Quantum theory of spontaneous and induced radiation of channeled electrons and positrons

V. V. Beloshitskiĭ and M. A. Kumakhov

Nuclear Physics Institute, Moscow State University
(Submitted 6 July 1977)
Zh. Eksp. Teor. Fiz. **74**, 1244–1256 (April 1978)

The quantum theory of spontaneous and induced radiation by channeled relativistic particles is developed. Formulas are obtained under various conditions of channeling for the spectral and angular characteristics of the radiation. A comparison is made with synchrotron radiation and important advantages of the radiation from channeled particles for a number of applications are shown. The possible existence of enhancement of induced radiation under channeling conditions is predicted theoretically. On this basis a laser operating over a wide wavelength range including the x-ray region is proposed. The basic mechanisms of line broadening due to multiple scattering, beam nonmonochromaticity, and lattice periodicity are discussed.

PACS numbers: 61.80.Mk

1. INTRODUCTION

In a series of recent articles[1-3] we have shown that in channeling of relativistic particles an intense radiation of quanta in the x-ray and γ-ray regions occurs. The spectral density of this radiation in the region of the maximum substantially exceeds the density of bremsstrahlung, which may result in important applications in physics. The radiation has a high degree of monochromaticity and polarization. This effect can be considered both classically and in quantum mechanics. In the previous studies[1-3] we calculated the principal parameters of this radiation by various means for energetic quanta (refractive index $n=1$).

In the present article we shall use the quantum approach for a rigorous calculation of the spectral and angular characteristics of spontaneous radiation over a wide wavelength range including the optical region. In addition, we shall give for the first time the quantum theory of induced (stimulated) radiation in the case of channeling. It is shown that it is possible in principle to obtain enhancement of this radiation in a beam of channeled particles. A laser model based on this effect over a wide wavelength range is discussed.

2. DERIVATION OF FORMULAS FOR INTENSITY OF RADIATION

Neglecting secondary processes such as dechanneling and energy loss, we shall discuss the passage of a channeled particle through a single crystal as motion in the averaged potential of atomic strings or planes. A relativistic particle with spin 1/2 in an electrostatic potential U is described by the Dirac wave equation in the two-component form:

$$(E-U-m_0c^2)\varphi=c(\sigma\hat{\mathbf{p}})\chi, \qquad (1)$$

$$(E-U+m_0c^2)\chi=c(\sigma\hat{\mathbf{p}})\varphi, \qquad (2)$$

where E is the particle energy, φ, χ are two-component functions, $\hat{\mathbf{p}}=-i\hbar\nabla$ is the momentum operator, m_0 is the rest mass of the particle, and the four-component wave function in this case is described by the formula

$$\varphi(r,t)=\begin{pmatrix}\varphi\\ \chi\end{pmatrix}e^{-iEt/\hbar}. \qquad (3)$$

Using Eq. (2) we can express one two-component function in terms of the other:

$$\chi=\frac{c\sigma\hat{\mathbf{p}}}{E-U+m_0c^2}\varphi \qquad (4)$$

and substitute into Eq. (1). We obtain

$$(E-U-m_0c^2)\varphi=(\sigma\hat{\mathbf{p}})\frac{c^2}{E-U+m_0c^2}(\sigma\hat{\mathbf{p}})\varphi. \qquad (5)$$

The following relation is valid for the Pauli matrices:

$$(\sigma\mathbf{a})(\sigma\mathbf{b})=\mathbf{ab}+i\sigma\cdot\mathbf{a}\times\mathbf{b} \qquad (6)$$

Multiplying Eq. (5) by $E-U+m_0C^2$, we obtain with accuracy to terms U/E

$$(E^2-2EU-m_0^2c^4)\varphi=-c^2\hbar^2\Delta\varphi. \qquad (7)$$

Separating variables in this equation, we obtain

$$\varphi=\varphi(x)\frac{1}{L}\exp(ik_zz+ik_yy), \qquad (8)$$

where $p_z=\hbar k_z$ and $p_y=\hbar k_y$ are the momentum components parallel to the atomic plane, the plane waves having been normalized in a box with sides L along the z and y axes, and $\varphi(x)$ satisfies the Schrödinger equation

$$(E^2-2EU-c^2p_z^2-c^2p_y^2-m_0^2c^4)\varphi(x)=-c^2\hbar^2\frac{d^2\varphi(x)}{dx^2}. \qquad (9)$$

Dividing it by $2E=2mc^2$, we obtain

$$\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2}=(U-\mathscr{E})\varphi(x), \qquad (10)$$

where

$$\mathscr{E}=[E^2-m_0^2c^4-c^2(p_z^2+p_y^2)]/2E,$$

and m is the relativistic mass. Since $E^2=m_0^2c^4+p^2c^2$, where p is the momentum, we have

$$\mathscr{E}=c^2(p^2-p_z^2-p_y^2)/2E\approx E-c(m_0c^2+p_z^2+p_y^2)^{1/2}.$$

In order that the wave function

$$\psi_{\mathscr{E}p_zp_y}(\mathbf{r})=A\begin{pmatrix}\varphi\\ \chi\end{pmatrix}=A\begin{pmatrix}\mathbf{u}\\ \dfrac{c\sigma\hat{\mathbf{p}}}{m_0c^2+E}\mathbf{u}\end{pmatrix}\varphi_{\mathscr{E}}(x)\frac{1}{L}\exp(ik_zz+ik_yy) \quad (11)$$

be normalized in accordance with the condition

$$\int\psi^+_{\mathscr{E}p_zp_y}\psi_{\mathscr{E}'p_z'p_y'}d\tau=\delta_{\mathscr{E}\mathscr{E}'}\delta_{p_zp_z'}\delta_{p_yp_y'},$$

we shall take the spin functions $\mathbf{u}$ to satisfy the usual normalization condition:

$$u^+u=u_1^*u_1+u_2^*u_2=1,$$

$$A\approx[(m_0c^2+E)/2E]^{1/2},\quad\int\varphi_{\mathscr{E}}(x)\varphi_{\mathscr{E}'}(x)dx=\delta_{\mathscr{E}\mathscr{E}'}.$$

(Here we have taken into account that $|p|^2\approx\hbar^2k_z^2+\hbar^2k_y^2$.)

