Black hole in an external magnetic field

D. V. Gal'tsov and V. I. Petukhov

Moscow State University
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A study is made of the motion and scalar radiation of particles moving near a spherically symmetric black hole embedded in an external asymptotically homogeneous and constant magnetic field. It is shown that the magnetic field produces a potential barrier which prevents the escape of both charged and neutral particles to infinity in a plane orthogonal to the magnetic field. Exceptions are massless particles for which the projection of the angular momentum onto the direction of the magnetic field is zero. There exist stable circular orbits of charged particles corresponding to almost 100% mass defect, also stable untrarelativistic trajectories. The scalar radiation of an ultrarelativistic particle moving in a stable orbit is γ^2 times (γ is the energy to mass ratio) greater than the radiation of a particle moving in an unstable ultrarelativistic geodesic in the Schwarzschild field.

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1. INTRODUCTION

The investigation of processes that take place in the neighborhood of a black hole embedded in an external electromagnetic field is of interest in connection with many astrophysical situations. Although an electrically neutral black hole cannot by itself have "magnetic hair", accretion of plasma creates a tendency for an external magnetic field to arise.[1] A magnetic field near a black hole may be due to the presence of a magnetic satellite, for example, a pulsar. Finally, for particles with large charge to mass ratio e/μ (in all that follows, we use a system of units in which $G=\hbar=c=1$) a magnetic field that is even as weak as the intergalactic field may appreciably modify the motion in the gravitational field. The distance at which the Newtonian force acting on an electron is equal to the Lorentz force is determined by

$$r=r_g(2\lambda_c r_g^{-1} v B_e B^{-1} \sin\theta)^{1/2}, \qquad (1.1)$$

where $B_e=\mu^2/e\approx 4.4\cdot 10^{13}$ Oe, λ_c is the electron Compton wavelength, v is the velocity, θ is the angle between the direction of the velocity and the magnetic field, and r_g is the gravitational radius of the black hole. For a relativistic electron moving in the neighborhood of a black hole of mass $M\sim 10M_\odot$, the Lorentz force is of the same order as the gravitational force at a magnetic field strength of order $B\sim 10^{-5}$ Oe.

The existence near a black hole of a large-scale magnetic field of even very low strength may appreciably change the picture of accretion of charged particles into the hole. We show that a magnetic field extends right down to the horizon the region in which there exist stable circular orbits around a nonrotating black hole in the plane perpendicular to the magnetic lines of force. In particular, one can have stable orbits on which the particle has an almost 100% mass defect. The magnetic field is a stabilizer, and makes possible, for example, stable motion in circular trajectories around a nonrotating black hole with ultrarelativistic velocity. In this connection, we note that the problem of transforming the energy of relativistic particles moving in the neighborhood of a black hole into electromagnetic and gravitational radiation, which has been actively discussed in recent years,[2-5] becomes a more realistic proposition. Indeed, both the so-called geodesic synchrotron radiation in the Schwarzschild field[2] and a Kerr field[3,4] arises as a result of the motion of particles along unstable ultrarelativistic geodesics; in addition, the proximity of these trajectories to a closed null geodesic leads to an additional lowering of the radiation intensity. As we shall see below, an external magnetic field makes possible ultrarelativistic motion in stable circular orbits that are not close to a closed null geodesic, and this, in particular, increases the radiation intensity by γ^2 times compared with the geodesic synchrotron radiation, and it also extends the allowed range of initial data for realization of the effect.

A number of interesting effects arise when allowance is made for the influence of the magnetic field on the metric of spacetime. It is easy to estimate the characteristic strength of a magnetic field that affects the spacetime geometry. We shall assume that the magnetic field is constant and homogeneous (and directed along the polar axis of a spherical coordinate system). By virtue of the cylindrical symmetry, it is clear that to estimate the relative contribution of the magnetic field energy and the mass M of the black hole to the metric, it is necessary to compare the energy of the magnetic field in a cylinder of radius r and height r_g, which is equal to $\pi r^2 r_g B^2/8\pi$, with the mass M itself. We find then that a homogeneous magnetic field begins to distort the metric at a distance from the singularity of order B^{-1} (a quantity that has the dimensions of a length for $G=c=1$). From this we deduce a characteristic gravitational scale of the magnetic field strength (in oersteds) near the black hole:

$$B_M=1/M=2.4\cdot 10^{19}\, M_\odot/M, \qquad (1.2)$$

where $M_\odot$ is the solar mass. A magnetic field of order B_M significantly distorts the metric already near the horizon (although it does not change the form of the horizon nor the so-called surface gravity). However, even a field that satisfies $B\ll B_M$ but extends to a sufficiently great distance begins to influence the metric at large r.

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Ernst[6] has obtained an exact solution of the system of Einstein–Maxwell equations corresponding to a Schwarzschild black hole embedded in a constant and homogeneous magnetic field (magnetic universe). Investigation of the motion of particles in this metric shows that the magnetic field produces a potential barrier for not only charged particles but also neutral particles and prevents their escaping to infinity except for massless particles which have zero projection of their angular momentum onto the direction of the magnetic field. This, in particular, leads to the following picture of the evaporation of mini black holes in a magnetic universe. Photons and other massless particles in a state with $m=0$ (m is the azimuthal quantum number) escape to infinity; the remaining particles created near the horizon form an atmosphere around the black hole, this having the shape of a surface of revolution. The atmosphere contains charged particles, and in the equatorial plane the charges of opposite signs move in opposite directions.

The rotation of a black hole in an external magnetic field in the case when the direction of the rotation axis does not coincide with the magnetic field direction is an unstable situation since, by Hawking's well-known theorem,[7] a stationary black hole must be axisymmetric. Because of this, the rotation axis of the black hole begins to turn, and ultimately is established along the magnetic field,[8,9] which guarantees the required axial symmetry. It is convenient to express the characteristic time of this process in terms of the gravitational unit B_M:

$$\tau={}^3/_4 r_g (B_M/B)^2. \tag{1.3}$$

Another effect resulting from the rotation of a black hole in a magnetic field is a tendency for the hole to acquire electric charge,[10,11] which can happen if there is plasma in the neighborhood of the hole. The amount of charge corresponding to a stable equilibrium with the plasma is

$$Q_B=2aMB, \tag{1.4}$$

where $a=J/M$ is the ratio of the hole's angular momentum to its mass. In this connection, we mention the possibility of the following situation: a black hole forming a binary system with a magnetic satellite acquires an electric charge whose magnitude varies in general with the variation of the magnetic field strength and, therefore, with the orbital frequency of the pair. The ratio of the Coulomb and Newtonian forces exerted by the hole on an electron is

$$\frac{F_Q}{F_N}=\frac{Q_B e}{M\mu}=2\frac{a}{M}\frac{e}{\mu}\frac{B}{B_M}. \tag{1.5}$$

This ratio is of order unity (for $a/M\sim\frac{1}{2}$) for a hole of mass $10M_\odot$ if $B\sim10^{-3}$ Oe. Thus, this mechanism of charging a black hole in a magnetic field may modulate the process of accretion into the hole at the orbital frequency of the binary system.

