Nonlinear theory of stationary strata in dissipative systems

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(Submitted 27 June 1977; resubmitted 26 October 1977)
Zh. Eksp. Teor. Fiz. 74, 1675–1697 (May 1978)

Examples of stratification of homogeneously heated electron-hole plasmas in semiconductors and in chemical substances involved in biochemical reactions (Türing model) are used to develop a nonlinear theory of nonequilibrium systems whose properties depend on two parameters with different spatial dispersion, i.e., the distribution of one of them is characterized by a small length l and that of the other by a large length L. Possible inhomogeneous stationary states of such systems are investigated. The stability of the stationary states is examined and it is established that spatially periodic distributions in the form of narrow (of the order of l) strata of high-temperature excitons (or enhanced concentration of the chemical material in the Türing model) separated by a distance of the order of L are stable. It is shown that the number of such strata in the specimen is determined by its geometric size and the level of excitation of the system. Several stable inhomogeneous states will appear for a given level of excitation, and the number of such states is found to increase with the length of the specimen. Thus, one or two strata may be formed at the ends of the specimen or at its center even when the specimen length is less than L. The formation and disappearance of a distribution with a given number of strata are not uniquely determined by the level of excitation, i.e., the process exhibits hysteresis. When stability is lost, the system goes over in a random fashion (depending on the type of fluctuation) into one of the stable states that it can support for a given level of excitation.

PACS numbers: 05.90.+m, 71.35.+z

1. INTRODUCTION

The homogeneity of many systems taken out of thermodynamic equilibrium by external disturbances is often found to be violated by fluctuations with a particular value of the wave vector k_0. This type of instability, i.e., stratification, is encountered in biological systems and in chemical reactions,[1] and also during the homogeneous heating of excitons in semiconductors,[2] electron-hole plasmas,[3] weakly ionized gases,[4] and semiconductor structures.[5,6] The common feature of all these systems is that their properties depend on two (or more) parameters, Θ and η, with different spatial dispersion, i.e., the lengths l and L that, respectively, characterize the distribution of Θ and η in space are very different ($L \gg l$). The phenomenon of stratification in such systems is connected with the spatial decoupling of the parameters.[5]1) In fact, the parameter η cannot follow a fluctuation $\delta\Theta$ with $k_0 \approx (lL)^{-1/2}$ because $k_0 L \gg 1$. In other words, for inhomogeneous fluctuations of this kind, the parameter η does not actually vary in space. Stratification sets in if, for given η, the fluctuation $\delta\Theta$ grows because of the presence of positive feedback in the system. The stratification process in the systems under consideration may be accompanied by the appearance of stationary, spatially periodic, structures.

We shall use the examples of stratification in electron-hole plasmas during homogeneous heating in the course of photo-generation processes, and in chemical materials involved in biochemical reactions, to develop a nonlinear theory of stratification, i.e., of the development of one-dimensional periodic distributions.

2. SPECIFICATION OF SYSTEMS AND BASIC EQUATIONS

When nonequilibrium electrons and holes produced in a semiconductor as a result of photoexcitation or injection combine to form bound states, the accompanying energy release $\Delta\epsilon$ is transferred mainly to the resulting excitons. Even at relatively low concentrations, the hot excitons succeed in thermalizing quite rapidly. In fact, the exciton-exciton collision time is $\tau_{ee} = (\sigma v_T n)^{-1}$, where v_T is the thermal velocity and $\sigma \approx \pi a_B^2$ is the exciton-exciton scattering cross section. Substituting typical values for the exciton Bohr radius, $a_B \approx 10^{-6} - 10^{-7}$ cm,[7] we find that, for exciton concentration $n = 10^{15}$ cm^{-3} and $T = 4.2$°K, the collision time is $\tau_{ee} = 10^{-8} - 10^{-9}$ sec. The relaxation time of the exciton kinetic energy is determined by the exciton-phonon interaction, and is greater by several orders of magnitude than the exciton-momentum relaxation time τ_p. For example, in germanium at $T \approx 4.2$°K, experimental data indicate that $\tau_p = 10^{-8} - 10^{-10}$ sec.[7,8]

It follows that the hydrodynamic approximation $\tau_\epsilon \gg \tau_{ee}$ is valid for exciton concentrations in the range $n \approx 10^{14} - 10^{15}$ cm^{-3}. The effective temperature T of the gas of noninteracting excitons and their concentration n for $\tau_{ee} > \tau_p \sim T^\alpha$ is then described by the following set of equations.[2]

$$\frac{\partial n}{\partial t} = \frac{\partial^2}{\partial x^2}[nD(T)] - \frac{n}{\tau_r} + R, \qquad (1)$$

$$\frac{3}{2}\frac{\partial}{\partial t}(nT) = \left(\frac{5}{2} + \alpha\right)\frac{\partial^2}{\partial x^2}[nTD(T)] - n\frac{T - T_0}{\tau_\epsilon} + W, \qquad (2)$$

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where

$$D(T)=D_0(T/T_0)^{1+\alpha},\quad \tau_\varepsilon=\tau_\varepsilon^0(T/T_0)^s$$

is the exciton diffusion coefficient and kinetic energy relaxation time, respectively, R and τ_r are the rates of exciton generation and the recombination time, T_0 is the lattice temperature, and $W=\Delta\epsilon R$ is the power input into the exciton system when electrons and holes are bound into excitons. We emphasize that Eqs. (1) and (2) are the fundamental hydrodynamic equations: the first of them expresses the conservation of the number of particles and the second expresses the energy balance in the system. These equations describe the properties of many physical systems such as gas-discharge[4] or electron-hole[3] plasmas.

In particular, Eq. (1) and (2) provide the exact description of the properties of heated quasineutral electron-hole plasma for equal effective masses of electrons and holes.[3] The presence of the Coulomb interaction between the carriers ensures that the conditions for the hydrodynamic approximation are satisfied at lower concentrations than for hot excitons. The essential point is that the momentum-relaxation time of hot carriers at low temperatures is, as a rule, an increasing function of T.

It will be convenient to rewrite the set of equations given by (1) and (2) in the form

$$\tau_r\frac{\partial}{\partial t}(\eta\Theta^{-1-\alpha})=L^2\frac{\partial^2\eta}{\partial x^2}-\eta\Theta^{-1-\alpha}+1, \tag{3}$$

$$\frac{3}{2}\tau_\varepsilon^0\frac{\partial}{\partial t}(\eta\Theta^{-\alpha})=l^2\frac{\partial^2}{\partial x^2}(\eta\Theta)-\eta\cdot\frac{\Theta-1}{\Theta^{1+\alpha+s}}+J, \tag{4}$$

where

$$\Theta=T/T_0,\quad \eta=nD(T)/n_0D_0=n\Theta^{1+\alpha}/n_0,\quad n_0=\tau_rR,$$
$$J=\Delta\varepsilon\tau_\varepsilon^0/T_0\tau_r,$$

and $L=(\tau_r D_0)^{1/2}$, $l=[(5/2+\alpha)D_0\tau_\epsilon^0]^{1/2}$ are the diffusion length and the energy mean free path (cooling) of hot carriers. It is important to note that, as a rule, $\tau_r \gg \tau_\epsilon$ so that $L\gg l$. (For example, in germanium, $\tau_r \approx 10^{-4}-10^{-6}$ sec and τ_ϵ does not exceed 10^{-7} sec[7,8] even for $T=4.2\,^\circ$K.

Consider the stratification of plasma while it is being heated in the absence of extraneous currents, i.e., the temperature and concentration gradients on the lateral surfaces of the specimen are zero.[3]2) Hence, we have

$$\frac{\partial\eta}{\partial x}(0,l_x)=\frac{\partial\Theta}{\partial x}(0,l_x)=0, \tag{5}$$

where l_x is the specimen length. By linearizing (3)–(5) in the inhomogeneous fluctuations. $\delta\eta$ and $\delta\Theta$, we can easily verify[2,3] that, for $l_x \gtrsim L\gg l$, the homogeneous distribution becomes aperiodically unstable with respect to fluctuations with

$$k_0=(lL)^{-1/2}(1+\alpha)^{1/4}(\alpha+s)^{1/4(s+2+2\alpha)}(1+\alpha+s)^{-1/4(s+3+2\alpha)},$$

when

$$\Theta>\Theta^0=(1+\alpha+s)/(\alpha+s)\quad (J>J_0=(\alpha+s)^{s-1}(1+\alpha+s)^{-s}). \tag{6}$$

Hence, it follows that the necessary condition for the stratification of the exciton gas (electron-hole plasma) is that $\alpha+s>0$.[2,3] This condition is satisfied at low enough temperatures when the electrons and holes dissipate their momenta on charged centers since, under these conditions, $\alpha=3/2$ and $s\geqslant -1/2$ for all hot-carrier energy dissipation mechanisms.[9,10]

From the standpoint of the general approach to the analysis of such systems, which we are attempting, the stratification of the electron-hole plasma can be elucidated as follows.[3] Suppose a fluctuation $\delta\Theta$, i.e., a temperature fluctuation (Fig. 1a), appears in a region of size of the order of $(lL)^{1/2}$. It then follows from (3) that $\delta\eta$ will change little (roughly by l/L), i.e., the parameter η will remain virtually constant in space for such local fluctuations. Moreover, it follows from (4) that, for a given η, the fluctuation $\delta\Theta$ will grow for $\Theta>\Theta^0$. Fluctuations with $k>k_0$ will be damped out because they produce large diffusion currents, and those with $k<k_0$ will decay because the feedback mechanism producing the growth of $\delta\Theta$ with $k=k_0$ is reduced as a result of the damping change in η.

