vector **M** we can construct just one invariant M^2, which does not change under spin rotations. Taking the magnetic field into account as above by adding the term $\mathfrak{M}\cdot\mathbf{H}$ to the Lagrangian, we obtain

$$L=\frac{b}{\gamma}(\mathbf{M}\dot{\omega})^2+\frac{2}{\gamma}\mathbf{M}\omega-\frac{2aM}{\gamma}\left\{\omega_i^2-\frac{1}{M^2}(\mathbf{M}\omega_i)^2\right\}- \\ -\frac{2\chi_{\parallel}c^2}{\gamma^2M^2}(\mathbf{M}\omega_i)^2+\mathbf{MH}+b(\mathbf{MH})(\mathbf{M}\dot{\omega})-U_{\mathrm{an}}\cdot \tag{18}$$

where a and c are positive constants and $\chi_{\parallel}$ is the susceptibility in the direction of **M**. From (18) we easily find the magnetic moment with allowance for the magnetic field:

$$\mathfrak{M}=\mathbf{M}+b(\mathbf{M}\dot{\omega})\mathbf{M}+{}^1/{}_2b\gamma\mathbf{M}(\mathbf{MH}),$$

whence it can be seen that $b = 2\chi_{\parallel}/\gamma M^2$.

Taking (17) and (8) into account, we can rewrite the Lagrangian (18) in the following, final form:

$$L=\frac{2\chi_{\parallel}}{\gamma^2M^2}\left(\mathbf{M}_0\frac{\dot{\varphi}+[\dot{\varphi}\times\varphi]}{1+\varphi^2}\right)^2+\frac{2}{\gamma}\mathbf{M}_0\frac{\dot{\varphi}+[\dot{\varphi}\times\varphi]}{1+\varphi^2} \\ -\frac{2aM}{\gamma}\left(\frac{\varphi_i+[\varphi\times\varphi_i]}{1+\varphi^2}\right)^2-2\frac{\chi_{\parallel}c^2-a\gamma M}{\gamma^2}\left(\frac{\mathbf{M}_0}{M}\frac{\varphi_i+[\varphi_i\times\varphi]}{1+\varphi^2}\right)^2 \\ +\left(\mathbf{M}_0\mathbf{H}+\frac{2}{1+\varphi^2}((\varphi\mathbf{M}_0)(\varphi\mathbf{H})-\varphi^2(\mathbf{M}_0\mathbf{H})+\varphi[\mathbf{M}_0\times\mathbf{H}])\right). \\ \times\left(1+\frac{2\chi_{\parallel}}{\gamma M^2}\mathbf{M}_0\frac{\dot{\varphi}+[\dot{\varphi}\times\varphi]}{1+\varphi^2}\right)-\frac{\alpha\varphi^2+\beta\varphi^4}{(1+\varphi^2)^2}. \tag{19}$$

We write the linearized (about the equilibrium state $\mathbf{M}_0 \parallel \mathbf{H}$) equations of motion obtained from (19):

$$M\gamma[\mathbf{v}\dot{\varphi}]-\chi_{\parallel}(\mathbf{v}\ddot{\varphi})\mathbf{v}+a\gamma M\Delta\varphi+(\chi_{\parallel}c^2-a\gamma M)(\mathbf{v}\Delta\varphi)\mathbf{v} \\ -\gamma^2MH(\varphi-(\varphi\mathbf{v})\mathbf{v})-{}^1/{}_2\alpha\gamma^2\varphi=0; \tag{20}$$

here, $\nu = \mathbf{M}_0/M$ and we have neglected the term $\chi_{\parallel}\gamma H[\nu\times\boldsymbol{\phi}]$, which is small compared with the first term.

From (20) we obtain the spectra of the longitudinal spin waves:

$$\omega_l^2=c^2k^2+\alpha\gamma^2/2\chi_{\parallel}$$

and of the transverse spin waves:

$$\omega_t=ak^2+\gamma H+\alpha\gamma/2M,$$

Which, for $H = \alpha = 0$, coincide with the formulas given in Ref. 8.