Something interesting must happen when a Kerr mini black hole evaporates in a magnetic field. Since, as we have noted above, an atmosphere containing charged particles is formed around the evaporating hole, the conditions are created for a black hole bomb[12] due to the trapping of the radiation and its "superamplification".[13,14]

2. SCHWARZSCHILD BLACK HOLE IN A MAGNETIC UNIVERSE

As has been shown by Ernst[15] and Kinnersley[16] the system of Einstein–Maxwell equations for axisymmetric configurations in the electrovacuum are invariant under transformations of the group SU(2, 1) acting on the space of certain complex functions related to the components of the metric tensor and the electromagnetic potentials. Application of transformations in this group to the well-known solutions of the Einstein–Maxwell equations enables one to construct new solutions. One of these transformations, which is defined by a real parameter B, leads when applied to a flat empty world to Melvin's magnetic universe,[17] and the parameter B here plays the role of the magnetic field strength. As a result of this transformation, the Schwarzschild spacetime goes over into the spacetime with the interval[6]

$$ds^2=\Lambda^2\left(\frac{\Delta}{r^2}dt^2-\frac{r^2}{\Delta}dr^2-r^2d\theta^2\right)-\frac{r^2\sin^2\theta\, d\varphi^2}{\Lambda^2},$$
$$\Delta=r^2-2Mr,\quad \Lambda=1+{}^1/_4B^2r^2\sin^2\theta, \tag{2.1}$$

which coincides with the Schwarzschild spacetime as $B\to0$ and the magnetic universe as $M\to0$. From this we obtain the natural interpretation of (2.1) as a solution describing a nonrotating black hole embedded in a constant and homogeneous magnetic field. The vector potential of this magnetic field has the form

$$A_\mu=-\frac{Br^2\sin^2\theta}{2\Lambda}\delta_{\mu3}. \tag{2.2}$$

The spacetime (2.1) is not asymptotically flat, in agreement with our expectation for a homogeneous magnetic field extending to infinity. Its characteristic feature is the difference between the spatial sections $\theta=$ const and $\varphi=$ const. For example, the circumference of a circle whose center coincides with the singularity increases anomalously in the plane $\varphi=$ const for large r, whereas the length of the circumference in the plane $\theta=\pi/2$ is $2\pi r(1+B^2r^2/4)^{-1}$, and this tends to zero as $r\to\infty$.

It can be seen from (2.1) that the form of the horizon surface is not distorted by the magnetic field. As before, the horizon is nonsingular, as can be seen by calculating the invariants of the curvature tensor. In a local inertial frame, the acceleration of a particle held at rest on the horizon is

$$w=\sqrt{g^{11}}\frac{du_1}{ds}=\sqrt{g^{11}}(g_{00})^{-1}\frac{\partial g_{00}}{\partial r}=\frac{M}{r\Delta^{1/2}\Lambda(\theta)}, \tag{2.3}$$

where u_1 is the radial component of the four-velocity $u^\mu=dx^\mu/ds$. The way in which w tends to infinity as the horizon is approached differs depending on the polar angle θ, but for $\theta=0,\ \pi$ the expression (2.3) coincides with the Schwarzschild value. It follows from this, in particular, that the so-called surface gravity of a black hole is not changed by the magnetic field; for in accordance with the well-known theorem[18] the surface gravity $\varkappa$ is constant on the horizon. Since the expressions (2.3) and (2.1) coincide for $\theta=0$ with the corresponding Schwarzschild quantities, the assertion follows. This is also readily seen directly by using the following definition of $\varkappa$:

$$\varkappa u^\mu=\frac{du^\mu}{dt},\qquad \varkappa=(g_{00}g^{11})^{1/2}\frac{du_1}{ds}=g_{00}^{1/2}w=\frac{1}{4M}. \tag{2.4}$$

Let us consider the picture of magnetic lines of force corresponding to the vector potential (2.2). The nonzero Schwarzschild components of the electromagnetic field tensor are

$$F_{13}=-\frac{Br}{\Lambda^2}\sin^2\theta,\quad F_{23}=-\frac{Br^2\sin\theta\cos\theta}{\Lambda^2}. \tag{2.5}$$

We go over to a local Lorentz frame:

$$d\xi^{\hat 0}=\Lambda\frac{\Delta^{1/2}}{r}dt,\quad d\xi^{\hat 1}=\Lambda\frac{r}{\Delta^{1/2}}dr,\quad d\xi^{\hat 2}=\Lambda r\,d\theta,\quad d\xi^{\hat 3}=\frac{r\sin\theta}{\Lambda}d\varphi. \tag{2.6}$$

The tetrad components of $F_{\mu\nu}$ in this frame are

$$B_r=-F_{\hat 2\hat 3}=B\Lambda^{-2}\cos\theta,\quad B_\theta=F_{\hat 1\hat 3}=-B\frac{\Delta^{1/2}}{r}\Lambda^{-2}\sin\theta. \tag{2.7}$$

As $r\to 2M$, the value of B_θ becomes zero, and the lines of force become orthogonal to the horizon. This however does not contradict the concept of a homogeneous field. A two-dimensional spatial section of the Schwarzschild geometry can be perspicuously represented as a surface of revolution in three-dimensional space.[19] The "visible" homogeneity of the magnetic field corresponds to the projection onto a plane of the lines of force on this surface. Therefore, being orthogonal to some curve on the surface of revolution, the lines of force preserve homogeneity in the projection onto the plane.[20] The factor Λ^{-2} in (2.7), which leads to an apparent inhomogeneity of the field, also has a purely geometrical origin. Indeed, the element of a two-dimensional spherical surface in the space (2.1) has the form $r^2\Lambda^2 d\Omega$, so that the factor Λ^{-2} in (2.7) is a geometric factor guaranteeing constancy of the flux.

In the case when $B/B_M\ll 1$, there is outside the black hole a certain region $Br\ll 1$ in which the spacetime is approximately Schwarzschild. In this region, the vector potential (2.2) coincides with the solution of Maxwell's equations on a given Schwarzschild background, this solution having the nature of a homogeneous magnetic field at infinity.[20]

We construct the nonvanishing invariant of the electromagnetic field tensor (2.5):

$$I=F_{\mu\nu}F^{\mu\nu}=B^2\Lambda^{-4}\left(1-\frac{2M\sin^2\theta}{r}\right). \tag{2.8}$$

This expression has a singularity at the point $r=0$; it disappears on the transition $M\to 0$ to the magnetic universe, which does not contain a black hole. To elucidate the nature of this singularity, we go over to a tetrad which is nonsingular on the horizon, which corresponds to a certain Lorentz transformation of the tetrad (2.6):

$$\begin{aligned} d\eta^{\hat 0}&=\left(1+\frac{2M}{r}\right)^{-1/2}\left(dt-\frac{2Mr}{\Delta}dr\right)\Lambda,\\ d\eta^{\hat 1}&=\left(1+\frac{2M}{r}\right)^{-1/2}\left(-\frac{2M}{r}dt+\frac{r^2}{\Delta}dr\right)\Lambda,\\ d\eta^{\hat 2}&=d\xi^{\hat 2},\quad d\eta^{\hat 3}=d\xi^{\hat 3}. \end{aligned} \tag{2.9}$$

The corresponding tetrad projections of the electromagnetic field tensor which are nonzero have the form

$$\begin{aligned} \tilde B_r&=-\tilde F_{\hat 2\hat 3}=\Lambda^{-2}B\cos\theta,\\ \tilde B_\theta&=\tilde F_{\hat 1\hat 3}=-B[\Lambda^2(1+2M/r)^{1/2}]^{-1}\sin\theta,\\ \tilde E_\varphi&=\tilde F_{\hat 0\hat 3}=-2MB[r\Lambda^2(1+2M/r)^{1/2}]^{-1}\sin\theta. \end{aligned} \tag{2.10}$$

As a consequence of the Lorentz transformation, an electric field arises in this system. Near the singularity, $\tilde B_\theta\to 0$, $\tilde B_r\to$ const, and $\tilde E_\varphi\to\infty$. The flux of magnetic lines of force through the surface of a hemisphere near the singularity:

$$2\pi\int_0^1 d\cos\theta\, r^2\Lambda^2\tilde B_r=\pi r^2 B \tag{2.11}$$

is equal to the analogous quantity for flat spacetime. On the other hand, for $r=0$ the field invariant (2.8) can be represented in the form $I=(\tilde E_\varphi)^2$. We conclude that the singularity of the invariant (2.8) at $r=0$ is not due to a "sucking in" of the magnetic field (as can be seen from (2.11), this does not occur at all) but has a purely geometrical origin since the singularity of the electric field is related to a feature of the tetrad (2.9) as $r\to 0$.