The small variation in $\eta\sim nD(T)$ in the region of growing Θ is due to the fact that the heat flux ensures that hot carriers are removed from this hotter region, so that their concentration n in this region is reduced (Fig. 1a), and this, in turn, produces a diffusion current in the opposite direction. In other words, the thermal current of carriers is balanced by the diffusion current. The small change in η means that $nT^{1+\alpha}\approx$ const and, therefore, changes in the concentration and temperature are in antiphase but have the same characteristic length (Fig. 1a). We note that the condition $L\gg l$ is satisfied because $\tau_r\gg\tau_\epsilon$. The latter ensures that the carrier concentration n cannot follow the fluctuation in the temperature (see the first footnote above). How-

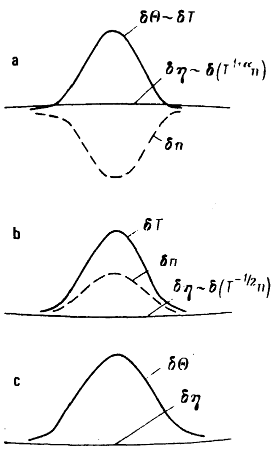

FIG. 1. Illustration of the stratification mechanism: the form of inhomogeneous fluctuations $\delta\Theta$ and $\delta\eta$ in a region of size $(L1)^{1/2}$ for a hot electron-hole plasma (a) and for plasma in thermodynamic equilibrium with the lattice (b); c—Türing model.

ever, the difference between the frequency dispersion of Θ and n does not result in stability.[3] In fact, it follows from (1) and (4) that the fluctuations $\delta\Theta$ are damped out for n =const, but are found to grow for $\eta \sim \eta\Theta^{1+\alpha}$. The latter condition is satisfied only because of the different spatial dispersion of η and Θ.

We emphasize that a change in the concentration, δn, need not necessarily be in antiphase with the change in temperature. Stratification may appear even when δn is in phase with δT (Fig. 1b). This situation occurs at higher temperatures when the exciton gas (or electron-hole plasma) comes to thermal equilibrium with the lattice, and the power supplied to the specimen under illumination is expended mainly by heating the lattice (phonons) whose temperature T is given by the thermal conduction equation averaged over the thickness of the specimen:

$$c\rho\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left[\varkappa(T)\frac{\partial T}{\partial x}\right]+\frac{nE_g}{\tau_r}-\frac{(T-T_0)\varkappa(T)}{l^2}, \tag{7}$$

where c, ρ, and $\varkappa$ are, respectively, the specific heat, density, and thermal conductivity of the lattice, E_g is the band gap (exciton energy), l is the characteristic length of temperature variation,[8,11] and, in contrast to (2), T_0 is the temperature of the ambient medium. At higher temperatures, the momentum of excitons (electrons and holes) is, as a rule, dissipated on phonons. Under these conditions, $\alpha=-3/2$ when excitons succeed in thermalizing with the lattice, i.e., $D(T) \sim T^{-1/2}$. The thermal conductivity of the lattice is a decreasing function of temperature above a certain sufficiently low temperature (for example, in the case of germanium for $T>20°$K), so that, to be specific, we shall suppose that $\varkappa(T)=\varkappa_0(T/T_0)^{-\beta}$, where $\beta\approx1$.[12] Substituting

$$\eta=n\left(\frac{T}{T_0}\right)^{1+\alpha},\quad \Theta=(1-\beta)(\varkappa_0T_0)^{-1}\int^T\varkappa(T')dT'=\left(\frac{T}{T_0}\right)^{1-\beta}, \tag{8}$$

in (7), we may rewrite it in the form

$$(1-\beta)\tau_T{}^0\frac{\partial}{\partial t}(\Theta^{1/(1-\beta)})=l^2\frac{\partial^2\Theta}{\partial x^2}+\frac{l^2E_g(1-\beta)}{\varkappa_0T_0\tau_r}\eta\Theta^{-(1+\alpha)/(1-\beta)}$$
$$-(1-\beta)(\Theta-\Theta^{-\beta/(1-\beta)}), \tag{9}$$

where $\tau_T^0=c\rho l^2\varkappa_0^{-1}$ is the characteristic time for lattice-temperature relaxation. It is easily verified[5] by linearizing (1) and (9) in the inhomogeneous fluctuations that, for $L\gg l$, the homogeneous distribution becomes unstable with respect to fluctuations with

$$k_0=(lL)^{-1/2}(\beta-\alpha-2)^{1/4(1+\alpha)}(\beta-\alpha-1)^{-1/4(2+\alpha)}(-\alpha-1)^{1/4}$$

for

$$J=R\tau_r>J_0=\tau_r\varkappa_0T_0(\beta-\alpha-2)^{\beta-1}(\beta-\alpha-1)^{-\beta}l^{-2}E_g{}^{-1},$$

when

$$\Theta>\Theta^0=(T^0/T_0)^{1-\beta}=(\beta-\alpha-1)^{1-\beta}(\beta-\alpha-2)^{\beta-1},$$

and it follows from this that the necessary condition for stratification in the present case is[3)] $\beta-\alpha>2$. This condition is satisfied for the system we are considering because $\alpha=-3/2$ and $\beta\approx1$. Here, the instability is due to the fact that the quantity $\eta\sim nD(T)$ in (8) remains, as in the previous case, practically constant in space for fluctuating $\delta\Theta$ with $k=k_0$ (Fig. 1b). However, since, in the present case, $D(T)\sim T^{-1/2}$, it follows from $\eta \sim \eta T^{-1/2}\approx$const that δn and δT are in phase (Fig. 1b).

We note that, in the above systems, the parameter $\eta\sim nD(T)$, which varies smoothly in space, has been introduced artifically, whereas the true physical variables, namely, the carrier concentration n and their temperature T vary in space with the same characteristic length (Fig. 1). Moreover, there are systems whose properties depend on two real physical parameters with different spatial dispersion. This situation occurs, for example, in heated semiconducting structures[5,6] and in biochemical reactions.[1,14]

Many biological systems in which the concentration of chemical materials exhibits stratification are described by the Türing equations. In the simplest case (assuming that the kinetic coefficients of the chemical reaction[1] are equal), these equations can be written in the form[15]

$$\tau\frac{\partial\eta}{\partial t}=L^2\frac{\partial^2\eta}{\partial x^2}+J\Theta-\Theta^2\eta, \tag{10}$$

$$\tau\frac{\partial\Theta}{\partial t}=l^2\frac{\partial^2\Theta}{\partial x^2}+A+\Theta^2\eta-(J+1)\Theta, \tag{11}$$

where η and Θ are the concentrations of the intermediate materials participating in the biochemical reactions, L and l are the corresponding diffusion lengths, A and J are the concentrations of the initial materials, which are held constant during the reaction, and τ is the characteristic time of the chemical reaction. In this particular model, the homogeneous distribution of the concentrations η and Θ of the intermediate materials for $L\gg l$ and the boundary conditions given by (5) turn out to be aperiodically unstable[1] with respect to fluctuations with

$$k_0=(lL)^{-1/2}A^{1/2}\quad\text{for}\quad\Theta>\Theta^0=2A(J+1)^{-1},$$

i.e., $J>J_0=1$ (practically independently of A for $1<A\ll L/l$[1,15]).

Thus, the phenomenon of stratification can be formulated in a unified fashion for a broad class of physical and biological systems whose properties depend on two parameters with different spatial dispersion. The stratification of such systems is due to the nonunique dependence of the rapidly-varying parameter Θ (with the smaller characteristic distribution length) on the level of excitation J of the system when the slowly-varying parameter η is fixed in space [this is clear, in particular, from (4), (9), and (11)]. The result of this is that the form of the stationary states, their stability, and other characteristic features are common for this class of systems. To be specific, we shall largely confine our discussion to heated electron-hole plasmas. The set of equations (10)–(11) corresponding to the Türing model then has the desirable feature of simplicity and can be used for illustration and comparison.

3. STATIONARY STATES AND THEIR STABILITY

The fact that the systems under consideration have the small parameter $\epsilon = l/L$ enables us to use the concept of "slow" and "fast" motion,[16] in our case, slowly and rapidly varying distributions, and to perform a qualitative analysis of the stationary states. These states are conveniently investigated as the phase paths in four-dimensional phase space which, as can be seen from (3) and (4), are described by the following set of equations:

$$\partial\rho/\partial x=\eta\Theta^{-1-\alpha}-1, \quad \partial\eta/\partial x=\rho, \tag{12}$$

$$\varepsilon\frac{\partial\mu}{\partial x}=\eta\frac{\Theta-1}{\Theta^{1+\alpha+s}}-J, \quad \varepsilon\frac{\partial(\eta\Theta)}{\partial x}=\mu, \tag{13}$$

where x is measured in units of L. According to general theory,[16] when $\epsilon \ll 1$, all the phase paths of this system are close to the path corresponding to the slowly-varying or rapidly-varying distributions, and to their combinations.