To simplify the formulas we discuss here a single potential well. The effect of periodicity of the potential, i.e., of the band structure of the resulting levels, will be discussed below.

The probability of a transition from state b to state a with emission of quanta with polarization β_λ and momentum $\boldsymbol{\varkappa}$ is given by the following formula of quantum electrodynamics [4]:

$$w_{ab}=\frac{e^2}{\hbar}\frac{4\pi^2}{L^3}\sum_{\varkappa}\frac{c\delta(\omega_{ab}-\omega)}{\varkappa}\Phi_\lambda[N_\lambda(\varkappa)+1], \quad (12)$$

where

$$\Phi_\lambda=(\alpha_{ab}^*\beta_\lambda)(\alpha_{ab}\beta_\lambda),\quad\hbar\omega_{ab}=E_b-E_a, \quad (13)$$

$\varkappa=\omega/c$, $N_\lambda(\varkappa)$ is the total number of quanta with polarization λ and momentum $\hbar\varkappa$. The first term in the square brackets corresponds to induced radiation, and the second to spontaneous radiation. The matrix element

$$\alpha_{ab}=\int e^{-i\varkappa r}\psi_a^+\alpha\psi_b\,d\mathbf{r},\quad\alpha=\begin{pmatrix}0&\sigma\\ \sigma&0\end{pmatrix}, \quad (14)$$

using Eq. (11), reduces to the form

$$\alpha_{ab}=\frac{c}{2E}\int e^{-i\varkappa r}\varphi_a^+[\sigma(\sigma\mathbf{p})+(\sigma\mathbf{p})\sigma]\varphi_b\,d\mathbf{r}. \quad (15)$$

From this, using the relation $\sigma(\sigma\mathbf{p})+(\sigma\mathbf{p})\sigma=2\mathbf{p}$, we obtain

$$\alpha_{ab}=\frac{c}{E}\int e^{-i\varkappa r}\varphi^+_{\mathscr{E}_a}\hat{\mathbf{p}}\varphi_{\mathscr{E}_b}d\mathbf{r}. \quad (16)$$

Integration over the variables y and z in Eq. (16) gives the laws of conservation of the y and z projections of momentum:

$$(\alpha_{ab})_x=\frac{c}{E}\int\exp(-i\varkappa_xx)\varphi^*_{\mathscr{E}_a}\hat{p}_x\varphi_{\mathscr{E}_b}dx\,\delta_{k_{by},k_{ay}+\varkappa_y}\delta_{k_{bz},k_{az}+\varkappa_z},$$

$$(\alpha_{ab})_y=\frac{c}{E}\int\exp(-i\varkappa_xx)\varphi^*_{\mathscr{E}_a}\varphi_{\mathscr{E}_b}dx\,\hbar k_{by}\delta_{k_{by},k_{ay}+\varkappa_y}\delta_{k_{bz},k_{az}+\varkappa_z} \quad (17)$$

$$(\alpha_{ab})_z=\frac{c}{E}\int\exp(-i\varkappa_xx)\varphi^*_{\mathscr{E}_a}\varphi_{\mathscr{E}_b}dx\,\hbar k_{bz}\delta_{k_{by},k_{ay}+\varkappa_y}\delta_{k_{bz},k_{az}+\varkappa_z}.$$

Assuming in what follows that these laws are satisfied, we shall drop the δ symbols.

We shall use the dipole approximation, i.e., in Eq. (17) we shall expand $\exp(-i\varkappa_xx)$ in series and retain non-vanishing terms linear in x in the matrix element. The condition for the dipole approximation $\varkappa_x\langle x\rangle\ll1$ in this case has the form $\langle x\rangle\ll c\gamma/\Omega_{ab}$, where $\Omega_{ab}=\mathscr{E}_b-\mathscr{E}_a)/\hbar$ and γ is the relativistic factor. This condition is equivalent in order of magnitude to $\theta_c\gamma\ll1$, where θ_c is the critical channeling angle, well known from Lindhard's theory. It is satisfied up to energies of several GeV. The dipole approximation condition has been investigated in more detail recently by Bazylev and Zhevago.[5] In this approximation we obtain

$$(\alpha_{ab})_x=\frac{c}{E}\int\varphi^*_{\mathscr{E}_a}p_x\varphi_{\mathscr{E}_b}dx,$$

$$(\alpha_{ab})_y=-\frac{i\hbar c}{E}\varkappa_xk_{by}x_{ab},\quad(\alpha_{ab})_z=-\frac{i\hbar c}{E}\varkappa_xk_{bz}x_{ab}, \quad (18)$$

$$\hbar\omega_{ab}=E_b-E_a=\mathscr{E}_b-\mathscr{E}_a+c[(p_{bz}+p_{by}+m_0^2c^2)^{1/2}-((p_{bz}-\hbar\varkappa_z)^2+(p_{by}-\hbar\varkappa_y)^2+m_0^2c^2)^{1/2}].$$

To simplify the further calculations we shall assume that the axis is directed along the longitudinal motion, i.e., $P_{by}=k_{by}=0$. We shall also use the well known relation following from the Schrödinger equation (9):

$$(p_{ab})_x=-im\Omega_{ab}x_{ab}. \quad (19)$$

Equation (18) then takes the form

$$(\alpha_{ab})_x=-\frac{i\Omega_{ab}}{c}x_{ab},\quad(\alpha_{ab})_y=0,\quad(\alpha_{ab})_z=-i\beta_\parallel\varkappa_xx_{ab}, \quad (20)$$

$$\omega_{ab}=\Omega_{ab}+\frac{c}{\hbar}[(p_{bz}^2+m_0^2c^2)^{1/2}-((p_{bz}-\hbar\varkappa_z)^2+\hbar^2\varkappa_y^2+m_0^2c^2)^{1/2}], \quad (21)$$

where $\beta_\parallel=cp_{bz}/E$ is the ratio of the particle's longitudinal velocity to the velocity of light. Assuming that the energy of the quanta and the transverse energy are much less than the total energy of the particle, Eq. (21) can be written approximately with accuracy to terms of order $(\hbar\varkappa/p)^2$ in the form

$$\omega_{ab}=\Omega_{ab}+c\varkappa\beta_\parallel\cos\theta, \quad (22)$$

where θ is the angle between $\varkappa$ and the z axis, i.e., between the direction of radiation and the z axis.