3. MOTION OF NEUTRAL PARTICLES IN ERNST'S METRIC

To investigate geodesics in the spacetime (2.1), we use the Hamilton–Jacobi equation

$$\frac{r^2}{\Delta\Lambda^2}\left(\frac{\partial S}{\partial t}\right)^2-\frac{\Delta}{r^2\Lambda^2}\left(\frac{\partial S}{\partial r}\right)^2-\frac{1}{r^2\Lambda^2}\left(\frac{\partial S}{\partial\theta}\right)^2-\frac{\Lambda^2}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial\varphi}\right)^2=\mu^2, \tag{3.1}$$

where μ is the mass of the particle. By virtue of the axial symmetry, the solution of this equation can be represented in the form

$$S=-Et+L\varphi+S_1(r,\theta), \tag{3.2}$$

where E is the energy of the particle and L is the projection of the angular momentum; however, the variables in the equation for S_1 do not separate in the general case. Nevertheless, it follows from the symmetry of the problem that one can have purely radial motion along the polar axis that coincides with motion in the Schwarzschild field, and also planar motion for $\theta=\pi/2$, to which we turn below. For trajectories in the plane $\theta=\pi/2$, we obtain from (3.1)

$$S_1(r)=\int\frac{\Lambda r}{\Delta^{1/2}}\left[\frac{r^2E^2}{\Delta\Lambda^2}-\mu^2-\frac{(\Lambda L)^2}{r^2}\right]^{1/2}dr. \tag{3.3}$$

Using (3.2) and (3.3), we write down the first integrals of the equations of motion of the particle:

$$\mu\Lambda_0^2\frac{dt}{ds}=\frac{Er^2}{\Delta},\quad \mu\Lambda_0^2\frac{dr}{ds}=\pm(E^2-U_{eff}^2)^{1/2},\quad \mu\Lambda_0^{-2}\frac{d\varphi}{ds}=\frac{L}{r^2}, \tag{3.4}$$

$$U_{eff}^2=\frac{\Delta\Lambda_0^2}{r^2}\mu^2\left(1+\frac{L^2\Lambda^2}{\mu^2r^2}\right), \tag{3.5}$$

where $\Lambda_0=\Lambda(\theta=\pi/2)$, and in what follows we shall omit the subscript zero.

In contrast to the case of the Schwarzschild metric, the effective potential energy of the radial motion (3.5) increases unboundedly as $r\to\infty$, and this feature is also preserved for massless particles provided $L\neq 0$. It follows that the escape of massive particles to infinity in the plane $\theta=\pi/2$ is altogether impossible, while in the case $\mu=0$ it is possible only for purely radial motion ($L=0$).

For massless particles we introduce the impact parameter $\rho=L/E$ and, dividing the first two of Eqs. (3.4), we obtain

$$\frac{dr}{dt}=\frac{\Delta}{r^2}\left(1-\frac{\Delta\Lambda^4\rho^2}{r^4}\right)^{1/2}. \quad (3.6)$$

The vanishing of the radicand determines the turning points. Apart from the value of r_{t1}, which is the coordinate of the turning point that goes over into the corresponding Schwarzschild value as $B\to 0$, there is one further turning point, whose coordinate for $r\gg M$, $B\rho\ll 1$ is

$$r_{t2}=2^{4/3}/(B^{4/3}\rho^{1/3}), \quad (3.7)$$

corresponding to reflection from infinity for $\rho\neq 0$.

Differentiating the right-hand side of the expression (3.6) with respect to the parameter r, we obtain as a result of simultaneous solution of the equations

$$\frac{\partial}{\partial r}\left(\frac{dr}{dt}\right)=\frac{dr}{dt}=0,$$

a condition for determining the radii of closed circular null geodesic orbits:

$$r-3M={}^1/_4(3r-5M)B^2r^2. \quad (3.8)$$

Equation (3.8) for $B=0$ has one root $r=3M$, while for sufficiently large B there are no roots at all in the physical region $r>2M$ since the right-hand side of (3.8) is a curve that goes to infinity too rapidly. To find the "critical" magnetic field strength B_{cr} for which there is one closed null geodesic, we bear in mind that in this limiting case the curve corresponding to the right-hand side of (3.8) touches the straight line $r=3M$ at some point r_0, and, therefore, the value of the derivative of the right-hand side of (3.8) at this point is equal to the value of the derivative of the left-hand side, i.e., unity. In conjunction with Eq. (3.8), this condition enables us to find r_0:

$$r_0={}^1/_3M(8+\sqrt{19}), \quad (3.9)$$

and the corresponding value of B_{cr} is

$$B_{cr}=2\cdot 3^{1/2}B_M(169+38\sqrt{19})^{-1/2}. \quad (3.10)$$

For $B>B_{cr}$, there are no closed circular null geodesics; for $B=B_{cr}$ there is one circular null geodesic; for $B<B_{cr}$ there exist two null geodesic circles with radii r_1 and r_2, for which, with allowance for the smallness of the ratio B_{cr}/B_M, one can obtain the following approximate expressions:

$$r_1\approx 3M+9\left(\frac{B}{B_M}\right)^2,\quad r_2\approx\frac{2}{3^{1/2}}\frac{1}{B}. \quad (3.11)$$

Note that as $B\to 0$ the value of the radius r_1 goes over into the Schwarzschild value $r_1=3M$, and $r_2\to\infty$.

For massive particles when $B<B_{cr}$ in the region $r_1<r<r_2$ there exist circular orbits whose parameters are determined by the conditions $U_{\text{eff}}=E$, $\partial U_{\text{eff}}/\partial r=0$:

$$\gamma=\frac{E}{\mu}=\frac{\Delta^{1/2}\Lambda}{r}\left[(r-2M)\left(1+\frac{2}{\Lambda}\right)\right]^{1/2}\left[r-3M-4(r-2M)\left(1-\frac{1}{\Lambda}\right)\right]^{-1/2},$$

$$\frac{L}{\mu M}=\pm\frac{1}{\Lambda}\left[1+2\left(\frac{r}{M}-2\right)\left(1-\frac{1}{\Lambda}\right)\right]^{1/2}\times\left[1-\frac{3M}{r}+4\left(1-\frac{2M}{r}\right)\left(1-\frac{1}{\Lambda}\right)\right]^{-1/2}. \quad (3.12)$$

As the null geodesics (3.11) are approached, these orbits become ultrarelativistic, and

$$\gamma(r\to r_1)\to\left[\frac{M}{3(r-r_1)}\right]^{1/2},\quad \gamma(r\to r_2)\to\frac{2\sqrt{2}}{3^{5/4}}\left(\frac{M}{r-r_2}\right)^{1/2}. \quad (3.13)$$

Note that the factor Λ in the region of existence of circular orbits lies, as follows from (3.10) and (3.11), in the interval $1\leq\Lambda\leq 4/3$.

