Calculation of the Maxwell stress tensor of a turbulent medium

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A calculation is made of the Maxwell stress tensor on the basis of symmetry considerations. An expression is obtained for the force acting on a plasma in the presence of isotropic and anisotropic inhomogeneities. It is shown that a negative pressure may appear in the anisotropic case. This pressure may give rise to a modulation instability. The reaction of magnetic inhomogeneities to large-scale flow of a plasma is calculated. The specific case of magnetohydrodynamic waves in a homogeneous magnetic field is discussed.

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INTRODUCTION

Magnetohydrodynamic turbulence is a phenomenon which is widely encountered in astrophysics and in laboratory experiments. There are a number of problems related to this turbulence which are being solved one way or another. They include the range of problems associated with weak fluctuations of magnetic fields in the presence of a strong hydrodynamic turbulence (dynamo instability), and those associated with magnetohydronamic (MHD) waves and their interaction. The particularly difficult cases are those of strong turbulence when the magnetic and kinetic energies are comparable and when the magnetic energy includes the energy of a homogeneous magnetic field, if such is present.

Nevertheless, calculation of the average electromagnetic force in the case of an MHD turbulence—which need not be weak—is a relatively simple matter. The present paper deals with such a calculation. Let us first of all consider what results can be expected from qualitative physical considerations. The theory of the internal structure of stars frequently involves the problem of the magnetic pressure which appears even in the absence of a large-scale magnetic field and is simply due to small-scale inhomogeneities. In fact, it is clear that a set of such inhomogeneities should have elasticity. However, the problem becomes more complicated if we bear in mind that the electromagnetic force acting on a plasma does not consist of the magnetic pressure alone. There is also a component which is known as the line-of-force stress (LFS). Because of this stress we may expect elasticity in nonpotential motion. Moreover, the LFS makes its own contribution to the magnetic pressure but with the opposite sign. The question arises whether the LFS can compensate completely the magnetic pressure or make it negative. We shall show that such a situation is indeed possible.

We may expect a medium with a negative magnetic pressure to be unstable. This instability is similar to the modulation type. In fact, let us assume that there is a fluctuation of the mean-square field intensity, i.e., that the average energy of the field varies slowly over distances much greater than the dimensions of inhomogeneities (in other words, let us assume that there is a modulation of inhomogeneities). The action of a negative pressure begins to drive a plasma toward the region where the magnetic pressure is higher and this enhances the fluctuation. Thus, qualitative considerations show that the average electromagnetic force may give rise to a number of effects with interesting applications.

§1. GENERAL RELATIONSHIPS

We shall begin from the Maxwell stress tensor

$$\sigma_{ij}=\frac{1}{4\pi}\left(\frac{1}{2}\delta_{ij}H^2-H_iH_j\right) \tag{1}$$

(see, for example, the book by Landau and Lifshitz).[1] The tensor does not include the electric component because in the range of validity of magnetohydrodynamics we have $E/H\ll 1$, where **E** is the electric field intensity. In fact, direct substitution shows that $-\operatorname{div}\sigma=[\operatorname{curl}\mathbf{H}\times\mathbf{H}]/4\pi$ is the force acting on a plasma, which occurs in the equation of motion

$$\frac{\partial}{\partial t}\rho v_i+\partial_j\rho v_i v_j=-\partial_i p-\partial_j\sigma_{ij}, \tag{2}$$

where p is the pressure. We shall now average σ_{ij} over small-scale pulsations, i.e., we shall calculate the average (large-scale) force acting on a plasma. Clearly,

$$\langle\sigma_{ij}\rangle=\frac{1}{4\pi}\left(\frac{1}{2}T_{ll}\delta_{ij}-T_{ij}\right), \tag{3}$$

where $T_{ij}=\langle H_iH_j\rangle$. The tensor $\hat{T}$ is the correlation tensor of the magnetic field and the values of the fluctuations H_i and H_j are taken at coincident points. We shall use the nonnegative definite form of the matrix T_{ij}:

$$T_{ij}C_iC_j^*\geqslant 0, \tag{4}$$

which follows from the fact that T_{ij} is a correlation tensor.[2] This also follows directly from the definition of T_{ij}.

In the simplest case of an isotropic random process, we have $T_{ij}=\frac{1}{3}\delta_{ij}\langle H^2\rangle$. Then, $\langle\sigma_{ij}\rangle=\delta_{ij}\langle H^2\rangle/3\cdot 8\pi$. Consequently, the average magnetic pressure is one-third of the "usual" pressure $H^2/8\pi$. This reduction is due to the compensating effect of the LFS. In fact, the LFS in

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the tensor (3) is represented by the term $\sim T_{ij}$. In the absence of this term, we have $\langle\sigma_{ij}\rangle=\delta_{ij}\langle H^2\rangle/8\pi$.

The next case of increasing complexity is the existence, in a random process, of some preferred direction parallel to a unit vector λ. Then, obviously,

$$T_{ij}=A\delta_{ij}+B\lambda_i\lambda_j=\frac{\langle H^2\rangle}{3+\varepsilon}(\delta_{ij}+\varepsilon\lambda_i\lambda_j). \tag{5}$$

Reducing the tensor (5) to the principal axes, we now find that the requirements (4) lead to

$$A\geqslant 0,\ A+B\geqslant 0, \tag{6}$$

or

$$\langle H^2\rangle\geqslant 0,\ 1+\varepsilon\geqslant 0. \tag{6'}$$

If $\varepsilon=0$, we return back to the isotropic case; ε may be positive or negative. In particular, we can have the case $\varepsilon=-1$ (the maximum, in the absolute sense, negative value of ε).