The slowly-varying distributions correspond to solutions with $\epsilon = 0$. The characteristic length for such distributions is unity (L). It then follows from (12) and (13) that the connection between η and Θ is local (Fig. 2a):

$$\eta=J\Theta^{1+\alpha+s}(\Theta-1)^{-1}, \tag{14}$$

and the distribution of η is described by

$$\frac{\partial^2\eta}{\partial x^2}+\frac{dU_\eta}{d\eta}=0, \text{ where } U_\eta=\int\left(1-J\frac{\Theta^s(\eta)}{\Theta(\eta)-1}\right)d\eta. \tag{15}$$

Rapidly-varying distributions correspond to solutions of (12) and (13) for $\eta(x) = \text{const}$. The characteristic length for such distributions is ϵ – (l) and the form of $\Theta(x)$ is determined by

$$\varepsilon^2\frac{\partial^2\Theta}{\partial x^2}+\frac{dU_\Theta}{d\Theta}=0, \quad U_\Theta=\int^\Theta\left(\frac{J}{\eta}-\frac{\Theta-1}{\Theta^{1+\alpha+s}}\right)d\Theta, \quad \eta=\text{const}. \tag{16}$$

Equations (15) and (16) have the form of the equations for the conservative motion of particles in fields described by potentials U_η and U_Θ, respectively. We emphasize that a similar situation occurs for other systems of this kind. Thus, in Türing's model, the equations for the slowly-varying distributions are

$$\eta=\frac{(J+1)\Theta-A}{\Theta^2}, \quad \frac{\partial^2\eta}{\partial x^2}+\frac{dU_\eta}{d\eta}=0, \quad U_\eta=\int^\eta[A-\Theta(\eta)]d\eta, \tag{17}$$

and, for the rapidly-varying distributions,

$$\eta=\text{const}, \quad \varepsilon^2\frac{\partial^2\Theta}{\partial x^2}+\frac{dU_\Theta}{d\Theta}=0, \quad U_\Theta=A\Theta+\frac{1}{3}\Theta^3\eta-\frac{1}{2}\Theta^2(J+1). \tag{18}$$

We note that the potential U_Θ corresponding to the rapidly-varying distributions has two extremal points Θ_1 and Θ_2 for each $\eta > \eta_{0J}$, where η_{0J} corresponds to the extremum of the function $\eta(\Theta)$ in (14), and these are the roots of (14) which, in turn, depend on the level of excitation J (Fig. 2). The point $\Theta_1 < \Theta_2$ is a saddle point and Θ_2 is a center. For $\eta \to \eta_{0J}$, the points Θ_1 and Θ_2 merge into a single point Θ^0, which is a point of inflection of U_Θ. It is clear from Fig. 2 that, for the system under investigation and for the Türing model with given J, there is an infinite number of potentials U_Θ corresponding to different values of η. Since, for $\eta > \eta_{0J}$, the potential U_Θ (Fig. 2b) has the form of a potential well, it follows that $\Theta(x)$ can have oscillatory solutions with characteristic length l and the number of half-oscillations is determined by the specimen length l_x.

Without leaving the class of rapidly-varying distributions for the fluctuations $\delta\Theta(x,t)$, it can be shown that distributions in the form of several rapid oscillations are unstable. In other words, such distributions are unstable with respect to the rapidly-oscillating fluctuations $\delta\Theta$ (with characteristic length ϵ) for which, as in the stationary case, we may suppose that $\eta = \text{const}$. In fact, substituting (3) in (4), and using $\epsilon \ll 1$, we obtain

$$\frac{3}{2}\eta\Theta^{-1-\alpha}\frac{\partial\Theta}{\partial t}=\varepsilon^2\eta\frac{\partial^2\Theta}{\partial x^2}-\eta\frac{\Theta-1}{\Theta^{1+\alpha+s}}+J. \tag{19}$$

Here and henceforth, the time will be measured in units of τ_ϵ^0. Linearizing (19) and the boundary conditions (5) for $\eta = \text{const}$ in the inhomogeneous perturbation $\delta\Theta = \delta\Theta(x)e^{-\gamma t}$, we have

$$\hat{H}_\Theta\delta\Theta\equiv\left(-\varepsilon^2\frac{d^2}{dx^2}+V_\Theta\right)\delta\Theta=\left[-\varepsilon^2\frac{d^2}{dx^2}-\frac{(\alpha+s)}{\Theta^{2+\alpha+s}}\left(\Theta-\frac{1+\alpha+s}{\alpha+s}\right)\right]\delta\Theta=\gamma\frac{3}{2}\Theta^{-1-\alpha}\delta\Theta, \tag{20}$$

$$\frac{d(\delta\Theta)}{dx}(x=0,l_x)=0. \tag{21}$$

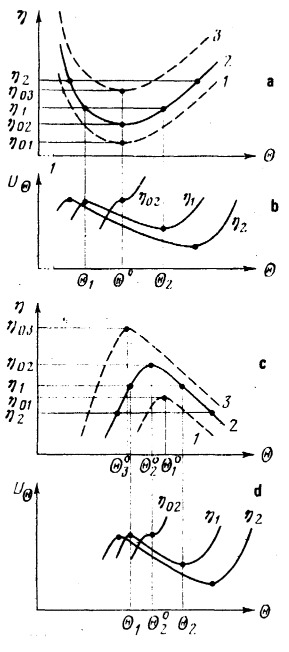

FIG. 2. The function $\eta(\Theta)$ in the case of local coupling (a) and the potential U_Θ (b) for rapidly-varying distributions in the case of $J = J_2$ and several values of $\eta \geq \eta_{02}$. Curves 1, 2, and 3 correspond to different heating levels J ($J_1 < J_0 < J_2 < J_3$); c and d give the corresponding functions for the Türing model.

To obtain the actual form of $V_\Theta(x)$, we must substitute the solution of (16), which describes the distribution of $\Theta(x)$, whose stability has been investigated in the literature,[9,17,18] into (20). The solution is unstable if at least one of the eigenvalues γ of (20) is less than zero. Differentiating (16) with respect to x, we have

$$\hat{H}_\Theta d\Theta/dx=0, \qquad (22)$$

Hence, it follows that $d\Theta(x)/dx$ is an eigenfunction of the operator H_Θ, corresponding to $\gamma=0$ but satisfying boundary conditions other than (21).

Thus, analysis of the stability of rapidly-oscillating solutions is the exact analog of the analysis of the stability of stationary inhomogeneous states in systems with an S-shaped current-voltage characteristic. The latter has been examined in the literature[9,17,18] and it has been shown that distributions in the form of two or more half-oscillations are unstable with respect to fluctuations $\delta\Theta$ when the number of nodes in the latter is less by one (or more) than the number of nodes in the function $d\Theta(x)/dx$ in the interval $(0, l_x)$. The growth of such fluctuations leads to an increase in the temperature in one layer due to the reduction in temperature in a neighboring layer. The result of this is that the number of layers is reduced, and the separation between them increases. Hence, it follows, in particular, that complicated stationary states containing at least one segment of the rapidly-varying distribution in the form of two (or more) layers is unstable (this is examined in greater detail in the next section).

The potential U_η, corresponding to the slowly-varying distribution, has two branches (Fig. 3) with $\Theta > \Theta^0$ (6) and $\Theta \leqslant \Theta^0$, respectively, (Fig. 2), which combine at the point $\eta=\eta_{0J}$ ($\Theta=\Theta^0$). For $J>J_0$, the right-hand branch of U_η (broken curve in Fig. 3) has the form of a potential well, i.e., it allows periodic solutions with $\Theta(x)>\Theta^0$ in the form of oscillations with characteristic length of the order of unity (L). Such oscillating distributions are known[3] to be unstable with respect to short-wave inhomogeneous fluctuations $\delta\eta$, for which the number of nodes is greater than the number of nodes of $d\eta(x)/dx$ in the interval $(0, l_x)$, where $\eta(x)$ is the

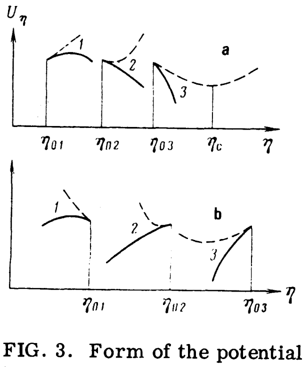

FIG. 3. Form of the potential U_η for slowly-varying distributions: a—heated electron-hole plasma; b—Türing model. Broken curves correspond to $\Theta \geqslant \Theta^0$, solid curves $\Theta \leqslant \Theta^0$, boundary point $\eta=\eta_{0J}$ corresponds to $\Theta=\Theta^0$, where Θ^0 is the magnitude of the parameter Θ, for which stratification sets in. Curves 1, 2, and 3 correspond to different values of J ($J_1<J_0<J_2<J_3$).

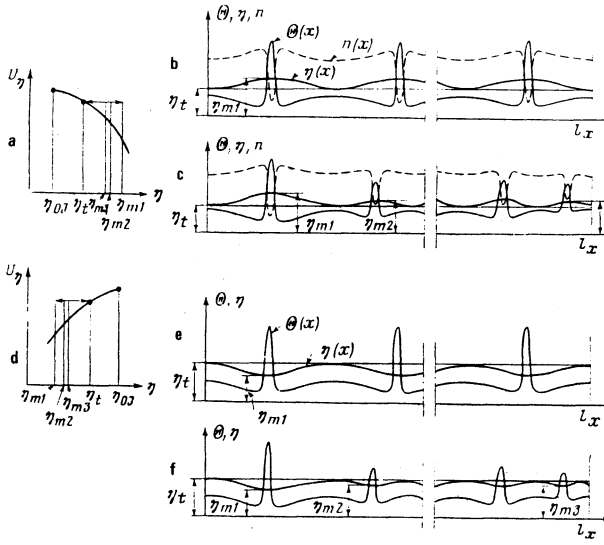

FIG. 4. Stationary states in the form of successive combinations of rapidly- and slowly-varying distributions $\Theta(x)$ and $\eta(x)$: a—potential U_η for slowly-varying distributions for $\Theta \leqslant \Theta^0$ and heating level $J>J_0$; the distribution of Θ and η in the form of periodic (b) and nonperiodic (c) combinations (broken curve shows the concentration distribution); d, e, and f—the corresponding functions for the Türing model.

slowly-varying distribution whose stability is being investigated. The instability of the slowly-varying distributions corresponding to $\Theta(x)>\Theta^0$ is connected with the fact that the parameters Θ and η are practically constant within a region with linear dimensions less than L. Moreover, linear theory shows that such a distribution is unstable against short-wave fluctuations with $k \gtrsim k_0 \approx (lL)^{-1/2}$ for $\Theta>\Theta^0$, which tend to split such slowly-varying distributions into shorter-wave layers. It also follows from this analysis that any complicated solutions $\Theta(x)$ and $\eta(x)$ that are combinations of the slowly- and rapidly-varying distributions turn out to be unstable if they contain a segment of the slowly-varying distributions corresponding to $\Theta>\Theta^0$.