4. MOTION OF CHARGED PARTICLES

To describe the motion (nongeodesic) of charged particles in the Ernst field (2.1)–(2.2) it is necessary in Eqs. (3.1), (3.3)–(3.5) to make the substitution $L\to L-(eBr^2/2\Lambda)\sin^2\theta$, where e is the charge of the particle, and the parameter L now plays the role of the generalized momentum corresponding to the azimuthal coordinate. Right-handed and left-handed rotation in the equatorial plane are now inequivalent since the Lorentz force has a different direction. For the same reason, radial motion for fixed value of L is different for particles with charges of opposite sign. For values of L satisfying the inequality

$$L<|e|BM^2=|e|MB/B_M, \quad (4.1)$$

the effective potential of the radial motion increases monotonically with increasing r for any sign of the charge. For values of L sufficiently large compared with the right-hand side of (4.1), U_{eff} has a minimum, which for $LB_M\gg|e|MB$ is situated at the point

$$r_m=(2LB_MM/|e|B)^{1/3}, \quad (4.2)$$

and the curve corresponding to the sign of the charge for which the Lorentz force is directed away from the hole passes everywhere lower than the curve corresponding to the opposite sign of the charge (Fig. 1). We conclude that the radial potential barrier for particles of one sign of the charge is lowered under the influence of the magnetic field, while for particles of the opposite charge it is raised, which makes possible radial penetration of particles in the first case at an energy lower than the corresponding Schwarzschild value.

The parameters of circular trajectories of charged particles in the plane $\theta=\pi/2$ also depend on the sign of

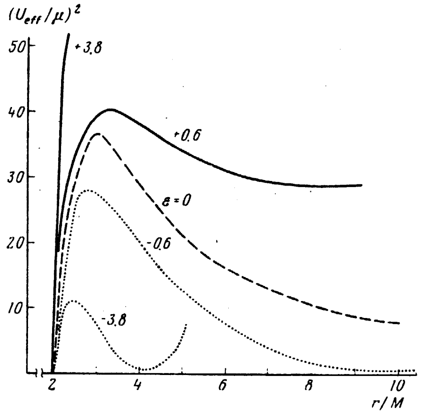

FIG. 1. Effective potential for radial motion of a charged particle $(U_{eff}/\mu)^2 = (1-2M/r)(1+(L/\mu r - \epsilon r/2M)^2)$ for different values of the parameter ϵ characterizing the influence of the magnetic field. The curves are constructed for the case $L = 30M\mu$.

the charge. The energy and generalize angular momentum for circular orbits depend on their radius as follows:

$$L=\frac{\mu r}{\Lambda}\left(\lambda-\frac{eBr}{2\mu}\right),\quad E=\mu\gamma=\frac{\Delta^{1/2}}{r}\Lambda(1+\lambda^2)^{1/2},\tag{4.3}$$

$$\lambda=\lambda_{(\pm)}=-\frac{1}{2}\left(\eta\pm\left\{\eta^2+4\left[1+2\left(\frac{r}{M}-2\right)\left(1-\frac{1}{\Lambda}\right)\right]\right.\right.$$
$$\left.\left.\times\left[\frac{r}{M}-3-4\left(\frac{r}{M}-2\right)\left(1-\frac{1}{\Lambda}\right)\right]\right\}^{1/2}\right)$$
$$\times\left[\frac{r}{M}-3-4\left(\frac{r}{M}-2\right)\left(1-\frac{1}{\Lambda}\right)\right]^{-1},\quad \eta=\frac{e}{\mu}\frac{B}{B_M}\frac{\Delta}{M^2\Lambda},\tag{4.4}$$

while the condition for the existence of circular orbits is that the radicand be non-negative.

Note that whereas in the case of neutral particles the influence of the magnetic field was determined by the ratio B/B_M, in the case of charged particles the factor $\epsilon = eB/\mu B_M$, which is associated with the Lorentz force, becomes decisive. Even for a very small value of the ratio B/B_M, the parameter $\epsilon = eB/\mu B_M$ need not be small for particles with large ratio e/μ (for an electron $e/\mu \sim 10^{21}$). In what follows, we shall altogether ignore the "geometrical" influence of the magnetic field on charged particles, setting $\Lambda = 1$. In this case, instead of (4.4) we have

$$\lambda_{(\pm)}=-\frac{\Delta}{2M(r-3M)}\left(\varepsilon\pm\left[\varepsilon^2+4\frac{(r-3M)M^2}{\Delta^2}\right]^{1/2}\right).\tag{4.5}$$

It is readily seen that $\lambda_{(+)}$ has a singularity at $r = 3M$, whereas $\lambda_{(-)}$ does not. The region of existence of circular trajectories, which is determined by the condition that the radicand in (4.5) be positive, extends right to the horizon for sufficiently large values of the ratio $\epsilon = eB/\mu B_M$ (Fig. 2).

We show that for $r > 3M$ one can have rotation in both directions, but in the region $2M < r \leq 3M$ one can only have rotation for which the Lorentz force is directed away from the hole. The radial equation of motion is

$$\frac{r^2}{\Delta}\frac{d^2r}{dt^2}-3\frac{Mr^2}{\Delta^2}\left(\frac{dr}{dt}\right)^2=-\frac{M}{r^2}+\Omega^2r+\omega_H\Omega r,\quad \Omega=\frac{d\varphi}{dt},\tag{4.6}$$

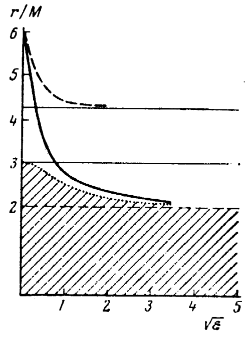

FIG. 2. Regions of existence and stability of circular orbits. The hatched region is the region of parameter values for which circular orbits do not exist. The continuous curve bounds the region of radial stability of anti-Larmor orbits; the dashed curve is the region of stability of the Larmor orbits.

where $\omega_H = eB/\mu u^0$ is the cyclotron frequency in the gravitational field $(u^0 = dt/ds = \gamma r^2/\Delta)$. The first term on the right-hand side of (4.6) is the force of the gravitational attraction; the second, the centrifugal force; the third, the Lorentz force. Denoting the frequency of circular geodesic motion in the Schwarzschild field by $\omega_s = (M/r^3)^{1/2}$, for circular orbits $(d^2r/dt^2 = dr/dt = 0)$ we find

$$\Omega_{\binom{a}{l}}=\tfrac{1}{2}\omega_H[\pm(1+4\omega_s^2/\omega_H^2)^{1/2}-1].\tag{4.7}$$

The lower sign in (4.7) corresponds to the Lorentz force directed toward the hole ("Larmor" rotation); the upper sign, to Lorentz force directed away from the hole ("anti-Larmor" rotation). For $\omega_H \to 0$, the frequencies (4.7) tend to $\pm\omega_s$, and for $\omega_s \ll \omega_H$ we find that $\Omega_a \approx -\omega_s^2/\omega_H$, $\Omega_l \approx -\omega_H$. From the condition that the square of the four-velocity be equal to unity, we obtain

$$g_{\mu\nu}u^\mu u^\nu=1=(u^0)^2\left(1-\frac{3M}{r}+r^2\omega_H\Omega\right).\tag{4.8}$$

It can be seen from this formula that Larmor motion $(\Omega_l < 0)$ is possible only in the region $r > 3M$, whereas anti-Larmor motion is possible in the region $r > 3M$ and the region $2M < r \leq 3M$. Combining Eqs. (4.3), (4.5), (4.7), we readily conclude that for $r > 3M$ Larmor rotation corresponds to the value $\lambda_{(+)}$ of the expression (4.5), and anti-Larmor rotation to the value $\lambda_{(-)}$. The angular velocity (4.7) after substitution of the corresponding values of the energy can also be written in the form

$$\Omega_{\binom{a}{l}}=\pm\frac{1}{r}\frac{\sqrt{\Delta}}{r}\lambda_{(\mp)}(1+\lambda_{(\mp)}^2)^{-1/2}.\tag{4.9}$$

For anti-Larmor rotation, the point $r = 3M$ is not a singularity; for Larmor rotation $\gamma_{(+)}(r \to 3M) \to \infty$, as in the case of geodesic motion. Sufficiently far from the point $r = 3M$, namely for

$$\varepsilon\Delta/M^2 \ll 2(r/M-3)^{1/2},\tag{4.10}$$

the difference between the energies corresponding to the two rotations in opposite direction at given radius is expressed by

$$E_l - E_a = \mu(\gamma_{(+)} - \gamma_{(-)}) = \mu \frac{r-2M}{r-3M} \varepsilon \sqrt{\frac{r}{M}}; \tag{4.11}$$

for positive charge, the energy corresponding to Larmor rotation is greater. At even larger radii, the Lorentz force become predominant, and we obtain

$$E_l \approx \mu[1+(eBr/\mu)^2]^{1/2}, \quad L_l = -eBr^2/2, \\ E_a \approx \mu, \quad L_a \approx 0, \tag{4.12}$$

whence for Larmor orbits we obtain the usual value of the cyclotron radius. Thus, Larmor rotation can be regarded as cyclotron rotation distorted by the gravitational field, while anti-Larmor rotation is possible only in the combined field of a black hole and the magnetic field.