Taking into account the δ function in the expression for the probability (12) we obtain the Doppler formula for the frequency of radiation,

$$\omega=\Omega_{ab}/(1-\beta_\parallel\cos\theta). \quad (23)$$

We take the linear polarization vectors such that β_1 lies in the $\varkappa$, z plane. Then β_2 will be perpendicular to this plane and will lie in the x, y plane. Then, taking into account that $(\alpha_{ab})_y=0$, we have

$$\Phi_1=|(\alpha_{ab})_x\cos\theta\cos\zeta-(\alpha_{ab})_z\sin\theta|^2,\quad\Phi_2=|(\alpha_{ab})_x|^2\sin^2\zeta,$$

where ζ is the polar angle of the vector $\varkappa$ in the x, y plane. Using Eqs. (20) and (23), we obtain

$$\Phi_1=x_{ab}^2\Omega_{ab}^2c^{-2}\cos^2\zeta(\cos\theta-\beta_\parallel)^2/(1-\beta_\parallel\cos\theta)^2, \quad (24)$$

$$\Phi_2=x_{ab}^2\Omega_{ab}^2c^{-2}\sin^2\zeta. \quad (25)$$

From these relations it can be seen that the radiation observed in the x, z plane is linearly polarized in this same plane; radiation observed in the y, z plane, parallel to the channel planes, is linearly polarized perpendicular to this plane. In the remaining directions the radiation is elliptically polarized. In the x, z plane in the direction $\theta = \cos^{-1}\beta_{\parallel}$, there is no radiation.

Let us consider first spontaneous radiation. Replacing the summation by integration in accordance with the well known rule in Eq. (12), we obtain

$$w_{ab} = \frac{ce^2}{2\pi\hbar}\int \Phi \frac{\delta(\omega_{ab}-\omega)}{\varkappa} d^3\varkappa \qquad (26)$$

where $\Phi = \Phi_1 + \Phi_2$ is the sum over the polarizations:

$$\Phi = \frac{x_{ab}^2 \Omega_{ab}^2}{c^2}\left[\cos^2\zeta \left(\frac{\cos\theta - \beta_{\parallel}}{1-\beta_{\parallel}\cos\theta}\right)^2 + \sin^2\zeta\right]. \qquad (27)$$

Making use of Eq. (22), we obtain the angular distribution of the spontaneous radiation:

$$\frac{dw_{ab}}{d\Omega} = \frac{e^2 x_{ab}^2 \Omega_{ab}^3 \{(1-\beta_{\parallel}\cos\theta)^2 - (1-\beta_{\parallel}^2)\sin^2\theta\cos^2\zeta\}}{2\pi\hbar c^2 (1-\beta_{\parallel}\cos\theta)^4}. \qquad (28)$$

The intensity distribution of the spontaneous radiation is then

$$\frac{dI_{ab}}{d\Omega} = \frac{e^2 x_{ab}^2 \Omega_{ab}^4 \{(1-\beta_{\parallel}\cos\theta)^2 - (1-\beta_{\parallel}^2)\sin^2\theta\cos^2\zeta\}}{2\pi c^3 (1-\beta_{\parallel}\cos\theta)^5}. \qquad (29)$$

The total intensity of spontaneous radiation I_{ab} and the probability A_{ab} are found after integration over $d\Omega$:

$$I_{ab} = 4e^2 x_{ab}^2 \Omega_{ab}^4 \gamma^4 / 3c^3, \qquad (30)$$

$$A_{ab} = 4e^2 x_{ab}^2 \Omega_{ab}^3 \gamma^2 / 3\hbar c^3. \qquad (31)$$

The intensity differs from the usual dipole radiation formula by the presence of the relativistic factor γ^4. However, the energy levels $\mathscr{E}_a$ and $\mathscr{E}_b$ and consequently also Ω_{ab} depend on energy and therefore the dependence of the intensity on γ is found to be weaker. Similarly, by integrating over angles in Eq. (26), we obtain the frequency distribution of the probability w_{ab} and intensity I_{ab}:

$$\frac{dw_{ab}}{d\omega} = \frac{3I_{ab}}{\hbar\omega_m^2}\left[1 - 2\left(\frac{1+\beta_{\parallel}}{1+\beta_{\parallel}^2}\right)\frac{\omega}{\omega_m} + \frac{(1+\beta_{\parallel})^2}{1+\beta_{\parallel}^2}\left(\frac{\omega}{\omega_m}\right)^2\right],$$

$$\frac{dI_{ab}}{d\omega} = \frac{3I_{ab}\omega}{\omega_m^2}\left[1 - 2\left(\frac{1+\beta_{\parallel}}{1+\beta_{\parallel}^2}\right)\frac{\omega}{\omega_m} + \frac{(1+\beta_{\parallel})^2}{1+\beta_{\parallel}^2}\left(\frac{\omega}{\omega_m}\right)^2\right],$$

where $\omega_m = \Omega_{ab}/(1-\beta_{\parallel})$ is the maximal frequency of radiation. The probability distribution has the form of a parabola symmetric about the minimum, which occurs at the point $\omega_m/(1+\beta_{\parallel}) \approx \omega_m/2$. The probability is maximal at $\omega = 0$ and $\omega = \omega_m$ is equal to $3I_{ab}/\hbar\omega_m^2$ at $\beta_{\parallel} \approx 1$, while at the point of the minimum it is smaller by a factor of two. The intensity also has a maximum at $\omega = \omega_m$ and an inflection point at $\omega \approx \omega_m/3$.