4. THE DISORDERED ANTIFERROMAGNET

For a disordered antiferromagnet it is convenient to measure the angles φ^{α} from the state in which the moments are equal to $M_i^{\alpha} = \delta_i^{\alpha}$. In the state characterized by the functions $\varphi^{\alpha}(x_i, t)$, according to (6) the moments acquire the following values:

$$M_i^{\alpha}(x_i,t)=\delta_i^{\alpha}+\frac{2}{1+\varphi^2}\{\varphi^{\alpha}\varphi^{i}-\delta_i^{\alpha}\varphi^2+e_{i\alpha\beta}\varphi^{\beta}\}. \tag{21}$$

They satisfy the orthogonality and completeness relations

$$M_i^{\alpha}M_k^{\alpha}=\delta_{ik}, \quad M_i^{\alpha}M_i^{\beta}=\delta_{\alpha\beta}. \tag{22}$$

In the expression for the Lagrangian of a disordered antiferromagnet the spatial derivatives can occur in the following combinations:

$$\omega_i^2, \quad (\omega_i\mathbf{M}_i)^2, \quad (\omega_i\mathbf{M}_k)(\omega_i\mathbf{M}_k), \quad (\omega_i\mathbf{M}_k)(\omega_k\mathbf{M}_i).$$

By virtue of (22), the third of these reduces to the first, which appeared in the spin-glass Lagrangian. By virtue of the relations

$$\frac{\partial\mathbf{M}_i}{\partial x_k}\frac{\partial\mathbf{M}_k}{\partial x_i}=4[\omega_k\times\mathbf{M}_i][\omega_i\times\mathbf{M}_k]=4\omega_i^2-4(\omega_i\mathbf{M}_i)^2,$$

$$\left(\frac{\partial\mathbf{M}_i}{\partial x_i}\right)^2=4\omega_i^2-4(\omega_k\mathbf{M}_i)(\mathbf{M}_k\omega_i)$$

we can use in place of the second and fourth combinations the expressions

$$\frac{\partial\mathbf{M}_i}{\partial x_k}\frac{\partial\mathbf{M}_k}{\partial x_i}, \quad \left(\frac{\partial\mathbf{M}_i}{\partial x_i}\right)^2.$$

Since these expressions differ from those in the case of the spin glass by an exact divergence

$$\frac{\partial}{\partial x_i}\left(\mathbf{M}_i\frac{\partial\mathbf{M}_k}{\partial x_k}-\mathbf{M}_k\frac{\partial\mathbf{M}_i}{\partial x_k}\right),$$

in the present case it is necessary to take into account only one combination, for which we choose

$$(\omega_i\mathbf{M}_i)^2=\left(\frac{\operatorname{div}\varphi+\varphi\operatorname{rot}\varphi}{1+\varphi^2}\right)^2.$$

In addition, in the antiferromagnet there appears an invariant that simultaneously contains space and time derivatives of the rotation angles:

$$\mathbf{M}_i[\dot{\boldsymbol{\omega}}\times\boldsymbol{\omega}_i]=\dot{\boldsymbol{\omega}}[\boldsymbol{\omega}_i\times\mathbf{M}_i]=\frac{1}{2}\dot{\boldsymbol{\omega}}\frac{\partial\mathbf{M}_i}{\partial x_i}.$$

This, however, need not be taken into account, since it is equal to the sum of an exact time derivative and an exact divergence:

$$\omega\frac{\partial M_i}{\partial x_i}=\operatorname{div}\left(\frac{\dot{\varphi}+[\dot{\varphi}\times\varphi]}{1+\varphi^2}\right)-\frac{\partial}{\partial t}\left(\frac{\operatorname{div}\varphi+\varphi\operatorname{rot}\varphi}{1+\varphi^2}\right).$$