To anti-Larmor rotation in the region $2M < r \leq 3M$ there correspond both signs in front of the root in (4.5). For $\epsilon \gg 1$ there exist circular trajectories whose radii are close to the radius of the horizon:

$$r \geq M(2+\varepsilon^{-1}). \tag{4.13}$$

As the horizon is approached, the energy $\mu\gamma_{(\pm)}$ of the anti-Larmor trajectories tends to zero, which is due to the gravitational mass defect. The energy measured in a local Lorentz frame (2.6) on the boundary determined by the equality sign in (4.13) is equal to the finite quantity $\bar{E} = \mu\sqrt{2}$. To this energy there corresponds the gravitational mass defect

$$\Delta\mu = (\mu - E)/\mu = (1-\varepsilon^{-1/2}), \tag{4.14}$$

which can be arbitrarily close to 100%.

The dependence of the energies of the Larmor and anti-Larmor trajectories on r for various values of the parameter ϵ is shown in Fig. 3. Note that for $r > 3M$ there exist Larmor trajectories characterized by an arbitrarily large value of γ, and that the ultrarelativistic nature of the motion is not due to the proximity to the closed null geodesic $r = 3M$. In particular, if the condition opposite to (4.10) is satisfied,

$$\gamma_{(+)}^2 = \left(1 - \frac{2M}{r}\right)\left[1 + \frac{\varepsilon^2\Delta^2}{M^2(r-3M)^2}\right] \gg 1. \tag{4.15}$$

We now investigate the stability of these circular orbits of charged particles in the equatorial plane. For stability in the radial direction, it is necessary that $\partial^2 U_{\text{eff}}/\partial r^2 > 0$. The resulting inequality can be represented in the two equivalent forms

$$\varepsilon^2 > \frac{\gamma^2(6M/r-1)}{(r/M-2)^3}, \quad \frac{\omega_{\Pi}^2}{\omega_s^2} > \frac{6M-r}{r-2M}. \tag{4.16}$$

It is clear from this that for $r > 6M$ the motion is radially stable irrespective of the magnetic field strength and the direction of the rotation, as is the case for $B = 0$.[19] For $r < 6M$, the regions of stability in the radial direction for Larmor and anti-Larmor rotations are different since for given r and ϵ the energies that must be substituted in (4.16) are not equal. For sufficiently large ϵ, the rotation corresponding to the energy value $\gamma_{(+)}$ are stable until $r \approx 4.3M$; the rotation with energy

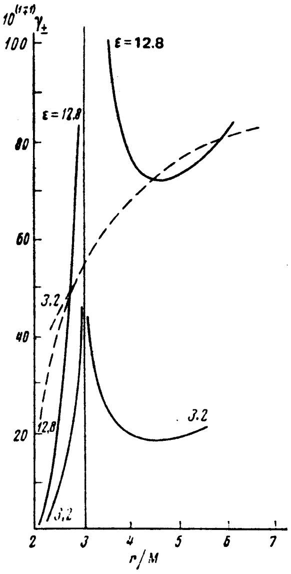

FIG. 3. Ratios of the energy to mass, $\gamma = E/\mu$, for circular orbits corresponding to the two signs in Eq. (4.5); the upper sign corresponds to continuous curves, the lower sign to the dashed curve.

$\gamma_{(-)}$ is stable right to the horizon. In particular, there exists a stable anti-Larmor orbit for which the mass defect is given by (4.14).

To investigate the stability in the vertical direction, in the Hamilton–Jacobi equation we make an expansion with respect to the angle $\alpha = \pi/2 - \theta$ near the plane $\theta = \pi/2$:

$$-\frac{r^2}{\Delta}\left(\frac{\partial S}{\partial t}\right)^2 + \frac{L^2}{r^2}(1+\alpha^2) + \frac{e^2B^2r^2(1-\alpha^2)}{4} + \mu^2 + eBL \\ + \frac{1}{r^2}\left(\frac{\partial S}{\partial \alpha}\right)^2 + \frac{\Delta}{r^2}\left(\frac{\partial S}{\partial r}\right)^2 = 0. \tag{4.17}$$

Collecting the terms proportional to α^2, we reduce the corresponding contribution to the form

$$\frac{\alpha^2}{r^2}\left(L^2 - \frac{e^2B^2r^4}{4}\right) = \frac{\alpha^2 r^6}{E^2\Delta^2}[\Omega(\Omega+\omega_{\Pi})]. \tag{4.18}$$

Taking into account the expression (4.7) for the frequencies, we readily see that the quantity in the square brackets is equal to ω_s^2 and, thus, is positive irrespective of the energy and the direction of the rotation, which in its turn means that small oscillations with respect to the angle θ take place with real frequency, so that the motion in the approximation of small oscillations is stable for all allowed r.

5. RADIATION OF SCALAR WAVES

One can distinguish three basic regimes of radiation of waves by particles moving in circular orbits in the Schwarzschild field: (a) radiation of the fundamental harmonic in nonrelativistic motion; (b) radiation of harmonics of order γ^2 relative to the fundamental harminic for motion in ultrarelativistic orbits close to the null geodesic $r = 3M$; (c) radiation of harmonics of order γ^3 for motion along nongeodesic trajectories sufficiently far from $r = 3M$ with ultrarelativistic velocity. The difference between the regimes (b) and (c) can be understood simply by using arguments about the formation length of radiation with given frequency.[21] For motion in an orbit close to the null geodesic, the high frequency beam emitted by a relativistic particle follows the particle because the trajectories of a photon and ultrarelativistic particle are nearly the same. As a result, radiation of a high frequency in a given direction is formed, not over a short section of the trajectory of order r/γ, as in the case of flat spacetime,[22] but over a length of the order of the radius r of the trajectory. Therefore, the spectral distribution of the radiation decreases exponentially, not at frequencies of the order of γ^3 relative to the fundamental harmonic, as in flat space, but at frequencies that are γ times lower. The radiation of relativistic particles in the regime (b) has been investigated in detail by a number of authors,[2-5,21] and we shall not consider this case further. The majority of the relativistic trajectories considered above, except for the Larmor orbits near $r=3M$, correspond to regime (c). Qualitative arguments relating to the raddiation in this case were considered in[3], but a quantitative theory was not constructed. Below, we give the complete theory of the radiation of scalar and electromagnetic waves in the regimes (a) and (c), and we also follow the transition from regime (c) to regime (b).

All the following treatment is given for the case $B/B_M \ll 1$. In this case, there exists outside the hole a region $Br \ll 1$ in which we have an approximately Schwarzschild spacetime, and to it all the conclusions derived below apply. This region must be sufficiently large to encompass the wave zone, for which [in regime (a)] we require that $B/B_M \ll \omega M (\ll 1)$. In all that follows, the passage to the limit $r\to\infty$ denotes transition to the wave zone.