Thus, of all the stationary solutions corresponding to the above system for $L \gg l$, the only stable solutions are those that are the successive combinations of rapidly-varying distributions in the form of single layers (strata) and slowly-varying distributions (Fig. 4) corresponding to the second branch of U_η with $\Theta \leqslant \Theta^0$ (solid curves in Fig. 3). Since U_η does not form a potential well for $\Theta<\Theta^0$ (Fig. 4a), the motion of a "particle" in this potential can only be an infinite motion. However, such slowly-varying distributions may become periodic (Fig. 4b) when combined with the rapidly-varying distributions. In point of fact, since $\eta(x)$ varies only slightly in space, we may suppose that the value $\eta=\eta_m$ remains practically constant in the region of the rapidly-varying distribution $\Theta(x)$, i.e., the enhanced-temperature stratum. Outside the stratum, $\eta(x)$ at first varies with the characteristic length L up to $\eta=\eta_t$, which corresponds to the turning point (Fig. 4), where $d\eta(x)/dx=0$, and then again increases up to $\eta=\eta_m$ at the point where the new stratum is located. Since there is an infinite set of such solutions for a given excitation level, we may select those (Fig. 4) for which

$$\frac{d\eta}{dx}(\eta=\eta_m)=0.$$

Since this condition can be satisfied for many values of η_m, there is, in principle, an infinite number of nonperiodic successive combinations (Fig. 4c) of rapidly-varying distributions with different values of η_m and slowly-varying distributions corresponding to the same turning point η_t of the potential U_η (Fig. 4a).

To analyze the stability of stationary states forming the successive alternation of rapidly- and slowly-varying distributions, we must investigate the complete set of equations describing both the slowly- and rapidly-varying distributions. In the case of the electron-hole plasma, these equations are the equations given by (3) and (19), whereas, for the Türing model, we have (10) and (11). We recall that (19) follows from (3) and (4) to within terms proportional to ϵ. Analysis of the exact set (10) and (11) is then analogous to the analysis of (3) and (19), and leads to the same qualitative results.

By linearizing (3) and (19) in the inhomogeneous fluctuation

$$\delta\eta=\delta\eta(x)e^{-\gamma t},\quad \delta\Theta=\delta\Theta(x)e^{-\gamma t},$$

we obtain

$$\left[\hat{H}_\eta-\gamma\left(\frac{5}{2}+\alpha\right)\Theta^{-1-\alpha}\right]\delta\eta\equiv\left[-\frac{d^2}{dx^2}+\varepsilon^2\Theta^{-1-\alpha}-\gamma\left(\frac{5}{2}+\alpha\right)\Theta^{-1-\alpha}\right]\delta\eta$$
$$=(1+\alpha)\eta\left[\varepsilon^2-\gamma\left(\frac{5}{2}+\alpha\right)\right]\Theta^{-2-\alpha}\delta\Theta, \qquad (23)$$

$$\left[\hat{H}_\Theta-\gamma\cdot\frac{3}{2}\Theta^{-1-\alpha}\right]\delta\Theta\equiv\left[-\frac{d^2}{dx^2}+V_\Theta-\gamma\cdot\frac{3}{2}\Theta^{-1-\alpha}\right]\delta\Theta$$
$$=-\left[\frac{(\Theta-1)}{\Theta^{1+\alpha+s}}-\frac{d^2\Theta}{dx^2}\right]\frac{1}{\eta}\delta\eta, \qquad (24)$$

where lengths are measured in units of l and the times in units of τ_ϵ^0. To obtain the specific form of the operators $\hat{H}_\eta$ and $\hat{H}_\Theta$, and, in particular, the potential V_Θ given by (20), we must use the stationary solution of the set (12), (13), whose stability we are investigating. Let μ_k and $\delta\eta_k$ be the eigenvalues and eigenfunctions of (23) with right-hand side equal to zero and normalized with weight $l_x^{-1}(5/2+\alpha)\Theta^{-1-\alpha}$, and let λ_n and $\delta\Theta_n$ be the eigenvalues and eigenfunctions of (24) with the right-hand side again equal to zero and normalized with the weight $(3/2)l_x^{-1}\Theta^{-1-\alpha}$. Expanding $\delta\eta$ into a series in terms of the functions $\delta\eta_k$, we obtain from (23)

$$\delta\eta=\sum_{k=0}^{\infty}\delta\eta_k(1+\alpha)[\varepsilon^2-\gamma(5/2+\alpha)]\langle\eta\Theta^{-2-\alpha}\delta\eta_k^*\delta\Theta\rangle(\mu_k-\gamma)^{-1}, \qquad (25)$$

where the angle brackets represent averaging over the specimen length l_x. We now substitute (25) in (24), and into the resulting equation we insert the expansion of $\delta\Theta$ in terms of the functions $\delta\Theta_n$. The result is a set of algebraic equations for the expansion coefficients

$$\sum_{n=0}^{\infty}C_n\left\{(\lambda_n-\gamma)\delta_{nm}+\sum_{k=0}^{\infty}P_{knm}\right\}=0; \qquad (26)$$

$$P_{knm}=(1+\alpha)\left[\varepsilon^2-\gamma\left(\frac{5}{2}+\alpha\right)\right]\left\langle\delta\eta_k\delta\Theta_m^*\eta^{-1}\left(\frac{\Theta-1}{\Theta^{1+\alpha+s}}-\frac{d^2\Theta}{dx^2}\right)\right\rangle\langle\eta\Theta^{-2-\alpha}\delta\eta_k^*\delta\Theta_n\rangle(\mu_k-\gamma)^{-1}. \qquad (27)$$

Thus, the question of the stability of a particular inhomogeneous stationary state reduces to the question of whether the roots of the determinant of (26), i.e., the spectrum of values of γ, include at least one negative value.

It follows from (10) and (11) that for the Türing model,

$$P_{knm}=\varepsilon^2\langle\Theta^2\delta\eta_k\delta\Theta_m^*\rangle\langle(2\Theta\eta-J)\delta\eta_k^*\delta\Theta_n\rangle(\mu_k-\gamma)^{-1}, \qquad (28)$$

where $\delta\eta_n$ and $\epsilon^2\mu_n$ are the eigenfunctions, normalized to $\epsilon^{-2}l_x$, and the eigenvalues of the operator

$$\hat{H}_\eta\equiv(-d^2/dx^2+V_\eta),\quad V_\eta=\Theta^2(x)\varepsilon^2, \qquad (29)$$

and $\delta\Theta_n$ and λ_n are the eigenfunctions and eigenvalues of the operator

$$\hat{H}_\Theta\equiv(-d^2/dx^2+V_\Theta),\quad V_\Theta=-2\Theta\eta+1+J. \qquad (30)$$

(Here, $\delta\Theta_n$ are normalized to l_x and x is measured in units of l.) Physically, $\Theta(x)>0$ and, therefore, it is clear from (29) that $V_\eta>0$ for any distribution. Hence, it follows that $\mu_n>0$. Finally, according to the oscillation theorem, μ_n increases with increasing number of nodes of $\delta\eta_n$.

We therefore conclude that all the fluctuations $\delta\eta_n$ are damped and the damping rate increases with increasing n. This conclusion is valid for all the systems under investigation. In point of fact, by transforming (23) with the right-hand side equal to zero to the normal Liouville form, we can easily verify that the spectrum of μ_n is determined by an equation whose form is analogous to the Schrödinger equation with potential $V_\eta>0$ practically everywhere, with the exception of, possibly, a narrow region of size less than l. The relaxation of the fluctuations $\delta\eta_n$ follows directly from the physics of the stratification process for which we have already noted that the long-wave fluctuations $\delta\Theta$, which are locally "followed" by the $\delta\eta$, are attenuated by the damping effect of the variation in η. Short-wave fluctuations $\delta\Theta$ with $k>k_0$, for which the change in η is small, are found to grow. Since the $\delta\Theta_n$ describe fluctuations in Θ for $\delta\eta=0$, one would expect that the values of λ_n include negative values. This is also indicated by the form of the potential V_Θ, which, as can be seen from (24) and (30), is less than zero in the region of the rapidly-varying distribution. In further analysis, it is convenient to consider short ($l_x<L$) and long ($l_x\gg L$) specimens separately.