One goes over to the classical formulas from Eqs. (29) and (30) in the usual manner[6] by replacement of the dipole matrix element $4x_{ab}^2 \to d_n^2$ and of the frequency $\Omega_{ab} \to \omega_n$, where d_n and ω_n are the amplitude and frequency of harmonic oscillation of the particle. Here we obtain expressions for the intensity which coincide with the results of Refs. 1–3.

The expressions obtained also coincide with the formulas for radiation of a moving oscillator.[7] The quasiclassical nature of the problem in this case is due, as usual, to the condition $\hbar\omega \ll E$. Equation (30) can be written in the form

$$I_{ab} = 4e^2 \Omega_0^4 x_{ab}^2 / 3c^3, \qquad (32)$$

where $\Omega_0 = \Omega_{ab}/(1-\beta_{\parallel}^2)^{1/2}$ is the frequency in the moving reference frame. It is usually assumed that the latter does not depend on the motion of the reference frame. Then the intensity I_{ab} does not depend on the longitudinal motion of the oscillator, since the transverse dipole moment does not depend on the velocity. In the case of channeling, however, in the transition to the moving reference frame the potential of the atomic planes increases by a factor γ. For a harmonic potential the frequency increases by a factor $\gamma^{1/2}$, i.e., $\Omega_0 \sim \gamma^{1/2}$, $\Omega_{ab} \sim 1/\gamma^{1/2}$, and correspondingly $I_{ab} \sim \gamma^2$.[1-3] This is valid for channeling of positrons, since in this case the potential is close to harmonic (at least near the center of the channel). The transition probability in the moving frame

$$A_0 = A_{ab}\gamma = 4e^2 \Omega_0^3 x_{ab}^2 / 3\hbar c^3 \qquad (33)$$

in this case is proportional to $\gamma^{3/2}$, and the fixed frame we have $A_{ab} \propto \gamma^{1/2}$. These conclusions may change when refraction of light in the crystal is taken into account. For a refracting medium it is necessary to use the dispersion relation $\varkappa = n\omega/c$, i.e., to replace the velocity of light c by c/n, where n is the refractive index.[8] In this case the Doppler formula, as usual, is replaced by

$$\omega = \Omega_{ab}/(1-\beta_{\parallel} n \cos\theta), \qquad (34)$$

and the formula for the spontaneous radiation intensity becomes the following (c^2 in the denominator is due to the relativistic motion of the particle, so that it is not replaced by c^2/n^2):

$$\frac{dI_{ab}}{d\Omega} = \frac{e^2 \Omega_{ab}^4 x_{ab}^2 n \{(1-\beta_{\parallel} n\cos\theta)^2 - (1-\beta_{\parallel}^2 n^2)\sin^2\theta\cos^2\zeta\}}{2\pi c^3 |1-n\beta_{\parallel}\cos\theta|^5} \qquad (35)$$

Hence the total power is

$$I_{ab} = 4e^2 \Omega_{ab}^4 x_{ab}^2 n / 3c^3 (1-\beta_{\parallel}^2 n^2)^2 \qquad (36)$$

and the probability is

$$A_{ab} = 4e^2 \Omega_{ab}^3 x_{ab}^2 n / \hbar c^3 (1-\beta_{\parallel}^2 n^2). \qquad (37)$$

The last two formulas are valid only for $n\beta_{\parallel} < 1$. In the region of velocities greater than the velocity of light in the medium, i.e., for $n\beta_{\parallel} > 1$, it is necessary in obtaining similar formulas to take into account the dispersion of the medium in integration over angles or frequencies, where the denominator in Eq. (35) goes to zero (i.e., near the Čerenkov radiation angle). We do not give these formulas, since they are very cumbersome. Note, however, that Eq. (35) remains valid in this case; it is sufficient to take into account in it the dependence $n(\omega)$.

We see that without taking into account dispersion for $\beta_{\parallel} n < 1$ the intensity depends on energy as $\gamma^2/[n^2 + \gamma^2$

$(1-n^2)]^2$, i.e., it increases for $n>1$ and decreases for $n<1$, $\gamma>n/(1-n^2)^{1/2}$. In the case $\beta_\parallel n>1$ it is necessary to take into account dispersion and the dependence on energy becomes still more complicated. In the x-ray and γ-ray regions, in which the particles begin to radiate at energies of the order of tens of meV, Eq. (30) is valid, i.e., the intensity increases as γ^2.

When dispersion is taken into account, the Doppler effect becomes complicated. This question has been discussed in detail in papers by Frank[7] and Ginzburg[9] and recently in papers by Baryshevskiĭ.[10] The radiation of the anomalous Doppler branch, for which $\beta_\parallel n\cos\theta>1$, occurs mainly in the optical region, where $n>1$, and in a narrow frequency region near the x-ray absorption edges (as follows from Eq. (34), in this case $\Omega_{ab}<0$, i.e., the transverse energy increases). The radiation in the normal Doppler branch ($n\beta_\parallel\cos\theta<1$) is most interesting in the high-frequency region where $n\approx1$, since the intensity, as follows from Eqs. (36) and (37), will then increase with energy.

We now consider induced radiation under the influence of a monochromatic wave with wave vector $\varkappa$ and polarization λ. From Eq. (12) we find

$$w_{ab}=\frac{4\pi^2e^2N_\lambda(\varkappa)c\delta(\omega_{ab}-\omega)}{L^3\hbar\varkappa}\Phi_\lambda(\varkappa). \tag{38}$$

Taking into account that $N_\lambda/L^3=I_\lambda/c$, where I_λ is the photon flux intensity, we obtain the cross section for induced radiation:

$$\sigma_{ab}=\frac{4\pi^2e^2\delta(\omega_{ab}-\omega)}{\hbar\varkappa}\Phi_\lambda(\varkappa). \tag{39}$$

Let us see in what directions the maximum of the induced radiation will occur. It follows from Eq. (24) that the maximum of radiation polarized in the $\varkappa, z$ plane lies at $\theta=0$. However, the frequency depends on θ. For a fixed θ the maximal radiation will be in the y, z plane, i.e., parallel to the atomic planes of the channel, and will be polarized perpendicular to them. For a level of finite width Γ it is necessary in Eq. (39) to make the substitution

$$\delta(\omega_{ab}-\omega)\to\frac{1}{2\pi}\frac{\Gamma}{(\omega_{ab}-\omega)^2+\Gamma^2/4}. \tag{40}$$