We now turn to the calculation of the scalar radiation of charged particles moving in circular trajectories of radius r_0 (the subscript zero is introduced to distinguish the orbital radius from the coordinate r at which the radiation field is considered). We write down the scalar wave equation in the Schwarzschild metric:

$$\frac{r^4}{\Delta}\frac{\partial^2\psi}{\partial t^2}-\frac{\partial}{\partial r}\Delta\frac{\partial\psi}{\partial r}-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}=4\pi r^2 fT,$$
$$T=\mu\int ds\,\delta^{(4)}(x,x(s))/\sqrt{-g}=\frac{\mu}{u^0 r_0^2}\delta(r-r_0)\delta\left(\theta-\frac{\pi}{2}\right)\delta(\varphi-\Omega t), \qquad (5.1)$$

where f is a coupling constant. Separating the variables in Eq. (5.1), we find that

$$\psi=\sum_{lm}R_{l\omega}(r)Y_{lm}(\theta,\varphi)e^{-i\omega t},\quad \omega=m\Omega, \qquad (5.2)$$

where Y_{lm} are spherical functions, and $R_{l\omega}$ satisfies the equation

$$\Delta\frac{d}{dr}\left(\Delta\frac{dR_{l\omega}}{dr}\right)+(\omega^2r^4-l(l+1)\Delta)R_{l\omega}=-\frac{4\pi f\mu\Delta_0}{u^0}Y_{lm}\left(\frac{\pi}{2},0\right)\delta(r-r_0). \qquad (5.3)$$

We agree to normalize the two linearly independent solutions of the homogeneous equation without right-hand side corresponding to (5.3) by the asymptotic conditions

$$\vec{R}_{l\omega}(r\to\infty)\approx\frac{1}{(2\omega)^{1/2}}\frac{1}{r}e^{i\omega r},$$
$$\overleftarrow{R}_{l\omega}(r\to\infty)\approx\frac{1}{(2\omega)^{1/2}}\frac{1}{r}(e^{-i\omega r}+\beta e^{i\omega r}), \qquad (5.4)$$

where β is the reflection coefficient, its value being unimportant in what follows. The solution $\vec{R}$ on the horizon must contain only a wave falling into the hole. Using these functions, we obtain a retarded solution of Eq. (5.3) in the form

$$R_{l\omega}=4\pi i f\mu Y_{lm}\left(\frac{\pi}{2},0\right)\frac{1}{u^0}\frac{m}{|m|}[\theta(r-r_0)\vec{R}_{l\omega}(r)\overleftarrow{R}_{l\omega}(r_0)+\theta(r_0-r)\vec{R}_{l\omega}(r_0)\overleftarrow{R}_{l\omega}(r)], \qquad (5.5)$$

where $\theta(r-r_0)$ is the Heaviside step function.

The intensities of the radiation going to infinity P^{out} and falling into the hole P^{in} can now be expressed in terms of the values of the functions $\vec{R}$ and $\overleftarrow{R}$ at the point r_0 corresponding to the orbital position of the particle:

$$P^{\text{out}}=\sum_{l=0}\sum_{m=0}4\pi m\Omega\left(\frac{f\mu}{u^0}\right)^2Y_{lm}^2\left(\frac{\pi}{2},0\right)|\overleftarrow{R}_{l\omega}(r_0)|^2,$$

$$P^{\text{in}}=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}8\pi i m\Omega\left(\frac{f\mu}{u^0}\right)^2Y_{lm}^2\left(\frac{\pi}{2},0\right)|\vec{R}_{l\omega}(r_0)|^2\left(\Delta\frac{\partial\overleftarrow{R}_{l\omega}}{\partial r}\overleftarrow{R}^{*}_{l\omega}\right)\Big|_{r=2M}. \qquad (5.6)$$

The radial functions $\vec{R}_{l\omega}$ and $\overleftarrow{R}_{l\omega}$ can be obtained explicitly in the case of low frequencies $\omega r \ll 1$[14] and in the case $l,m\gg1$, corresponding to predominant radiation of high harmonics. In the first case, the solutions normalized in accordance with (5.4) have for $l=1$ the form

$$\overleftarrow{R}_{l\omega}=-\frac{\sqrt{2\omega}}{3}M\Omega\left(\frac{x}{x+1}\right)^{iQ}(1+2x),$$
$$\vec{R}_{l\omega}=\frac{3}{8M^3\omega^2}\frac{1}{(2\omega)^{1/2}}\left[\left(\frac{x}{x+1}\right)^{-iQ}\left(\frac{1}{1+2iQ}+2x\right)-\left(\frac{x}{x+1}\right)^{iQ}\left(\frac{1}{1-2iQ}+2x\right)\right], \qquad (5.7)$$
$$Q=-2M\omega,\ x=r/2M-1.$$

We consider the radiation produced by a particle moving with nonrelativistic velocity. In this case, we must assume in (4.9) that $\lambda\ll1$, which corresponds to the condition $\Omega r_0\ll1$ of applicability of Eqs. (5.7). Substituting (5.7) in (5.6), we obtain for the intensity of the long wavelength radiation that escapes to infinity and that falls into the hole the expressions

$$P^{\text{out}}={}^1/_3\Omega^2(\Delta_0/r_0^2)^2(f\mu)^2(1+2x)^2(M\Omega)^2,$$
$$P^{\text{in}}={}^3/_4\Omega^2(\Delta_0/r_0^2)^2(f\mu)^2[(1+2x)\sin\varphi+2Q\cos\varphi]^2. \qquad (5.8)$$

The intensity of the radiation that falls into the hole contains an oscillating factor which depends on $\varphi=Q\ln[x/(x+1)]$. Physically, this is due to the fact that, because of reflection from the barrier, the "outgoing" solution contains both infalling and reflected parts in the near zone, between which interference occurs. Since $M\Omega\ll1$ in the case considered, it can be seen from (5.8) that the main part of the radiation produced by a nonrelativistic particle moving near the gravitational radius falls into the hole except for the cases when the orbital radius of the particle is in a small neighborhood of the points r_n determined by the equation

$$Q\ln(1-2M/r_n)=\pi n,\ n=1,2\ldots \qquad (5.9)$$

We now consider the scalar radiation of an ultrarelativistic particle. In this case, high harmonics of the orbital frequency are emitted, and to find the intensity of the radiation P^{out} [the value of P^{in} in case (c) is exponentially small] one must know the radial function $\tilde{R}_{l\omega}$ in the approximation of high frequencies, for which it is appropriate to use the WKB method. After the change of argument $r^* = r - 2M + 2M\ln(r/2M - 1)$ and substitution $R_{l\omega} = u_{l\omega}/r$, the homogeneous radial equation is rewritten in the standard form for applying the WKB method:

$$d^2u_{l\omega}/dr^{*2} + (\omega^2 - V_{eff})u_{l\omega} = 0,$$
$$V_{eff} = (1-2M/r)(l(l+1)/r^2 + 2M/r^3). \quad (5.10)$$

Ultrarelativistic orbits around a black hole embedded in a homogeneous magnetic field exist for $r \geq 3M$. It is easy to see that the point r_0 corresponding to the position of an ultrarelatistic particle is always below the barrier produced by the curve $V_{eff}(r)$ near the right-hand wing. Indeed, the distance between the turning point determined by the intersection of the curve $V_{eff}(r)$ with the straight line $\omega^2 = m^2\Omega^2$ (see Fig. 4) and the point r_0 is determined by the equation

$$m^2\left(\frac{r_t}{r_0}\right)^2\left(1-\frac{2M}{r_0}\right)\left(1-\left(1-\frac{2M}{r_0}\right)\gamma^{-2}(r_0)\right) = \left(1-\frac{2M}{r_t}\right)\left(\frac{l(l+1)}{r_t^2}+\frac{2M}{r_t^3}\right) \quad (5.11)$$

in which $m^2 \leq l^2$. For $\gamma \gg 1$, $|m| \gg 1$ we find from (5.11)

$$r_t - r_0 \approx \frac{r_0}{2}\frac{r_0-2M}{r_0-3M}\left[\frac{2(l-m)}{m} + \frac{r_0-2M}{r_0}\gamma^{-2}(r_0)\right]. \quad (5.12)$$