4. STABLE STATIONARY STATES IN SHORT SPECIMENS

We shall show that a single stratum at the center of a short specimen ($l\ll l_x<L$), i.e., a stationary state in the form of a single oscillation in the rapidly-varying

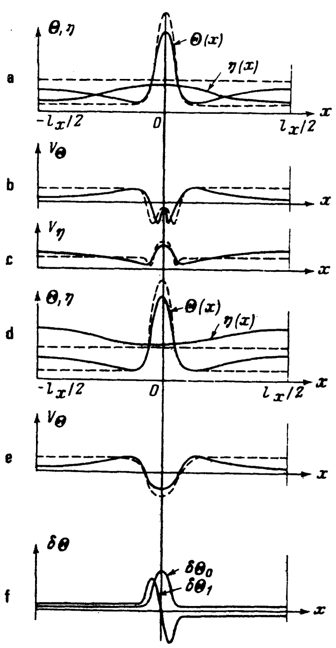

FIG. 5. Distribution in the form of a stratum at the center of a short specimen ($l \ll l_x < L$) and the corresponding functions: a—distributions of $\Theta(x)$ and $\eta(x)$, b and c—form of the potentials V_Θ and V_η determining the fluctuations of $\delta\Theta_n$ and $\delta\eta_k$ for heated electron-hole plasma; d and e—the corresponding functions for the Türing model (broken curve corresponds to $L = \infty$); f—eigenfunctions for the discrete spectrum corresponding to the "ground" and "first excited" states.

distribution, going over into a slowly-varying distribution on either side (Fig. 5a), is stable for a certain range of values of J. In a short specimen, $\eta(x)$ varies only slightly and, since Θ is related to η locally outside the stratum, it follows from (23) and (29) that the potential $V_\eta(x)$ will also vary only slightly everywhere with the exception of the narrow region of the stratum (Fig. 5c). For the longest-wave fluctuations $\delta\eta_k$, the region of rapid variation in the potential V_η is transparent (the particle corresponding to these $\delta\eta_k$ undergoes "tunneling"), and, therefore, such fluctuations "feel" only the average value of V_η evaluated over the region in which they vary. This enables us to take as the zero-order approximation for the potential V_η the average value of this potential, having replaced functions of $\Theta(x)$ by their average values in (23) with the right-hand side equal to zero. Using (5), we then have

$$\delta\eta_k = 2^{1/2}\left[\left(\frac{5}{2}+\alpha\right)\langle\Theta^{-1-\alpha}\rangle\right]^{-1/2}\cos\frac{\pi k l}{l_x}x,$$
$$\delta\eta_0 = \left[\left(\frac{5}{2}+\alpha\right)\langle\Theta^{-1-\alpha}\rangle\right]^{-1/2} \quad (31)$$

$$\mu_k = \varepsilon^2\left(\frac{5}{2}+\alpha\right)^{-1}[1+\pi^2k^2L^2l_x^{-2}(\langle\Theta^{-1-\alpha}\rangle)^{-1}]. \quad (32)$$

Since, for $l_x < L$, μ_k is much greater than μ_0 beginning with $k=1$, we can retain only the first term in the sum over k in (26). The condition that the déterminant of (26) must be equal to zero then reduces to

$$\prod_n (\lambda_n - \gamma)\left(1+\sum_{n=0}^{\infty}\frac{P_{0nn}}{\lambda_n - \nu}\right) = 0, \quad (33)$$

$$P_{0nn} = (1+\alpha)\left\langle \delta\Theta_n \eta^{-1}\left(\frac{\Theta-1}{\Theta^{1+\alpha+s}} - \frac{d^2\Theta}{dx^2}\right)\right\rangle \langle\delta\Theta_n\eta\Theta^{-2-\alpha}\rangle(\langle\Theta^{-1-\alpha}\rangle)^{-1}. \quad (34)$$

This result has a simple physical interpretation. For $l_x < L$, even the longest-wave inhomogeneous variations in η with the length l_x are rapidly damped out because of the large diffusion currents produced by them. Neglecting the inhomogeneous components of $\delta\eta$, we can determine the homogeneous change in η by averaging (3) linearized in the fluctuation $\delta\eta$ and $\delta\Theta$:

$$\langle\delta\eta\rangle = (1+\alpha)\langle\delta\Theta\eta\Theta^{-2-\alpha}\rangle/\langle\Theta^{-1-\alpha}\rangle. \quad (35)$$

Substituting $\delta\eta = \langle\delta\eta\rangle$ in (24) and performing the necessary calculations, we obtain (33) and (34).

We must now analyze the spectrum of λ_n. For the Türing model, it follows from (18) that a rapidly-varying distribution in the form of a single oscillation for $l_x \gg l$ is, to within $\exp(-l_x/l)$,

$$\Theta(x) = \frac{1+J}{2\eta} - \frac{2}{\eta}\xi^2 + \frac{6}{\eta}\xi^2\,\mathrm{ch}^{-2}\,\xi x, \qquad \xi^2 = \frac{1+J}{4}\left[1-\frac{4A\eta}{(1+J)^2}\right]^{1/2}, \quad (36)$$

and the corresponding potential well V_Θ (30) is

$$V_\Theta = 4\xi^2(1-3\,\mathrm{ch}^{-2}\,\xi x). \quad (37)$$

The eigenfunctions and eigenvalues corresponding to this well for $l_x \to \infty$ are well known.[19] The spectrum contains three discrete levels: $\lambda_0 = -5\xi^2$, $\lambda_1 = 0$, and $\lambda_2 = 3\xi^2$. Taking into account the boundary conditions (5) for the infinite specimen with $l_x \gg l$ and $\eta = \mathrm{const}$ ($L = \infty$), we obtain an exponentially small change in the above values of λ because the functions $\delta\Theta_n$ corresponding to the discrete spectrum are localized in a region of the order of l (Fig. 5f) and decay exponentially outside this region. By replacing the true well with a rectangular well, we can verify that λ is reduced by $\sim\exp(-l_x/l)$.

It follows that allowance for the finite length of the specimen for $\eta(x) = \mathrm{const}$ ($L = \infty$) may lead to the appearance of a second negative eigenvalue

$$\lambda_1 \approx -\exp(-l_x/l),$$

corresponding to the perturbation tending to shift the stratum toward one of the ends of the specimen (Fig. 5f). This conclusion applies to all the systems that we are considering. In fact, as already mentioned, $d\Theta/dx$ is an eigenfunction of the operator $\hat{H}_\Theta$ in (22) corresponding to $\lambda = 0$. For a single stratum, $d\Theta/dx$ has one node withing the interval $(0, l_x)$ and, therefore, the spectrum λ_n contains only one negative λ_0. The difference between the boundary conditions for $\delta\Theta$ and $d\Theta/dx$ in the case of the finite specimen[17] then ensures that λ_0 and λ_1 are modified by an exponentially small amount for $l_x \gg l$.

Since λ_1 is exponentially small when $\eta=\text{const}$ ($L=\infty$), it is natural to examine the influence of a small departure of $\eta(x)$ from a constant value by the amount λ_1 for $l_x \gg l$. For finite L in the Türing model, the functions $\eta(x)$ and $\Theta(x)$ within the slowly-varying distributions under consideration (Figs. 4d and 3) are shown by (17) to increase outside the stratum on average by

$$\Delta\eta\approx\eta(l_x/L)^2,\quad \Delta\Theta\approx\Theta(l_x/L)^2.$$

Averaging the stationary equations given by (10) and (11), we find that $\langle\Theta\rangle=A$, and hence the reduction in $\Theta(x)$ in the stratum is

$$\Delta\Theta_{st}\approx\Delta\Theta(l_x/l)\geqslant\Theta_{st}(l_x/L)^2.$$

According to (30), the potential well V_Θ rises by the amount $2\Delta\Theta_{st}$ and, therefore, the exponentially small negative level λ_1 becomes positive (Fig. 5e).

An analogous situation obtains for the heated electron-hole plasma. For this system, it follows from (15) that $\eta(x)$ decreases outside the stratum in the region of the slowly-varying distributions under consideration (Figs. 4a, b), whereas $\Theta(x)$ increases (Fig. 5a). Since $\eta \equiv n\Theta^{1+\alpha}n_0^{-1}$, it follows that $n(x)$ decreases toward the ends of the specimen (Fig. 4b). The reduction in $n(x)$ is due to the recombination of nonequilibrium carriers. For $l_x < L$, the average reduction in the carrier concentration outside the stratum is of the order of $\Delta n = n(l_x/L)^2$. Averaging the stationary equation (1) over the specimen length, we find that $\langle n(x)\rangle = R\tau_r=\text{const}$, and, therefore, the carrier concentration increases by $\delta n_{st} > n_{st}(l_x/L)^2$ within the region of the stratum. Integrating the stationary equation given by (19) over the stratum, we obtain

$$\int_{-l}^{l}\eta\frac{\Theta-1}{\Theta^{1+\alpha+s}}dx=\int_{-l}^{l}\frac{n(\Theta-1)}{n_0\Theta^s}dx=2lJ=\text{const},$$

and hence it follows that $\Theta(x)$ is reduced within the stratum (Fig. 5a) by the amount

$$\Delta\Theta_{st}\geqslant\Theta_{st}(l_x/L)^2.$$

The result of this is that the well V_Θ in (20) becomes narrower (Fig. 5b), and both λ_0 and λ_1 increase by an amount of the order of $(l_x/L)^2$. **Thus, even slowly varying functions $\eta(x)$, i.e., a slight damping effect associated with the parameter η, result in $\lambda_1 > 0$.**

Thus, the λ_n for one stratum at the center of a finite specimen include only one negative value ($\lambda_0 < 0$) corresponding to the ground-state function $\delta\Theta_0$ (Fig. 5f). This enables us to retain only the first factor within the product in (33). We note that the only small eigenvalue λ_1 is not present in (33). In point of fact, $\Theta(x)$ and $\eta(x)$ are, in this case, even functions with respect to the specimen center (Fig. 5a), whereas $\delta\Theta_1$ is an odd function (Fig. 5f), so that $P_{011}=0$. In view of the foregoing, we have from (33)

$$\gamma-\lambda_0-P_{000}-\sum_{n=2}^{\infty}\frac{P_{0nn}(\lambda_0-\gamma)}{\lambda_n-\gamma}=0, \tag{38}$$

where P_{000} reflects the contribution of the stratum to γ, and the sum on the right-hand side represents the contribution of the remainder of the specimen. In other words, the sum in (38) is determined by the fluctuations $\delta\Theta_n$ corresponding to the quasicontinuous spectrum of eigenvalues λ_n. Since these functions oscillate rapidly and are smeared out over the entire specimen, they provide an appreciable contribution only when the magnitude of Θ outside the stratum is comparable with the value Θ_m at its center.[9,18] For sufficiently large excitation levels ($J > J_0$), it is found that Θ_m is large (Fig. 5) and, therefore, the region of the stratum provides the main contribution. In other words, P_{000} exceeds the sum remaining in (38), and

$$\gamma=-|\lambda_0|+P_{000}, \tag{39}$$

from which it is clear that the necessary condition for the stability of the single stratum is that $P_{000} > |\lambda_0|$. Numerical calculations show that this condition is satisfied for $J > J_0$ and $l_x \gg l$. This is most easily verified for the Türing model by substituting (36) and the ground-state function for the potential well (37) in (28).

