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Solution of the Fokker-Planck equation for a laminar medium

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An exact analytic solution (Green's function) of the Fokker-Planck equation is found for the case in which the mean square of the multiple-scattering angle is a function of a longitudinal coordinate.

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The Fokker-Planck kinetic equation, which describes the spatial and angular distribution of particles subject to multiple scattering in the small-angle approximation, is of the well-known form

$$\frac{\partial w}{\partial z}+\theta\frac{\partial w}{\partial \mathbf{r}}=q\Delta_\theta w, \tag{1}$$

where $\mathbf{r}$ is the radius vector in the transverse direction, z is the coordinate of longitudinal displacement, $\boldsymbol{\theta}$ is a two-dimensional angular vector fixing the projection of the vector of the velocity's direction on a plane normal to the z axis, and q denotes one fourth of the mean square of the multiple scattering angle in unit path length along the longitudinal coordinate.

In applying this equation to actual physical problems it is necessary, as a rule, to take into account a dependence of q on the spatial coordinates. This is the situation, for example, when what is required is an estimate of the spatial and angular distribution of a beam of fast particles as they pass through an inhomogeneous medium. Examples are analysis of the passage of cosmic rays through the atmosphere, calculation of effectiveness of targets and of shielding arrangements in accelerators, investigation of the properties of charged-particle detectors, estimates of the effect of the spread in position and direction of travel on transition and Cherenkov radiation; there are many other cases. Moreover, if the particles are subject to continuous energy loss, one can take the effect of a dependence of q on the energy of a particle by introducing an appropriate dependence of the scattering properties of the medium on the coordinate. The case of dependence of q on the longitudinal coordinate z is of particular interest, since because the transverse displacements are small the dependence of the scattering properties of the medium on $\mathbf{r}$ are usually of little importance.

Therefore the aim of this paper is to find the distribution function in closed form (in terms of quadratures) in its dependence on the function $q(z)$.

We shall look for a solution of Eq. (1) in the form

$$w(\mathbf{r},\theta,z)=\frac{s}{4\pi^2}\exp(-p_1\mathbf{r}^2+p_2\mathbf{r}\theta-p_3\theta^2), \tag{2}$$

where p_1, p_2, p_3, and s are functions of the coordinate z.

Substitution of Eq. (2) in Eq. (1) gives the following system of equations:

$$\frac{1}{s}\frac{ds}{dz}=-4p_3q,\quad \frac{dp_1}{dz}=-p_2^2q,$$
$$\frac{dp_2}{dz}=2p_1-4p_2p_3q,\quad \frac{dp_3}{dz}=p_2-4p_3^2q. \tag{3}$$

Normalizing the function $w(\mathbf{r},\,\boldsymbol{\theta},\,z)$ to unity,

$$\iint w(\mathbf{r},\theta,z)\,d\mathbf{r}\,d\theta=1, \tag{4}$$

we find that $s=4p_1p_3-p_2^2$ and that the first equation is a consequence of the others. Successive use of the second, third, and fourth equations in combination with

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the first shows that the system (3) is equivalent to the following system:

$$\frac{d}{dz}\left(\frac{p_1}{s}\right)=q,\quad \frac{d}{dz}\left(\frac{p_2}{s}\right)=2\frac{p_1}{s},\quad \frac{d}{dz}\left(\frac{p_3}{s}\right)=\frac{p_2}{s}. \tag{5}$$

For the boundary condition

$$w(\mathbf{r},\theta,0)=\delta(\mathbf{r})\delta(\theta), \tag{6}$$

the solution is

$$\frac{p_1}{s}=\int_0^z q\,dz_1,\quad \frac{p_2}{s}=2\int_0^z q(z-z_1)\,dz_1,\quad \frac{p_3}{s}=\int_0^z q(z-z_1)^2dz_1. \tag{7}$$

From this we find the functions s, p_1, p_2, and p_3. Substitution of these in Eq. (2) and a bit of calculation give the following result:

$$w(\mathbf{r},\theta,z)=\frac{s}{4\pi^2}\exp\left\{-s\left[\mathbf{r}^2\int_0^z q\,dz_1-2\mathbf{r}\theta\int_0^z q(z-z_1)\,dz_1+\theta^2\int_0^z q(z-z_1)^2dz_1\right]\right\}, \tag{8}$$

where

$$s=\frac{1}{4}\left[\int_0^z q\,dz_1\int_0^z qz_1^2dz_1-\left(\int_0^z qz_1dz_1\right)^2\right]^{-1}. \tag{9}$$