As will be shown below, the range of values of the ratio $(l-m)/m$ for which the intensity of the radiation is appreciably nonzero is of order γ^{-2}, from which our assertion follows. The nature of the quasiclassical solution of Eq. (5.10) depends essentially on the order of magnitude of $\omega(r_0 - 3M)$. If this quantity is not large, then the quasiclassical solutions must be fitted at the top of the barrier. In the absence of a magnetic field, the corresponding problem was solved in[2-5]. In our case, the calculations are completely equivalent; it is only necessary to use the relations (4.3) and (4.4) for the energy. The spectral distribution of the radiation repeats the results of[2], and we shall not give it [regime (b)]. We merely note that the number m_{cr} of the harmonic above which the intensity decreases rapidly is equal to

$$m_{cr} = \frac{12}{\pi}\left(1-\frac{2M}{r_0}\right)\lambda_{(+)}^2(r_0) = \frac{12}{\pi}\gamma_{(+)}^2, \quad (5.13)$$

and for the major part of the radiation $l = m$.

We now consider the radiation of harmonics $m \gg [\Omega(r_0 - 3M)]^{-1}$ [regime (c)]. In this case, a large number of waves fit under the barrier $V_{eff}(r)$, and to match the solutions of Eq. (5.10) it is sufficient to retain the linear term in the expansion of the effective potential in the neighborhood of the turning point; as a result, we reduce this equation to the form

$$\frac{d^2u}{dq^2} - qu = 0, \quad q = -\left(\frac{2m^2\Omega^2(r_t-3M)}{r_t^2}\right)^{1/3}(r^* - r_t^*). \quad (5.14)$$

The Airy functions are a solution of this equation:

$$u = A\Phi(q) + B\Psi(q),$$
$$\Phi(q) = \frac{1}{\sqrt{\pi}}\int_0^\infty \cos\left(\frac{y^3}{3}+yq\right)dy, \quad \Psi(q) = \frac{1}{\sqrt{\pi}}\int_0^\infty \sin\left(\frac{y^3}{3}+yq\right)dy. \quad (5.15)$$

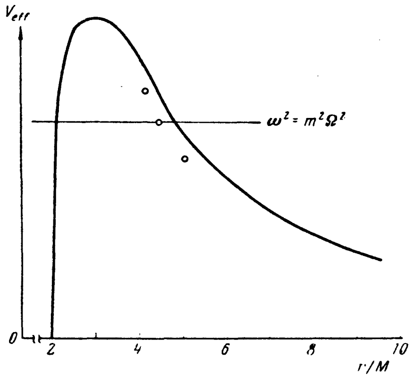

FIG. 4. Effective potential of the radial equation for the scalar field (5.10). The circles are the positions of the ultrarelativistic circular orbits of charged particles. The intersection of the horizontal lines with the curve V_{eff} determines the position of the turning points, at which the quasiclassical solutions are fitted.

As a result of fitting (5.15) to the infalling wave on the horizon, and far from the hole to the asymptotic expression (5.4), we find

$$B = 0, \quad A = 4[r_t^2/2m^2\Omega^2(r_t - 3M)]^{1/6}. \quad (5.16)$$

Going over from the Airy function to the MacDonald function $K_{1/3}$, we write the value of the radial function at the position of the particle in the final form

$$u(r_0) = \left[\frac{2}{3\pi}\frac{r_0}{\gamma_{(+)}^2}\frac{r_0-2M}{r_0-3M}(1+\psi^2)\right]^{1/2} K_{1/3}(z), \quad (5.17)$$

where

$$z = \frac{m}{3\gamma_{(+)}^3}\frac{(r_0-2M)^2}{r_0(r_0-3M)}(1+\psi^2)^{3/2}, \quad \psi^2 = 2\gamma_{(+)}^2\frac{l-m}{m}\frac{r_0}{r_0-2M}. \quad (5.18)$$

Bearing in mind that the MacDonald function decreases exponentially for values of the argument large compared with unity, we readily conclude that the main contribution to the radiation of the harmonic m of the fundamental frequency Ω is made by multipoles l satisfying the condition $l - m \lesssim m\gamma^{-2}$, and that the spectrum contains the harmonics

$$m \lesssim m_{max} = \gamma^3 r_0(r_0 - 3M)/(r_0 - 2M)^2. \quad (5.19)$$

As in the case of flat spacetime,[22] the maximal number of radiated harmonics is proportional to the third power of the energy, but, in addition, it depends strongly on how close the particle orbit is to the closed null geodesic $r_0 = 3M$. One can see that as the radius of the orbit approaches $3M$ the cubic dependence of m_{max} on γ is replaced by a quadratic dependence. Indeed, in the considered approximation one can set $r_0/M - 3 \ll 1$ if at the same time $m\Omega(r_0 \sim 3M) \gg 1$. In conjunction with (5.19), we obtain the restriction

$$\gamma^2(r_0/M - 3) \gg 1, \quad (5.20)$$

which must be satisfied for Larmor orbits if $\epsilon^2 \gg r/M - 3$. Taking into account (5.20), we find by means of Eqs. (4.3) and (4.5)

$$\gamma_{(+)}(r_0/M - 3) = 3^{1/2}\epsilon, \quad (5.21)$$

from which it follows that as $r_0 \to 3M$ the maximal number of the harmonic is

$$m_{max}(r_0 \to 3M) = 3^{3/2} \varepsilon \gamma_{(+)}^2. \tag{5.22}$$

Thus, the quadratic dependence of the maximal frequency of the radiation on the particle energy (5.13) characteristic for motion in orbits close to the null geodesic [regime (b)] is reproduced in the passage to the limit from the radiation regime (c) to the régime (b).

Substituting (5.17) in (5.6), we obtain the intensity of the radiation:

$$P^{out} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{8m\Omega}{3r_0 \nu_{(+)}^2} Y_{lm}^2\left(\frac{\pi}{2}, 0\right) \left(\frac{f\mu}{u^0}\right)^2 \frac{r_0 - 2M}{r_0 - 3M} (1+\psi^2) K_{1/3}^2(z). \tag{5.23}$$

For large values of l and small values of the ratio $(l-m)/m$, the following approximation is valid for spherical functions:

$$\left| Y_{lm}\left(\frac{\pi}{2}, \varphi\right) \right|^2 = \begin{cases} \dfrac{\psi}{\gamma} \dfrac{1}{\pi^2} \left(\dfrac{r_0}{r_0 - 2M}\right)^{1/2}, & l - m > 2n, \quad n = 0, 1, \ldots \\ 0, & l - m = 2n + 1. \end{cases} \tag{5.24}$$

Using this formula and taking into account the quasicontinuity of the spectrum, we go over from summation over l and m to integration over ψ and the parameter

$$y = \frac{2m}{3\gamma^3} \frac{(r_0 - 2M)^2}{r_0(r_0 - 3M)}, \tag{5.25}$$

as a result of which we obtain

$$\frac{d^2P}{dy\, d\psi} = \frac{9}{2\pi^2} \left(\frac{f\mu\gamma^2}{r_0}\right)^2 \left(\frac{r_0 - 3M}{r_0 - 2M}\right)^2 y^2 (1 + \psi^2) K_{1/3}^2(z). \tag{5.26}$$