For a more rigorous analysis of stability, we shall use the fact that $\gamma \ll |\lambda_n|$ near the point of loss of stability. This enables us to obtain from (38) the following stability condition:

$$-\frac{\gamma}{|\lambda_0|}=1+\sum_{n=0}\frac{P_{0nn}}{\lambda_n}<0. \tag{40}$$

The increase in η, averaged over the specimen for a small J, can be obtained from (19) (taking into account only $\delta\eta=\langle\delta\eta\rangle$), i.e., we can determine $\langle\delta\eta\rangle/\delta J$. Linearizing (19) in the perturbations

$$\delta\Theta=\delta\Theta(x)e^{i\omega t},\quad \delta\eta=\delta\eta(x)e^{i\omega t},\quad \delta J=\delta Je^{i\omega t},$$

we obtain (24) with the right-hand side augmented by $\delta J\eta^{-1}$. Substituting $\delta\eta=\langle\delta\eta\rangle$ into this equation, and performing the necessary transformations, we obtain

$$\frac{\langle\delta\eta\rangle}{\delta J}(i\omega)=\left(\sum_{n=0}^{\infty}\frac{B_n}{\lambda_n+i\omega}\right)\left(1+\sum_{n=0}^{\infty}\frac{P_{0nn}}{\lambda_n+i\omega}\right)^{-1}, \tag{41}$$

where

$$B_n=(1+\alpha)\langle\delta\Theta_n^*\eta^{-1}\rangle\langle\delta\Theta_n\Theta^{-2-\alpha}\eta\rangle/\langle\Theta^{-1-\alpha}\rangle. \tag{42}$$

It is clear from (40) that the stability of the distribution under consideration is determined by the sign of the denominator in (41) for $\omega=0$. For the Türing model, the corresponding static derivative is

$$\frac{\langle\delta\eta\rangle}{\delta J}=\frac{\langle\Theta\rangle}{\langle\Theta^2\rangle}\left(1+\sum_{n=0}^{\infty}\frac{B_n}{\lambda_n}\right)\left(1+\sum_{n=0}^{\infty}\frac{P_{0nn}}{\lambda_n}\right)^{-1}, \tag{43}$$

where

$$B_n=\langle(2\Theta\eta-J)\delta\Theta_n\rangle\langle\Theta\delta\Theta_n^*\rangle/\langle\Theta\rangle,$$

and the stability of the stratum for $J < -1+A^2$ is determined by the sign of the denominator in (43).

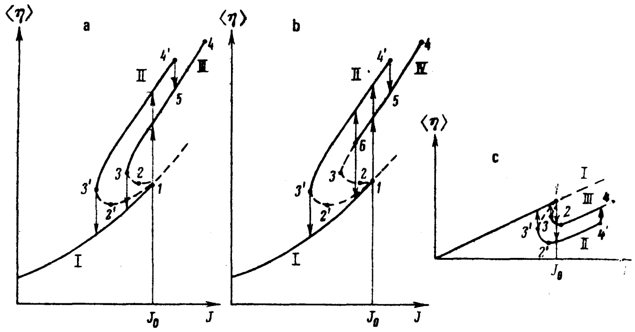

FIG. 6. Value of η averaged over the specimen with $l_x \lesssim L$ as a function of J: a and b—electron-hole plasma; c—Türing model (branch I corresponds to homogeneous distribution, II—distribution in the form of a single stratum at the end, III—single stratum at the center, IV—one stratum at each end of the specimen). Broken curve shows the regions of instability of the corresponding distribution.

Qualitative analysis of the dependence of $\langle\eta\rangle$ on J (Fig. 6) for a single stratum and $l \ll l_x < L$ can be based on the following considerations. At the branch point 1 (Fig. 6) of the solutions, where the inhomogeneous distribution goes over smoothly into the homogeneous distribution, the derivative $\langle\delta\eta\rangle/\delta J$ for $l_x \gg l$ should also go over smoothly into the derivative $d\eta/dJ$ for the homogeneous distribution, and should therefore be positive. For large J, when Θ_m is large (Fig. 5a), the quantity $\langle\eta\rangle$ exceeds η_{hom} corresponding to the homogeneous distribution (Figs. 6a, b). (For the Türing model, $\langle\eta\rangle$ is smaller, cf. Fig. 6c.) In fact, for $l_x < L$, the function $\eta(x)$ varies very little within the specimen and, as can be seen from Fig. 2a, the quantity η_m is determined by Θ_m and increases with increasing Θ_m. Since Θ_m is greater than Θ for the same level of excitation $J > J_0$ and a homogeneous distribution, the quantity $\langle\eta\rangle$ is also greater than η_{hom} (Fig. 5a). Comparison of (34) and (42) will show that $P_{0nn} \approx B_n$. As already noted in connection with (38), the sums in (41) are small for $J > J_0$ ($\Theta_m \gg 1$) in comparison with the other terms, and $P_{000} > |\lambda_0|$. Hence, it follows that both the numerator and the denominator in (41) are less than zero and, therefore, $\langle\delta\eta\rangle/\delta J > 0$. The fact that this condition is satisfied for $J > J_0$ has a simple physical interpretation. An increase in J for $\Theta_m \gg 1$ leads to an increase in the temperature at the center of the stratum and, consequently, to an increase in $\langle\eta\rangle$.

With decreasing level of heating J or, more precisely, decreasing $\langle\eta\rangle$, the distribution $\Theta(x)$ becomes less rapidly varying, so that the influence of the regions outside the stratum, i.e., the sums in (41), becomes greater. Hence, as J decreases (between 4 and 3 in Fig. 6a), the denominator in (41) initially passes through zero at 3, where $\langle\delta\eta\rangle/\delta J = \infty$, and then the numerator passes through zero at 2, where $\langle\delta\eta\rangle/\delta J = 0$ (Fig. 6a). Thus, according to (40), the segment 1–2–3 in Fig. 6a corresponds to an unstable state, whereas 3–4 corresponds to a stable state of a single stratum. In the Türing model (Fig. 6c), described by (43), all these conclusions can be rigorously justified in numerical fashion, as well, by using $\Theta(x)$ [in the form of (36)], the wave function $\delta\Theta_n$, and the eigenvalues λ_n in the potential (37).

Since the instability of the homogeneous distribution sets in near 1, and a single stratum becomes unstable at 3, the process of formation and disappearance of the stratum exhibits the phenomenon of hysteresis (Fig. 6a). At 1, the homogeneous distribution may go over as a result of instability into a number of stable inhomogeneous stationary states for given J (Figs. 6a, b), for example, a single stratum at the center (Fig. 5a) or on the lateral surface of the specimen (Fig. 7a). In point of fact, the distribution in the form of a single half-oscillation (Fig. 7a) corresponds to the $\Theta(x)$ of Fig. 5 for $x \geq 0$ (or $x \leq 0$). For this type of distribution, the boundary condition (5) is satisfied only by the fluctuations $\delta\Theta_n$ (Fig. 5f) with even n, i.e., the eigenvalue spectrum does not, in general, contain λ_1 and, therefore, even for $L \to \infty$, we only have $\lambda_0 < 0$. Hence, the analysis of stability is here analogous to the foregoing, and leads to the same results. Moreover, for a stratum at the end and the same $J(>J_0)$ and l_x, both Θ_m and, consequently, $\langle\eta\rangle$ are greater than for a stratum at the center. This is reflected in Fig. 6a.