To obtain the solution of Eq. (1) which satisfies the boundary condition

$$w(\mathbf{r},\theta,0)=\delta(\mathbf{r}-\mathbf{r}_1)\delta(\theta-\theta_1), \tag{10}$$

we need only replace θ with $\theta-\theta_1$ and $\mathbf{r}$ with $\mathbf{r}-\mathbf{r}_1-\theta_1 z$:

$$\begin{aligned} w(\mathbf{r},\theta,z)=\frac{s}{4\pi^2}\exp\Big\{-s\Big[&(\mathbf{r}-\mathbf{r}_1-\theta_1 z)^2\int_0^z q\,dz_1 \\ &-2(\mathbf{r}-\mathbf{r}_1-\theta_1 z)(\theta-\theta_1)\int_0^z q(z-z_1)\,dz_1+(\theta-\theta_1)^2\int_0^z q(z-z_1)^2dz_1\Big]\Big\}. \end{aligned} \tag{11}$$

From this we find the probability of a given transverse displacement of the particle without regard to the angular deflection,

$$\int w(\mathbf{r},\theta,z)\,d\theta=\left[4\pi\int_0^z q(z-z_1)^2dz_1\right]^{-1} \times\exp\left[-(\mathbf{r}-\mathbf{r}_1-\theta_1 z)^2\Big/4\int_0^z q(z-z_1)^2dz_1\right], \tag{12}$$

and the probability of a given angular deflection without regard to the transverse displacement,

$$\int w(\mathbf{r},\theta,z)\,d\mathbf{r}=\left[4\pi\int_0^z q\,dz_1\right]^{-1}\exp\left[-(\theta-\theta_1)^2\Big/4\int_0^z q\,dz_1\right]. \tag{13}$$

In the simplest special case $q=\text{const}$ our results go over, as they must, into the well known results obtained by Fermi.

To calculate the distribution function for a prescribed probability distribution of the initial values $\mathbf{r}_1$ and θ_1, we obviously have only to integrate the product of this distribution and the distribution function (11) over the variables $\mathbf{r}_1$ and θ_1. Thus our results solve the problem of the spatial and angular distribution of particles in an arbitrary medium made up of layers, and also the problem with energy loss taken into account, of course only if the fluctuations of the energy loss can be neglected.

As an illustration of these expressions let us consider the vertical passage of cosmic-ray particles through the atmosphere, on the assumption that the density of the air varies with the height h according to an exponential law: $\rho=\rho_0\exp(-h/L)$ (ρ_0 is the density at the surface of the Earth). Also for simplicity let us neglect the effects of a particle's loss of energy along its path and of the terrestrial magnetic field on the trajectory. Then $q=q_0\exp(-h/L)$, and the spatial-angular distribution of the particles at the Earth's surface is as given by Eq. (2) with

$$\begin{aligned} s=\frac{1}{4q_0^2}\lim_{h\to\infty}\Big\{&\int_{-h}^0 \exp(z_1/L)\,dz_1\int_{-h}^0\exp(z_1/L)(z_1+h)^2\,dz_1 \\ &-\left[\int_{-h}^{0}\exp(z_1/L)(z_1+h)\,dz_1\right]^2\Big\}^{-1}=\frac{1}{4q_0^2L^4}, \\ p_1=\frac{1}{4q_0L^3},\quad p_2&=\frac{1}{2q_0L^2},\quad p_3=\frac{1}{2q_0L}. \end{aligned} \tag{14}$$

In particular, the mean square displacement of a particle in a transverse direction is $\overline{\mathbf{r}^2}=8q_0L^3$; this is the same as for the passage through a distance $6^{-1/3}L$ in a homogeneous atmosphere with density ρ_0. The mean square of the angular deflection is $\overline{\theta^2}=4q_0L$, and the equivalent distance through a homogeneous atmosphere of density ρ_0 is L.

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