For radiation in the y, z plane we obtain

$$\sigma_{ab}=\frac{4\pi^2e^2x_{ab}^2\Omega_{ab}^2}{\hbar c\omega}\frac{1}{2\pi}\frac{\Gamma}{(\omega_{ab}-\omega)^2+\Gamma^2/4}. \tag{41}$$

Expressing x_{ab}^2 by means of Eq. (31) in terms of A_{ab}, we obtain

$$\sigma_{ab}=\frac{3\pi c^2A_{ab}\Gamma}{2\omega\Omega_{ab}\gamma^2[(\omega_{ab}-\omega)^2+\Gamma^2/4]}. \tag{42}$$

At resonance ($\omega=\omega_{ab}$, i.e., the Doppler condition (23) is satisfied)

$$\sigma_{ab}{}^0=12\pi c^2A_{ab}/\omega\omega_m\Gamma, \tag{43}$$

where $\omega_m=2\Omega_{ab}\gamma^2$. Equation (43) can be rewritten in a form suitable for use in a medium with $n\neq1$:

$$\sigma_{ab}{}^0=\frac{3}{\pi}\lambda\lambda_{min}\frac{A_{ab}}{\Gamma}, \tag{44}$$

where $\lambda_{min}=\pi c/\gamma^2\Omega_{ab}$. The enhancement coefficient G will be equal[11a] to $\sigma_{ab}^0\Delta N_e$, or

$$G=\frac{3}{\pi}\lambda\lambda_{min}\Delta N_e\frac{A_{ab}}{\Gamma}, \tag{45}$$

where ΔN_e is the population inversion of the levels of the transverse energy of the beam. It follows from this that it is necessary to take the minimal particle energy at which it is possible to obtain radiation of a given frequency ω (at an angle $\theta\approx0$). With increase of the energy, this wavelength will be radiated at a larger and larger angle, while λ_{min} will be decreased. However, the ratio A_{ab}/Γ depends on energy. In radiation of energetic quanta ($n\approx1$) it will increase with energy. The probability of a spontaneous transition increases as $\gamma^{1/2}$ with energy. The level width Γ is due to ionizing collisions and radiation loss. The frequency of Coulomb collisions falls off as $1/\gamma$, while the radiative width is approximately constant. Therefore up to energies $800(Z+1)$ MeV the value of Γ drops, and beyond this it remains constant. Further analysis of the possibilities of the use of induced radiation will be carried out in the next section.

In conclusion we mention briefly the difference of Eq. (43) from the well known formula for the cross section for induced radiation[11b]

$$\sigma_{ab}{}^0=c^2A_{ab}/2\pi\nu^2\Gamma,\quad \nu=\omega/2\pi. \tag{46}$$

If we set $\beta_\parallel=0$, $\gamma=1$, then it follows from Eq. (43) that

$$\sigma_{ab}{}^0=\frac{3}{2\pi}\left(\frac{c}{\nu}\right)^2\frac{A_{ab}}{\Gamma}$$

The difference is due to the definite orientation of the dipole and the polarization. Averaging over polarizations and directions of Eq. (39) gives immediately a factor 1/3.

3. RADIATION IN AXIAL CHANNELING

In the case of motion of particles in a two-dimensional string potential having axial symmetry, the wave function $\varphi(\mathbf{r})$ satisfies a two-dimensional Schrödinger equation analogous to (9) (well channeled particles are considered). Separating variables, it is easy to show that $\varphi(\mathbf{r})$, where q is the azimuthal angle and l is the orbital quantum number. In this case for $\Delta l=\pm1$ we obtain

$$(\alpha_{ab})_x=-i\Omega_{ab}r_{ab}/2c,\quad r_{ab}=\int\varphi_a{}^*(r)\varphi_b(r)r^2\,dr, \tag{47}$$

$$(\alpha_{ab})_y=\mp\Omega_{ab}r_{ab}/2c, \tag{48}$$

$$(\alpha_{ab})_z=-\frac{\beta_\parallel r_{ab}}{2}(\pm\varkappa_y+i\varkappa_x). \tag{49}$$

For right-handed and left-handed circular polarization the quantity $\Phi_\pm$ has the respective form

$$\Phi_{\pm1}=\frac{1}{2}[\varkappa_0\alpha_{ab}{}^*][\varkappa_0\alpha_{ab}]\mp\frac{i}{2}(\varkappa_0[\alpha_{ab}{}^*\alpha_{ab}]), \tag{50}$$

where $\varkappa_0$ is the unit vector specifying the direction of the radiation. The sum over polarizations is

$$\Phi=\frac{\Omega_{ab}{}^2\mathbf{r}_{ab}{}^2}{c^2}\left\{1+\frac{1}{2}\sin^2\theta\left[\left(\beta_\perp\frac{\omega}{\Omega_{ab}}\right)^2-\left(1+\beta_\parallel\frac{\omega}{\Omega_{ab}}\cos\theta\right)^2\right]\right\},$$

$$\mathbf{r}_{ab}{}^2=x_{ab}{}^2+y_{ab}{}^2=r_{ab}{}^2/2 \qquad (51)$$

and obviously is the sum of two expressions (27) for oscillations along the x and y axes with $x_{ab}^2=y_{ab}^2$. For circular polarization j we have

$$\Phi_j=\frac{\Omega_{ab}{}^2\mathbf{r}_{ab}{}^2(1\mp j\cos\theta)^2(1\pm j\beta_\parallel)^2}{4c^2(1-\beta_\parallel\cos\theta)^2}. \qquad (52)$$

Expression (51) can be rewritten in the form

$$\Phi=\frac{\Omega_{ab}{}^2\mathbf{r}_{ab}{}^2\{(1-\beta_\parallel\cos\theta)^2-(1-\beta_\parallel{}^2)\sin^2\theta/2\}}{c^2(1-\beta_\parallel\cos\theta)^2}. \qquad (53)$$