As J increases (from 3′ to 4′ in Fig. 6a), the temperature Θ, both in the stratum and at the opposite end of the specimen, is found to increase (Fig. 7a). This is due to the fact that the reduction in $\eta(x)$ and, consequently, the increase in $\Theta(x)$ in the region of the slowly-varying distribution, are faster as J increases. This follows from (14) and (15). As a result, the quantity $\Theta(l_x)$ becomes equal to Θ^0 for a certain value of J (Fig. 7a), provided the specimen is not too short ($l_x \gg l$), i.e., we reach the boundary branch point $\eta(l_x) = \eta$ for the potential U_η (Fig. 3a). For large J, a distribution $\Theta(x)$ in the form of a single stratum at the end (Fig. 7a), subject to the boundary conditions given by

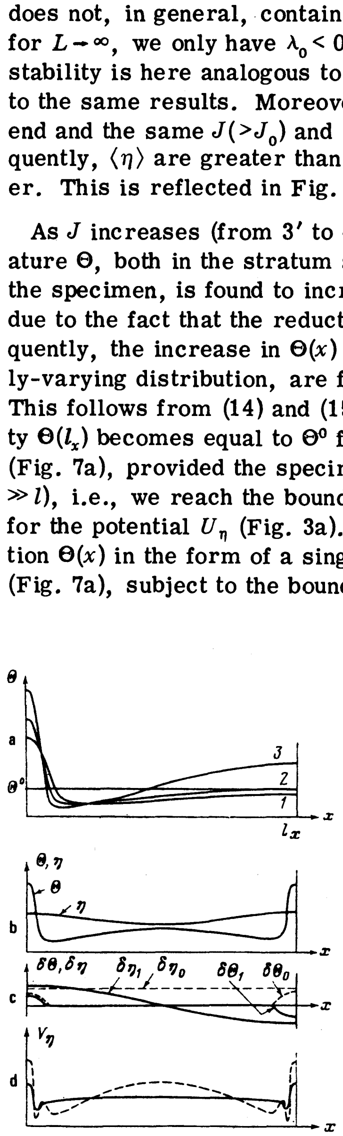

FIG. 7. Distributions in the form of a single stratum (a) and one stratum at each end of the specimen (b). Numbers shown against the curves correspond to increasing heating. The figure also shows the fluctuations $\delta\Theta_n$ and $\delta\eta_k$ (c) for two strata corresponding to the "ground" and "first excited" states in the potentials V_Θ and V_η (d). The broken curve in the last diagram shows V_η for large J.

(5), is possible only when $\eta(x)$ goes over to the solution of (15) corresponding to U_η for $\Theta \geqslant \Theta^0$ as $x = l_x$ is approached (broken curves in Fig. 3a). As was shown in Sec. 3, this solution for $\eta(x)$ is unstable, so that, beginning with a certain J (4′ in Fig. 6a), a single stratum on the end surface becomes unstable. This conclusion is valid for any distribution, including that in the form of a stratum at the center of the specimen (Fig. 5a), but the value of J for which this distribution becomes unstable is, of course, greater (4 in Fig. 6a) because the region in which $\eta(x)$ decreases in the case of a stratum at the center is smaller by a factor of two than for a stratum on the side surface (Figs. 5a and 7a).

Thus, for certain J, the distribution in the form of a stratum on the side surface (Fig. 7a) becomes unstable and the system may go over discontinuously to a stable state, for example, a stratum (Fig. 5a) at the center of the specimen (4′ → 5, Fig. 6a) which, as has been explained, is stable for the given J, or two strata (Fig. 7b) at the ends of the specimen (4′ → 5, Fig. 6b). It is clear that the dependence of $\langle\eta\rangle$ on J is the same for these distributions. It is possible to use the method developed below in Sec. 5 for the investigation of the stability of a large number of strata in a long specimen to show that two strata at the ends of the specimen (Fig. 7b) are unstable if L is not much greater than the specimen length l_x. However, because of the growth of fluctuations tending to increase the temperature in one stratum and reduce it in the other (see Sec. 5 and Fig. 7c), the value of J corresponding to the lower limit of stability of this distribution (6 in Fig. 6b) is then greater than for the single stratum at the center of the specimen (3 in Fig. 6a).

5. STABLE STATIONARY STATES IN LONG SPECIMENS

We shall now show that, in specimens with $l_x \gg L$, periodic combinations of rapidly and slowly varying distributions (Figs. 4b and e) i.e., strata separated by a distance $L_1 \gtrsim L$ (Fig. 8), are stable. We note that a long specimen in which this distribution is produced can be represented as a set of $N = l_x/L_1$ specimens of length L_1, in each of which the distribution corresponds to a single stratum (Fig. 5a). This means that the potentials V_η and V_Θ corresponding to Fig. 8a are the sum of N potentials (Figs. 8b and c), each of which corresponds to potentials V_η and V_Θ for a single stratum (Figs. 5b and c). It was shown in Sec. 4 that, among the eigenvalues corresponding to the potential V_Θ (Fig. 5b), only $\lambda_0 < 0$, whereas $\lambda_1 > 0$ but is close to zero.

The most convenient way to begin is to investigate the stability of the strata for a specimen in the form of a narrow ring for which the fluctuations $\delta\eta$ and $\delta\Theta$ correspond to the cyclic condition: $\delta\Theta(0) = \delta\Theta(l_x)$. Since λ_0 and λ_1 correspond to the discrete spectrum, i.e., their eigenfunctions $\delta\Theta_0^0$ and $\delta\Theta_1^0$ are localized in a region of the order of l (Fig. 5f), and the potential wells are at $L_1 \gg l$, it follows that, to determine the smallest eigenvalues and the eigenfunctions of the periodic potential V_Θ (Fig. 8c), we can use perturbation theory which, in

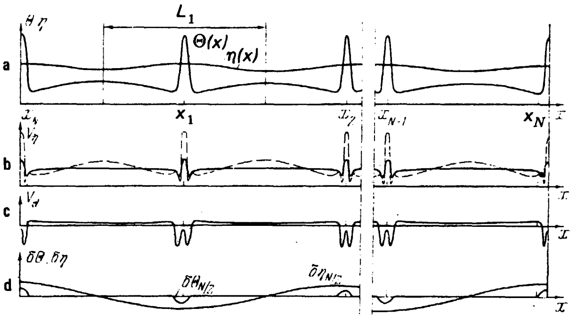

FIG. 8. Form of the stable periodic stationary state for heated electron-hole plasma in a long specimen ($l_x \gg L$) and the corresponding functions: a—$\Theta(x)$ and $\eta(x)$; b and c—V_η and V determining the fluctuations $\delta\eta_k$ and $\delta\Theta_n$, respectively (broken curve in b shows V_η for large J); d—shortest-wave fluctuations corresponding to the bottom band of values of λ_n.

solid-state physics, is commonly referred to as the approximation of strongly bound electrons.[20] In this approach, the eigenfunctions $\delta\Theta_n(x)$ in the periodic potential V_Θ (Fig. 8c) can be written in the form

$$\delta\Theta_n = \sum_{m=1}^{N} \exp(ik_n x_m)\,\delta\Theta_i^0(x - x_m), \tag{44}$$

and the discrete eigenvalues λ_0 and λ_1 split into N components in the form of narrow bands of width $|\Delta\lambda|$ and are shifted by $\Delta\lambda_0$, where, in the nearest-neighbor approximation,[20]

$$\lambda_{0,1}^{n} = \lambda_{0,1} + 2\Delta\lambda \cos k_n L_1 + \Delta\lambda_0; \qquad \Delta\lambda,\ \Delta\lambda_0 \sim \exp(-L_1/l). \tag{45}$$

In (44), $\delta\Theta_i^0(x)$ is the i-th eigenfunction normalized with the weight $(3/2)\,\Theta^{-1-\alpha} L_1^{-1}$ over the region of size L_1 in the single well (Fig. 5b), x_m is the coordinate of the center of the m-th stratum, and $k_n = 2\pi n/NL$, where $n = \pm 1, \pm 2, \ldots, \pm n/2$.

Since V_η (Fig. 8b) is a periodic and slowly-varying potential with the exception of narrow (of the order of l) regions, we can use the approximation of almost-free electrons[20] for sufficiently long-wave fluctuations $\delta\eta_k(x)$ with wave vector $k \lesssim L_1^{-1}$:

$$\delta\eta_k = \exp\{ik_k x\}\,u_k(x), \tag{46}$$

where $u_k(x)$ is the Bloch function with period L_1. When $L_1 \lesssim L$, the function $\eta(x)$ and, consequently, the potential V_η vary slowly with x, so that, as in the case of the stability of a single stratum in a short specimen, the potential V_η can be assumed to be constant in the zero-order approximation. When this is so, the μ_k are given by (32), and

$$u_k = [(5/2 + \alpha)\langle\Theta^{-1-\alpha}\rangle]^{-1/2}. \tag{47}$$

We note that the $\delta\Theta_i^0$ with $i > 2$ correspond to the quasicontinuous spectrum which, as described in Sec. 4 in connection with the derivation of (39), provides an appreciable contribution only for low heating levels corresponding to the lower boundary ($J < J_0$) of the stability of the corresponding distribution (Fig. 6). It follows that, when we analyze the stability of the distribution for $J > J_0$, we can confine our attention to the discrete values λ_0 and λ_1 which split, in this case, into the narrow bands of (45). Substituting (44) and (46) in (27), we obtain

$$P_{knm}^{ij} = \delta_{kn}\delta_{km}P_k^{ij}, \tag{48}$$

where

$$P_k^{ij} = \frac{l_x^2}{L_1^2}(1+\alpha)\left[\varepsilon^2 - \gamma\left(\frac{5}{2}+\alpha\right)\right]\left\langle u_k(x)\delta\Theta_i^0\frac{1}{\eta}\left(\frac{\Theta-1}{\Theta^{1+\alpha+s}} - \frac{d^2\Theta}{dx^2}\right)\right\rangle\frac{\langle u_k(x)\delta\Theta_j^0\eta\Theta^{-2-\alpha}\rangle}{\mu_k-\gamma}. \tag{49}$$

Since $\delta\Theta_1^0$ is an odd function of x (Fig. 5f), and the integrands in (49) are even functions on a segment of length L_1, it follows that, as in the case of the single stratum, all the p_k^{ij} with $j, i \neq 0$ are zero. Using the form of p_{knm}^{ij} given by (26), we find that the lowest values of γ are

$$\gamma_n^1 = \lambda_1 + 2\Delta\lambda\cos k_nL_1 + \Delta\lambda_0, \tag{50}$$

$$\gamma_n^0 = -|\lambda_0| + 2\Delta\lambda\cos k_nL_1 + \Delta\lambda_0 + P_{nnn}^{00}. \tag{51}$$