One can show that the distribution of the intensity with respect to the parameter ψ, which is related to the multipole order l by (5.18), is closely correlated to the angular distribution. Indeed, for $l \gg 1$ and $l - m \ll m$ the square of the modulus of the spherical function has a maximum at $\theta = \pi/2$, and for small angles $\alpha = \theta - \pi/2$ the following approximation in terms of Hermite polynomials holds[4]:

$$\left| Y_{lm}\left(\frac{\pi}{2} + \alpha, \varphi\right) \right|^2$$

$$= \begin{cases} m^{1/2} \exp(-m\alpha^2) \pi^{-3/2} 2^{m-l-1} [(l-m)!]^{-1} H_{l-m}^2 (m^{1/2}\alpha), & l - m = 2n \\ 0, \quad l - m = 2n + 1; \quad n = 0, 1, 2 \ldots \end{cases} \tag{5.27}$$

Calculating the mean value of the square of the angle α with the distribution (5.27), we find that the parameter ψ coincides to within a factor with the mean square value of the angle:

$$\langle \alpha^2 \rangle = \int d\Omega \left| Y_{lm}\left(\frac{\pi}{2} + \alpha, \varphi\right) \right|^2 \alpha^2 = \frac{l - m + 1/2}{m} \approx \frac{r_0 - 2M}{2r_0} \left(\frac{\psi}{\gamma}\right)^2. \tag{5.28}$$

Thus, the multipole and angular distribution of the radiation of an ultrarelativistic particle are intimately related. Taking into account (5.28), we find that the range of angles in which radiation is emitted is equal to

$$\Delta\theta \sim \gamma^{-1} \left(\frac{r_0 - 2M}{r_0}\right)^{1/2}. \tag{5.29}$$

The integration with respect to the parameter ψ in (5.26) is performed by means of the formulas given in [22], and leads to the following expression for the spectral distribution of the radiation:

$$\frac{dP}{dy} = \frac{3^{1/2}}{2\pi} \left(\frac{f\mu\gamma^2}{r_0} \frac{r_0 - 3M}{r_0 - 2M}\right)^2 y \int_y^{\infty} K_{5/3}(x)\, dx. \tag{5.30}$$

For small y, the intensity increases as $y^{5/3}$; for large y, it decreases exponentially. The spectral curve has a maximum corresponding to the critical harmonic

$$y \gg \gamma^{-3} [(r_0 - 2M)/(r_0 - 3M)]^2, \tag{5.31}$$

under which condition the treatment given above is valid [regime (c)], then the radiation of relativistic particles in the gravitational field qualitatively repeats the picture of the radiation in flat spacetime, though the parameters that characterize the angular and spectral distributions depend on how close the orbital radius is to the radius of the null geodesic.

Integrating in (5.30) with respect to y, we find the total intensity of the radiation:

$$P = \frac{2}{3} \left(\frac{f\mu\gamma^2}{r_0} \frac{r_0 - 3M}{r_0 - 2M}\right)^2. \tag{5.32}$$

This quantity is γ^2 times greater than the intensity of geodesic synchrotron radiation, but as r_0 tends to $3M$ the dependence on γ becomes quadratic in accordance with (5.21).

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Quantum interferometer as detecting element of a gravitational antenna

A. V. Gusev and V. N. Rudenko

Moscow State University
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The properties of a quantum interferometer as a probe of small acoustic perturbations of a gravitational detector are studied. The low-frequency fluctuations of a Josephson contact are calculated, and on this basis formulas are obtained for the limiting sensitivity of the gravitational antenna. It is shown that when quantum restrictions are taken into account it is possible in principle to achieve the resolution necessary for second-generation antennas.

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1. INTRODUCTION

Braginsky *et al.*[1] have formulated a very general prediction for the parameters of bursts of gravitational radiation which can be reasonably expected as the result of cosmic catastrophes occurring with participation of superdense stars ($r \sim r_g$). For a frequency of events no less than ten events per year and a duration of bursts $\hat{\tau} \sim 10^{-3}$–10^{-4} sec the upper limit of their energy density at the Earth lies in the range $W \sim 1$–10^4 erg/cm² (Braginsky's estimate takes into account components with $M \sim 3$–$30M_\odot$ and a range $\epsilon \sim 0.1$–10^{-3} for the fraction of the total energy converted to gravitational radiation). For a quadrupole gravitational detector (GD) with masses m, linear dimension $l_g \sim 10^2$ cm, and a mean frequency $\omega_\mu \sim 3\times10^4$ rad/sec this is equivalent to an external perturbation with a relative acceleration of the masses $F/m \sim 10^{-9}$ cm/sec² in the optimistic case ($W = 10^4$ erg/cm²) and $F/m \sim 3\times10^{-11}$ cm/sec² in the pessimistic case ($W = 1$ erg/cm², $\tau \sim 2\times10^{-4}$ sec). It is known that the potential sensitivity of GD permits detection of such excitations if the Brownian motion is reduced by cooling to $T_\mu \sim 3\times10^{-3}$ K or as the result of a high mechanical quality factor $Q_\mu \sim 10^{10}$.[2] The reason for the delay in construction of second-generation antennas is the lack of efficient detecting elements which measure small vibrations of the GD. Actually, in the best converters of the parametric type with a pumping frequency ω_e under matched conditions, their intrinsic fluctuations at temperature T_e limit the sensitivity of the antenna at the level[3]

$$\left(\frac{F}{m}\right)_{min} \geqslant \frac{3\pi^{1/2}}{\hat{\tau}}\left(\frac{kT_e\omega_\mu}{m\omega_e}\right)^{1/2}. \tag{1}$$

Substitution into Eq. (1) of the values $T_e = 2$ K, $m \approx 3\times10^4$ g, $\omega_e = 3\times10^{10}$, and $\hat{\tau} \approx 2\times10^{-4}$ sec gives $F/m \gtrsim 10^{-9}$ cm/sec², i.e., the sensitivity barely reaches the optimistic limit of the prediction. Increase of the mass of the GD to 5×10^6 g (Ref. 4) provides only $F/m \sim 10^{-10}$ cm/sec².

In addition there is an important limitation due to the possiblity of amplifying the probe signal. The amplifier noise temperature must satisfy the condition

$$T_n \leqslant \frac{2\pi G}{c^3 k}\omega_\mu\omega_e m l_g^2 \hat{\tau} W \sim 10^{-3} W. \tag{2}$$

For the best low-noise amplifiers in the frequency range considered, $T_n \sim 1$ K.[5] It is therefore clear that only $W \approx 10^3$ erg/cm² is accessible to detection; with increase of the mass $W \approx 10$ erg/cm².

It follows from Eqs. (1) and (2) that an increase of the pumping frequency ω_e would be a radical measure. However, this is hindered by the following considerations. The first, which is technical in nature, is the unavailability to experimenters of pumping generators with sufficient stability in the range $\omega_e > 3\times10^{10}$. A second, which is fundamental in nature, is the intrusion into the region of quantum limitations, according to which Eq. (1) is valid as long as $kT_e \gtrsim \hbar\omega_e$, and the maximum sensitivity of the antenna will not exceed the quantum limit[6]

$$\left(\frac{F}{m}\right)_{min} \geqslant \frac{1}{\hat{\tau}}\left(\frac{\hbar\omega_\mu}{m}\right)^{1/2}. \tag{3}$$

For $T_e \sim 2$–4 K the limiting value of the pumping frequency lies *a priori* somewhere near $\omega_e \sim 10^{11}$.

In recent years the hopes of a number of experimental groups have been based on the use of quantum magnetometers employing the Josephson effect—so-called SQUIDS.[1)] However, our analysis[7] has shown that single-contact SQUIDS with external pumping have the same

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