From this, substituting (53) into (26), we obtain the probability

$$\frac{dw_{ab}}{d\Omega}=\frac{e^2\mathbf{r}_{ab}{}^2\Omega_{ab}{}^3\{(1-\beta_\parallel\cos\theta)^2-(1-\beta_\parallel{}^2)\sin^2\theta/2\}}{2\pi\hbar c^3(1-\beta_\parallel\cos\theta)^4} \qquad (54)$$

and intensity of spontaneous radiation in the axial case,

$$\frac{dI_{ab}}{d\Omega}=\frac{e^2\Omega_{ab}{}^4r_{ab}{}^2\{(1-\beta_\parallel\cos\theta)^2-(1-\beta_\parallel{}^2)\sin^2\theta/2\}}{2\pi c^3(1-\beta_\parallel\cos\theta)^5}. \qquad (55)$$

As can be seen from (52), circularly polarized waves are radiated forward and backward and the direction of polarization is determined by the sign of ΔL, i.e., by the change in orbital angular momentum.

The last formula for the angular distribution of the intensity, after appropriate substitutions, also coincides, naturally, with the classical formula for dipole radiation of a particle on motion along a helical trajectory.[12]

Discussion of the axial channeling of positrons requires in the general case use of an axially asymmetric potential, but from comparison of (28) and (54) it is clear that in the most general case

$$\frac{dw_{ab}}{d\Omega}=\frac{e^2\Omega_{ab}\omega}{2\hbar\pi c^3}\left\{|\mathbf{r}_{ab}|^2-\frac{(1-\beta_\parallel{}^2)\sin^2\theta}{(1-\beta_\parallel\cos\theta)^2}|\mathbf{r}_{ab}\varkappa_0|^2\right\} \qquad (56)$$

$$\frac{dI_{ab}}{d\Omega}=\frac{e^2\Omega_{ab}\omega^3}{2\pi c^3}\left\{|\mathbf{r}_{ab}|^2-\frac{(1-\beta_\parallel{}^2)\sin^2\theta}{(1-\beta_\parallel\cos\theta)^2}|\mathbf{r}_{ab}\varkappa_0|^2\right\}. \qquad (57)$$

An error made by other authors[13,14] is apparently that they do not consider the longitudinal current $(\alpha_{ab})_z$ but set $\Phi_\lambda=\Omega_{ab}c^{-1}|\mathbf{r}_{ab}\beta_\lambda|^2$, which in this case is valid only if there is no longitudinal motion, i.e., $\beta_\parallel=0$. However, here there is also no channeling.

Terhune and Pantell[15] in a recent article also discuss this radiation and have confirmed our earlier results.[1-3] They state that Schiff[16] discussed similar radiation. However, analysis of Schiff's work shows that he discussed the inapplicability of the Born approximation in the theory of coherent bremsstrahlung. Coherent bremsstrahlung differs substantially from the radiation of a channeled particle.[1-3] The properties of coherent radiation were investigated more than twenty years ago.[17-21] Walker *et al.*[22] and Tomesasu *et al.*[23] also discuss the results from the point of view of the theory of coherent radiation. However, it must be added that introduction of the concept of channeling and observation of this effect occurred after the publication of Schiff's work.

The main results obtained by us differ fundamentally from those of Vorob'ev *et al.*[13,14] In the first place, they assert that the intensity of radiation decreases as the square of the relativistic mass of the electrons and this condition limits the radiation of high energy electrons. In the second place, they also state that the radiated frequency decreases with increase of the particle energy. The angular distribution is assumed equal to the distribution of a stationary dipole. It is also assumed that positrons cannot radiate. The fact is that the intensity of radiation in the optical region depends in a complicated way on the energy (see the text of our article after Eq. (37)). In the very interesting case of radiation in the x-ray and γ-ray regions the intensity increases as the square of the energy, i.e., in our calculations it is higher by a factor γ^4. This gives a difference, for example, for $E=1$ GeV of thirteen orders of magnitude. The mistake made by these authors is apparently explained by the fact that in using for their intensity calculation the formula for a stationary dipole (see Eq. (3) in Ref. 13) they assumed a decrease in the frequency of oscillation as the result of the relativistic mass increase. In that case the intensity falls off as γ^{-2}.

This discussion, however, is not valid, since, in contrast to a dipole, a channeled particle moves in an external field and in the transition to a moving frame (where the longitudinal velocity of the particle is equal to zero) the potential barrier increases by a factor of γ.

An underestimate of the Doppler effect for the channeled particle led to erroneous equations (9) and (10) in Ref. 13 for the probability and frequency of radiation and to the neglect in that work of the possibility of a Doppler shift of frequencies in the x-ray and γ-ray regions at high particle energies.

Vorobiev *et al.*[13] measured the radiation of photons in the visible region on the passage of 2.26-MeV electrons through a single crystal of rock salt under conditions of channeling. The authors assume that the increase in radiation intensity in the visible region for motion of the beam along the axis is due to spontaneous transitions of channeled particles and agrees with their theoretical predictions. However, the angular distribution of photons falls off in their case with increase of the angle of radiation, while according to our theory it should rise, since the radiation occurs under conditions of the anomalous Doppler effect (see our formula (55)). In addition, for alignment of the direction of beam motion along the ⟨100⟩ axis of the single crystal, Vorobiev *et al.*[13] observed a maximum photon intensity, whereas it is known from the experiments of these same authors[24] on passage of electrons that under these conditions a minimum of channeled electrons is observed. Estimates show that under the conditions of this experiment the background of Čerenkov radiation from δ electrons was much greater than the intensity of spontaneous radiation under conditions of channeling. The observed increase in photon intensity under channeling conditions was due not to the mechanism discussed by us but, apparently, to the ordinary increase of bremsstrahlung on channeling of electrons, which is well known.[25]

4. ESTIMATES OF ENHANCEMENT COEFFICIENT

For simple estimates the enhancement coefficient can be taken in the form $G \approx \lambda^2 \Delta N_e / \tau\Delta\omega$, where $\Delta\omega$ is the real width of the line (λ_{min} can be made close to λ by choice of the particle energy). Let us first estimate $\tau\Delta\omega$ with allowance for broadening. In our case it is due to the following factors: multiple scattering, beam nomonochromaticity, and band broadening as the result of lattice periodicity. The value of τ can be estimated classically.[3] Here for positrons with $E = 100$ MeV in a $\{110\}$ channel of silicon we obtain $\tau \approx 5\times10^{-11}$ sec.