We have established, in connection with the stability of a single stratum, that $\lambda_1 > 0$ and is not exponentially small (for $L_1 \ll L$, we have $\lambda_1 \approx L_1^2/L^2$). Therefore, all the $\gamma_n^1 > 0$. We can estimate P_{nnn}^{00} by substituting (47) and (32) in (49), whence

$$P_{nnn}^{00} = P_{000}[\varepsilon^2 - \gamma_n^0(5/2+\alpha)][\varepsilon^2 + \varepsilon^2k_n^2L^2(\langle\Theta^{-1-\alpha}\rangle)^{-1} - \gamma_n^0(5/2+\alpha)]^{-1}, \tag{52}$$

where P_{000} is given by (34). It is clear from (52) that P_{nnn}^{00} is a minimum for $n = N/2$. This value of n corresponds to the most "dangerous" fluctuations $\delta\Theta_n$ that tend to transfer heat between neighboring strata (Fig. 8d), i.e., reduce their number. [For two strata in a short specimen (Fig. 7), this fluctuation is the anticoupling combination $\delta\Theta_1$.] As noted in Sec. 4, when $J > J_0$, we have $P_{000} > |\lambda_0|$ and, therefore, all the γ_n^0 are positive, whereas, for $n = N/2$,

$$P_{000} - P_{nnn}^{00} < P_{000} - |\lambda_0|.$$

It is clear from (52) that this result is satisfied for $L_1 > L$. Moreover, L_1 cannot exceed L by many orders of magnitude because, if this were so, $\Theta(x)$ would reach Θ^0 between the strata, and such a distribution is unstable (Sec. 4).

When the stability of inhomogeneous stationary states is investigated for long specimens that do not close into a ring, the above cyclic boundary conditions of the Born-von Karman type must be replaced with the boundary conditions given by (5). This is equivalent to a transition from the analysis of the energy spectrum of an infinite one-dimensional crystal to the spectrum of a finite specimen. It is known[21] that the Tamm surface states may then appear near the surface. In the one-dimensional case, this means that a surface level may appear at a distance $\Delta\lambda^0$ below the "band" given by (45) on one of the boundaries, and the corresponding eigenfunctions will be localized on this boundary. In the ap-

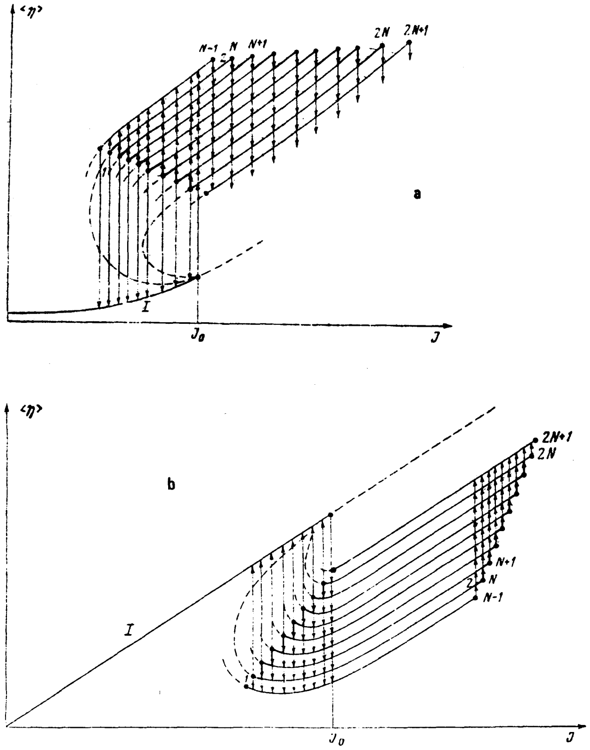

FIG. 9. Value of η averaged over the specimen ($l_x \gg L$) as a function of J. Branch I corresponds to the homogeneous contribution, whereas branches with numbers against them correspond to inhomogeneous distributions, the number indicating the number of strata in the distribution. Broken curves show unstable distributions. Arrows indicate possible jumps in $\langle\eta\rangle$ resulting from loss of stability in a given stationary state. The "upper sawtooth" represents situations when the number of strata increases by one on each instability, and the "lower sawtooth" corresponds to a reduction by one in the number of strata (a—hot electron-hole plasma; b—Türing model).

proximation of strongly bound electrons, the quantity $\Delta\lambda^0$ is given by the overlap integral for the wave functions of neighboring atoms[21] (neighboring potential wells, cf. Fig. 8), so that, in our case, $\Delta\lambda^0$, $\Delta\lambda$, and $\Delta\lambda_0$ (45) are exponentially small ($\sim e^{-L_1/l}$). Thus, our conclusions with regard to the stability of the distribution in the form of a large number of strata, obtained for cyclic boundary conditions, remain valid for the boundary conditions given by (5).

Since a long specimen can be represented by a set of N specimens of length L_1 each, we find that the dependence of $\langle\eta\rangle$ on J (Fig. 9) is precisely the same as the analogous result (Fig. 6b) for a specimen of length L_1 with a stratum at each end (Fig. 7b), i.e., 1 and 2 in Fig. 9a correspond to the boundary values of J corresponding to 6 and 4 in Fig. 6b. Thus, as J increases, the distribution at 2 in the form of N strata becomes unstable because Θ reaches the value Θ^0 between the strata. The result of this is that the number of strata increases. If the number of strata increases by one on each loss of stability, $\langle\eta\rangle$ will vary discontinuously as a function of J, forming the "upper sawtooth" in Fig. 9. When J is reduced at 1 (Fig. 9a), stability is lost as a result of the "transfer" of temperature between the strata, and the dependence of $\langle\eta\rangle$ on J when the number of strata is reduced by 1 is represented by the "lower sawtooth" in Fig. 9. Of course, the number of strata may change by 1, 2, 3, etc. on each loss of stability, and this is represented by the arrows in Fig. 9. The appearance of one or other stationary state is, in general, a random phenomenon unconnected with the development of one of the growing fluctuations $\delta\Theta$ appearing randomly in the specimen.

The number of stable strata for given l_x and J is bounded from above by their stability with respect to the transfer of heat between the strata (Fig. 8d), and from below by the fact that, as the separation between the strata increases, $\Theta(x)$ reaches Θ^0 in the region of the slowly-varying distributions. It is clear from Fig. 6 that, in a short specimen (of length L_1), several unstable inhomogeneous stationary states may coexist for a certain J, and the transitions II→III (Fig. 6a) and II→IV (Fig. 6b) formally signify an increase in the number of strata by a factor of two. Generalizing this result to a long specimen with $l_x = NL_1$, we may conclude that, for given J, at any rate, distributions corresponding to the number of strata between $N/2$ and N are stable.

As the number of strata increases, the separation L_1 between them (Fig. 8a) is reduced and, in accordance with (51) and (52), they become unstable with respect to heat transfer. We note, however, that, as the level of heating increases, the critical separation between the strata for which they are still stable decreases and may become much less than the value given by (52), which corresponds to the V_η = const approximation. In point of fact, as was noted in Sec. 4, an increase in J is accompanied by an increasingly rapid reduction in $\eta(x)$ and an increase in $\Theta(x)$ outside the strata (Fig. 7a). As a result, the potential V_η becomes more rapidly varying and increasingly resembles (for two strata) the shape of two separated potential wells (Fig. 7d). For a large number of strata (Fig. 8a), the potential V_η for large J assumes the form of a periodic combination of potential wells (Fig. 8b). In the limit of deep potential wells, μ_0 will split into N bands and, therefore, $\mu_{N/2}$ will not be very different from μ_0. The result of this will be that, for a sufficiently high level of heating, and in accordance with (49) and (51), the strata may turn out to be stable even for $L_1 < L$. (The values of Θ_m at the center of a stratum will then be much greater than Θ^0.) Extending this conclusion to the case of a short specimen (Fig. 7), we may conclude that, as l_x decreases, the level of heating necessary to maintain the stability of a stratum at each end of the specimen (Fig. 7b) must increase.

1)We note that, in real systems, the parameters exhibit not only spatial but also frequency dispersion, so that temporal instability may often arise[1,5] and may lead to periodic relaxation oscillations. However, this situation does not obtain in the systems discussed below.

2)Strictly speaking, the current of carriers on the boundary is determined by the rate of their surface recombination S. However, for properly prepared surfaces, $S \ll L\tau_r^{-1} \approx v_T\tau_\epsilon^{1/2}\tau_r^{-1/2}$, i.e., the carrier concentration even at the surface of the specimen is controlled by the bulk concentration and not by surface recombination. For example, for germanium, $S \approx 10$–10^2 cm·sec^{-1}, and for $T = 4.2$ °K we have $v_T \approx 2\times10^6$ cm·sec^{-1}, i.e., the condition quoted above is safely satisfied. The current of carriers on the boundary can, therefore, be assumed to be zero.

3)We emphasize that, when the instability of nonequilibrium carriers is investigated, the rate of generation R and the recombination time τ_r are assumed constant. If τ_r is a sharply decreasing function of temperature, the instability of the electron-hole plasma may set in earlier.[3,13] In particular, the above stratification condition turns out to be softer. However, when the system has this exotic function $\tau_r(T)$, it may exhibit a number of further features (for example, oscillations) which we shall not consider here.

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Translated by S. Chomet