All the reasoning is otherwise identical to the previous arguments. As a result we obtain the following Lagrangian:

$$L=\frac{2\chi}{\gamma^2}\left\{\frac{\dot{\varphi}^2-c_t^2\varphi_i^2}{1+\varphi^2}-\frac{(\varphi\dot{\varphi})^2-c_t^2(\varphi\varphi_i)^2}{(1+\varphi^2)^2}\right\}+\frac{2\chi}{\gamma}\dot{\varphi}\frac{\mathbf{H}+[\mathbf{H}\times\varphi]}{1+\varphi^2}$$
$$-\frac{2\chi}{\gamma^2}(c_l^2-c_t^2)\left(\frac{\operatorname{div}\varphi+\varphi\operatorname{rot}\varphi}{1+\varphi^2}\right)^2-\frac{A\varphi^2+B\varphi^4}{(1+\varphi^2)^2}. \qquad (23)$$

Here we have taken into account formula (1) for the anisotropy energy, the constants χ and γ have their usual meaning, and c_l and c_t are, as we shall see, the velocities of the longitudinal and transverse (with respect to the wave vector) spin waves in the isotropic phase. The magnetic moment **M** per unit volume of the antiferromagnet is determined by the formula

$$\mathbf{M}=\frac{2\chi}{\gamma}\frac{\dot{\varphi}+[\varphi\times\dot{\varphi}]}{1+\varphi^2}+\chi\mathbf{H}.$$

The formula (23) differs from the Lagrangian, found by Maki,[6] describing the spin dynamics of liquid He^3-*B* only in the fact that for He^3 a certain less-general set of constants occurs. In Ref. 6 the rotations were parametized by means of Euler angles, which greatly complicates the formulas.

The Lagrangian (23) describes all the antiferromagnetic phases. The only difference between them is that the equilibrium state of the isotropic phase corresponds to $\phi=0$, while for the other phases the equilibrium value of the angles can be written in the form $\phi=\mathbf{n}\varphi_0$, where **n** is the director. We write the linearized equations of motion of the isotropic phase ($\varphi\to 0$) that correspond to the Lagrangian (23):

$$\ddot{\varphi}-c_t^2\Delta\varphi-(c_l^2-c_t^2)\nabla\operatorname{div}\varphi+\gamma[\mathbf{H}\times\dot{\varphi}]+A\gamma^2\varphi/2\chi=0.$$

In the general case a cubic equation for the spin-wave frequencies is obtained. We have

$$\omega^2=A\gamma^2/2\chi+c_t^2k^2+\varkappa^2,$$

where κ satisfies the equation

$$\varkappa^6-\varkappa^4[(c_l^2-c_t^2)k^2+\gamma^2H^2]+\gamma^2H^2\varkappa^2[(c_l^2-c_t^2)k^2\cos^2\psi$$
$$-A\gamma^2/2\chi-c_t^2k^2]+\gamma^2H^2(A\gamma^2/2\chi+c_t^2k^2)(c_l^2-c_t^2)k^2\cos^2\psi=0.$$

Here ψ is the angle between the directions of the magnetic field and the wave vector **k**.

For $\mathbf{H}=0$ the frequencies of the longitudinal and transverse spin waves are equal to

$$\omega_l^2=A\gamma^2/2\chi+c_l^2k^2,\quad \omega_t^2=A\gamma^2/2\chi+c_t^2k^2.$$

For $\mathbf{k}=0$ there are three different magnetic-resonance frequencies:

$$\omega_1^2=A\gamma^2/2\chi,\quad \omega_{2,3}^2=\omega_1^2+\gamma^2H^2/2\pm[\gamma^4H^4/4+\gamma^2H^2\omega_1^2]^{1/2},$$

which coincide with the spin-glass magnetic-resonance frequencies obtained from (15) and (16).