Broadening of a line as the result of multiple scattering can be estimated from the formula $\Delta\omega = \beta_\parallel c/x_{1/2}$, where $x_{1/2}$ is the length for dechanneling of half of the particles. Another estimate of $\Delta\omega$ by means of the coefficient of absorption of particles from the beam due to inelastic scattering by electrons, derived by Ohtsuki[26] in complex potential theory, gives similar results. For positrons in silicon in this case for $E \approx 1$ MeV we have $x_{1/2} \approx (10-20)\ \mu$ and $\Delta\omega \approx 10^{13}$ sec^{-1}. It is evident from this that $\tau\Delta\omega \approx 10^2$ at low energies $E \approx 1$ MeV. However, for $E \approx 1$ GeV a value $x_{1/2} \gtrsim 1\ \mu$ is observed, which gives $\Delta\omega \approx 10^{11}$ sec^{-1} and $\tau\Delta\omega \approx 1$. For electrons with $E = 3$ MeV in the $\langle 111 \rangle$ direction of tungsten $x_{1/2} \gtrsim 1\ \mu$, which means $\Delta\omega \approx 10^{14}$ sec^{-1}. Electrons and heavy crystals are poorer in this respect as the result of more rapid dechanneling. Beam nomonochromaticity also produces broadening of a line as the result of Doppler effect, and here $\Delta\omega \approx \Omega_{ab}\Delta\gamma/\gamma$. It is evident from this that for $\Delta\gamma/\gamma \lesssim 10^{-4}$ (which is presently achievable[27]) $\Delta\omega \approx 10^{11}$ sec^{-1}. For transverse energies close to the barrier height U_{max} for relativistic particles with mass γm_0 the broadening can amount to $\hbar\Delta\omega \approx 10^{-2}-10^{-3}$ eV. However, for well channeled particles with $E_\perp \ll U_{max}$ the bands degenerate into discrete lines and the broadening as the result of this factor can be neglected. It can be seen from this analysis that the principal mechanism of broadening at high energies is beam nomonochromaticity, but it is possible already at the present time to obtain a value $\tau\Delta\omega \approx 10^2$ in the optical region and $\tau\Delta\omega \approx 1$ in the x-ray region. Hence it follows that to obtain $GL > 1$, where L is the length of the enhancement zone (or the path of the photons before absorption or scattering), is necessary, for $\lambda = 10^4-10^3$ Å, to have $\Delta N_e = 10^{10}-10^8$ cm^{-3} for $L \approx 1$ cm, i.e., with a current density $\gtrsim 10^4$ A/cm^2 it is possible to provide enhancement over a wide region of wavelengths up to $\lambda \sim 10^3$ Å. (In the case of channeling the difference in populations can amount to about one tenth of the beam density.)

At the present time electron current densities up to 10^8 A/cm^2 with energy ~10 MeV are achieved. However, at these particle energies a crystal can withstand beams only up to 10^6 A/cm^2. It is possible by means of storage rings to obtain such beam current densities even in the energy region of the order 1 GeV. If it is possible to achieve good beam monochromaticity ($\Delta\gamma/\gamma < 10^{-4}$), then it will be possible to obtain enhancement in the x-ray frequency region at currents ~10^8 A/cm^2. Heating and destruction of the crystal can be avoided by rapid scanning of the electron beam.

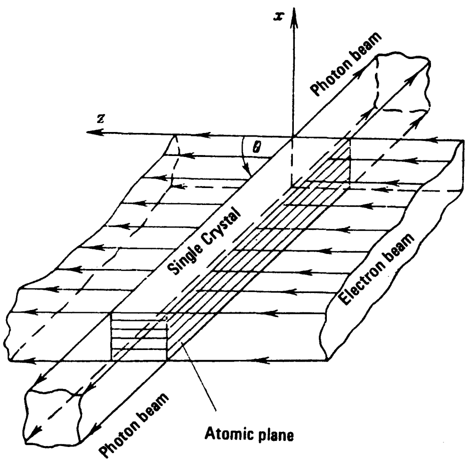

FIG. 1. Diagram of experiment.

An advantage of the lasers proposed is the possibility of retuning the generated frequency. This is achieved by simple rotation of the crystal with respect to the incident-beam direction (see the figure). Change in frequency can be obtained also by change of the crystallographic channeling direction and by replacement of the target (targets with larger Z correspond to higher frequencies). At high currents we have $GL \gg 1$, so that it is possible to work without mirrors. In order to enhance only the radiation forward and to reduce the effect of the beam on the crystal, it is necessary to scan the beam along crystallographic planes.

It should be noted that there is a definite analogy between enhancement of the radiation of channeled particles and the excitation of electrons in a magnetic field. The latter mechanism has been discussed in detail by Ginzburg and Frank,[28] who showed the instability of the motion of electrons in motion faster than the speed of light in a medium (the anomalous Doppler effect). In the case of channeling the enhancement is, in addition, also possible with the normal Doppler effect.

5. COMPARISON WITH SYNCHROTRON RADIATION

We shall present quantitative evaluations of the flux of spontaneous radiation of a channeled particle and compare them with the synchrotron radiation of the Pakhra accelerator now being built in the USSR[29] and with the DESY synchrotron in Germany.

TABLE I. Number of photons radiated at wavelength $\lambda = 1$ Å.

Particle energy	$N_\lambda \cdot 10^{-10}$, $\frac{\text{photons}}{\text{Å-sec-mA-mrad}}$	Remarks
	Pakhra synchrotron	
1.3 GeV	0.05	Projection mode
	2100	Storage mode with local curvature of orbit (wiggler)
	DESY synchrotron	
7.5 GeV	5300	
	Channeling	
150 MeV	10^9	Channeling of electrons in [110] direction in silicon crystal

Consider the probability of radiation of a photon forward, i.e., at $\theta=0$. From Eq. (56) we obtain

$$dw_{ab}/d\Omega=e^2\omega^2\Omega_{ab}r_{ab}^2/8\pi\hbar c^3. \quad (58)$$

The relative width of the spectral line, which was evaluated in the preceding section, we will take as of the order 10^{-4}. For a silicon single crystal the quantities are $r_{ab}\approx 0.3$ Å and $\Omega_{ab}\approx 10^{16}$ sec^{-1}. Proceeding from these values and choosing a crystal thickness of the order of 1 mm, we obtain the values given in the table. As can be seen from this table, for the same beam current at $\lambda=1$ Å we obtain an advantage of 5–11 orders of magnitude, depending on the mode of synchrotron operation. Here we must keep in mind two aspects: First, acceleration of the particles to significantly lower energies is required; second, it is possible to obtain radiation of shorter wavelengths $\lambda<1$ Å with still higher efficiency (see Eq. (58)), which is completely impossible with contemporary synchrotrons.

[1]M. A. Kumakhov, Phys. Lett. 57A, 17 (1976).
[2]M. A. Kumakhov, Dokl. Akad. Nauk SSSR 230, 1077 (1976) [Sov. Phys. Doklady 21, 581 (1976)].
[3]M. A. Kumakhov, Zh. Éksp. Teor. Fiz. 72, 1489 (1977) [Sov. Phys. JETP 45, 781 (1977)].
[4]A. A. Sokolov and I. M. Ternov, Relyativistskiĭ élektron (The Relativistic Electron), Nauka, 1974, Section 18, p. 239.
[5]V. Bazylev and N. Zhevago, Preprint IAÉ-2865, Institute of Atomic Energy, 1977; Zh. Éksp. Teor. Fiz. 73, 1697 (1977) [Sov. Phys. JETP 46, 891 (1977)].
[6]V. B. Berestetskiĭ, E. M. Lifshits, and L. P. Pitaevskiĭ, Relyativistkaya kvantovaya teoriya (Relativistic Quantum Theory), Part 1, Nauka, 1968, Section 45, p. 188. English trans., Addison. Wesley, 1971.
[7]I. M. Frank, Izv. AN SSSR, ser. fiz. 6, 3 (1942).
[8]V. L. Ginzburg, Teoreticheskaya fizika i astronomiya (Theoretical Physics and Astronomy), Nauka, 1975, Chapter 6.
[9]V. L. Ginzburg, Usp. Fiz. Nauk 69, 537 (1959).
[10]V. G. Baryshevskiĭ, Dokl. Akad. Nauk SSSR 231, 1335 (1976) [Sov. Phys. Doklady 21, 741 (1976)].
[11]A. L. Mikaélyan, M. L. Ter-Mikaélyan, and Yu. G. Turkov, Opticheskie generatory na tverdom tele (Solid-State Optical Generators), Sov. Radio, 1967, a) p. 16, b) p. 36.
[12]D. F. Alferov, Yu. A. Bashmakov, and E. G. Bessonov, Trudy FIAN 80, 100 (1975) [Proceedings of the Lebedev Institute].
[13]A. A. Vorobiev, V. V. Kaplin, and S. A. Vorobiev, Nucl. Instr. and Meth. 127, 265 (1975).
[14]S. A. Vorob'ev and V. V. Kaplin, Tezisy VII Vsesoyuznogo soveshchaniya po fizike vzaimodeĭstviya zaryazhennykh chastits s monokristallami (Abstracts of the Seventh All-Union Conf. on Physics of Interaction of Charged Particles With Single Crystals), Moscow State University, 1975, p. 41.
[15]R. W. Terhune and R. H. Pantell, Appl. Phys. Lett. 30, 265 (1977).
[16]L. I. Schiff, Phys. Rev. 117, 1394 (1960).
[17]M. L. Ter-Mikaélyan, Vliyanie sredy na élektromagnityne protsessy pri vysokikh énergiyakh (Influence of the Medium on Electromagnetic Processes at High Energies), AN Armenian SSR, Erevan, 1969.
[18]B. Ferreti, Nuovo Cimento 7, 118 (1950).
[19]H. Überall, Phys. Rev. 103, 1055 (1956).
[20]A. I. Akhiezer, P. I. Fomin, and N. F. Shul'ga, Pis'ma v Zh. Éksp. Teor. Fiz. 13, 713 (1971) [JETP Lett. 13, 506 (1971)].
[21]W. K. Panofsky and A. N. Saxena, Phys. Rev. Lett. 2, 219 (1959).
[22]R. L. Walker, B. L. Berman, and S. D. Bloom, Phys. Rev. A11, 736 (1975).
[23]T. Tomimasu, T. Mikado, and T. Yamazaki, Phys. Rev. B10, 2669 (1974).
[24]A. Y. Bobudaev, V. V. Kaplin, and S. A. Vorobiev, Phys. Lett. 45A, 71 (1973).
[25]K. A. Grishaev, B. M. Shramenko, G. L. Bochek, and G. D. Kovalenko, Trudy VI Vsesoyuznogo soveshchaniya po fizike vzaimodeĭstviya zaryazhennykh chastits s monokristallami (Proceedings of the Sixth All-Union Conf. on Physics of Interaction of Charged Particles With Single Crystals), Moscow State University, 1975, p. 256.
[26]Y. H. Ohtsuki, J. Phys. Soc. Japan 34, 473 (1973).
[27]H. Fleischmann, Phys. Today 28, No. 5, 34 (May 1975).
[28]V. L. Ginzburg and I. M. Frank, Dokl. Akad. Nauk SSSR 56, 583 (1947).
[29]D. F. Alferov, K. A. Belovintsev, L. V. Kelydysh, V. V. Mikhaĭlin, E. I. Tamm, and P. A. Cherenkov, Trudy FIAN, Nauka, 80, 125 (1975) [Proceedings of the Lebedev Institute].

Translated by Clark S. Robinson