The equations of motion linearized about the equilibrium position $\phi=\phi_0=\phi_0\mathbf{n}$ have the following form:

$$(1+\varphi_0^2)(c_t^2\Delta\varphi-\ddot{\varphi})-\varphi_0^2\{c_t^2\mathbf{n}(\mathbf{n}\Delta\varphi)-\mathbf{n}(\mathbf{n}\ddot{\varphi})\}-\gamma(1+\varphi_0^2)[\mathbf{H}\times\dot{\varphi}]$$
$$-\gamma\varphi_0(\mathbf{H}+[\mathbf{H}\times\varphi_0],\ \dot{\varphi})+\gamma(\mathbf{H}+[\mathbf{H}\times\varphi_0])(\varphi_0\dot{\varphi})+ \qquad (24)$$
$$+(c_l^2-c_t^2)\{\nabla(\operatorname{div}\varphi+\varphi_0\operatorname{rot}\varphi)-\operatorname{rot}\varphi_0(\operatorname{div}\varphi+\varphi_0\operatorname{rot}\varphi)\}-$$
$$-(\mathcal{A}\gamma^2/2\chi)\mathbf{n}(\mathbf{n}\varphi)=0.$$

Here the symbol ϕ denotes the deviation from equilibrium and we have introduced the notation

$$\mathcal{A}=\tfrac{1}{2}(1+\varphi_0^2)^2\partial^2U_{\text{an}}/\partial\varphi_0^2.$$

The general expression for the spin-wave frequencies in the nonisotropic phases have a cumbersome form. We shall give the results obtained from (24) in certain particular cases.

For $\mathbf{H}=0$ and $\mathbf{k}\perp\mathbf{n}$ there are three finite frequencies:

$$\omega^2=\mathcal{A}\gamma^2/2\chi+c_l^2k^2.$$

For $\mathbf{H}=0$ and $\mathbf{k}\parallel\mathbf{n}$ a finite frequency is obtained only for the longitudinally polarized wave:

$$\omega_1^2=c_t^2k^2,\quad \omega_2^2=c_t^2k^2,\quad \omega_3^2=c_l^2k^2+\gamma^2\mathcal{A}/2\chi.$$

The other frequencies vanish in the approximation under consideration.

In the spatially uniform case ($\mathbf{k}=0$) for $\mathbf{H}\parallel\mathbf{n}$, besides the longitudinal-resonance frequency

$$\omega_l^2=\mathcal{A}\gamma^2/2\chi$$

there is one more nonzero frequency, equal to γH. Finally, for $\mathbf{k}=0$ and $\mathbf{H}\perp\mathbf{n}$ there is just one nonzero frequency:

$$\omega^2=(\gamma H)^2+\mathcal{A}\gamma^2/2\chi.$$

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ERRATA

Erratum: Nonlinear cyclotron resonance in metals [Sov. Phys. JETP 45, 100 (1977)]

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In the formula that defines $I^{\beta}_{-l}(\mathbf{k}_1,\mathbf{k}_2)$ (p. 103) the factor $(k_1 v^y - \Omega)$ should be replaced by $(k_1 v^y - l\Omega)$. The expression for $\tilde{w}^{\beta}(t)$ in (32) and (36) must therefore be multiplied by $-il$, and the expression for $\tilde{w}^{\beta}(t)$ in the same formulas must be multiplied by -1. It should also be noted that formula (40), from which the amplitude of the reflected second harmonic was calculated, is generally speaking incorrect (the author is grateful Yu. A. Romanov for pointing this out). In the case of even resonance $\omega = l\Omega_m$, as well as far from resonance, formula (40) gives a correct estimate for the amplitude of the reflected second harmonic (the factors of order unity were of no interest at all to us in this case). For odd resonance $\omega = (l+1/2)\Omega_m$, formula (40) cannot be used under the assumption made in the paper. An exact formula for the amplitude of the reflected second harmonic can be obtained by using, for example, the reprocity theorem. Calculations show that in the case of the odd resonance $\omega = (l+1/2)\Omega_m$ considered by us it is necessary to introduce in (40) an additional factor of the order of $\delta(2\omega)/\delta(\omega)$ ($\delta(\omega)$ is the depth of the skin layer at the frequency ω), which decreases somewhat the sharpness of the resonance.

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