The other limiting case is $\varepsilon\gg 1$. Averaging of the tensor σ_{ij} gives

$$\langle\sigma_{ij}\rangle=\frac{1}{8\pi}(A+B)\delta_{ij}-\frac{1}{4\pi}B\lambda_i\lambda_j. \tag{7}$$

Two conclusions can now be drawn. Firstly, calculation of the force $F_i=-\partial_j\langle\sigma_{ij}\rangle$, shows that the tensor $\langle\sigma_{ij}\rangle$ corresponds not only to a potential force: there is also a term $\lambda(\lambda\nabla)B/4\pi$, which contains the nonpotential component. Secondly, we may have a situation in which a negative pressure appears. Let us, in fact, assume that $\varepsilon\gg 1$. Moreover, let us postulate that $\langle H^2\rangle=f(\lambda\cdot\mathbf{x})$, i.e., that the mean-square value $\langle H^2\rangle$ varies along the direction of λ, whereas λ itself is the selected direction of the tensor T_{ij}. Then,

$$\mathbf{F}=-\frac{1}{8\pi}\nabla B+\frac{1}{4\pi}\lambda(\lambda\nabla)B=\frac{1}{8\pi}\nabla B=\frac{1}{8\pi}\nabla\langle H^2\rangle. \tag{8}$$

A negative pressure appears because the LFS [the second term in Eq. (7)] may exceed the ordinary pressure [the first term in Eq. (7)] in the anisotropic case. The ponderomotive force which appears in a continuous medium because of fluctuations has been calculated in many papers.[3-6] This force can be expressed in terms of the permittivity tensor ε_{ij}. Washimi and Karpman[6] demonstrated, in particular, that the expression for the force can be obtained by direct averaging of the equation of motion. In the examples considered below (§§ 2–4) the assumption of weak fluctuations is not used and, therefore, the determination of the tensor ε_{ij} is difficult. For this reason, we shall average directly the equations of motion.

§2. TRANSVERSE AND LONGITUDINAL SECOND SOUND

Let us assume that small-scale magnetic fluctuations are distributed homogeneously in the statistical sense, i.e., that $\langle H^2\rangle$ is independent of the coordinates. In this case the average force naturally vanishes. A weak perturbation against the background of $\langle H^2\rangle$ causes perturbation of the large-scale velocity which in its turn affects the perturbation. We shall describe linear microscopic motion using Eq. (2):

$$\rho\frac{\partial\mathbf{u}}{\partial t}=-\partial_j\sigma_{ij} \tag{9}$$

(the plasma pressure is assumed to be low and there are no velocity fluctuations) and the influence of the plasma motion on the fluctuations of H will be assumed to be given by

$$\frac{\partial\mathbf{H}}{\partial t}=\operatorname{curl}[\mathbf{u}\times\mathbf{H}], \tag{10}$$

where $\mathbf{u}$ is the macroscopic velocity. Multiplying the i-th component of Eq. (10) by H_j, and the j-th component by H_i, averaging, and adding the two resultant equations, we obtain

$$\frac{\partial}{\partial t}T_{ij}=T_{il}\partial_l u_j+T_{jl}\partial_l u_i-u_l\partial_l T_{ij}-2T_{ij}\operatorname{div}\mathbf{u}. \tag{11}$$

In this averaging we have used the large-scale nature of the field $\mathbf{u}$, i.e., $\langle H_iH_ju_f\rangle=\langle H_iH_j\rangle u_f$. Let us assume that $T_{ij}=T^0_{ij}+T'_{ij}$, where $\hat{T}^0$ is the unperturbed tensor and $\hat{T}'$ is a fluctuation. Then,

$$\frac{\partial}{\partial t}T_{ij}'=T_{il}^0\partial_l u_j+T_{jl}^0\partial_l u_i-2T_{ij}^0\operatorname{div}\mathbf{u}. \tag{12}$$

Equation (9) contains only the perturbation $\hat{T}'$. Assuming next that all the perturbations vary as $\sim\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)]$, we obtain the equation for $\mathbf{u}$:

$$4\pi\rho_0\omega^2u_i=T_{jl}^0k_\alpha k_l(\delta_{\alpha j}u_i+\delta_{ij}u_\alpha-\delta_{\alpha i}u_j)+(T_{jj}^0k_i-2T_{ij}^0k_j)(\mathbf{ku}). \tag{13}$$

For the isotropic tensor T^0_{ij}, we have

$$\omega^2\mathbf{u}={}^1/_3(v_A)^2[k^2\mathbf{u}+\mathbf{k}(\mathbf{ku})], \tag{14}$$

where $v_A^2=\langle H^2\rangle/4\pi\rho_0$. Equation (14) corresponds to longitudinal oscillations $\omega^2(\mathbf{ku})=\frac{2}{3}k^2(\mathbf{ku})v_A^2$, i.e., $\omega=vk(\frac{2}{3})^{1/2}$, and two transverse oscillations: $\mathbf{k}\perp\mathbf{u}$, $\omega=v_Ak(\frac{1}{3})^{1/2}$. Each transverse wave has a definite polarization: there are two linearly independent velocities $\mathbf{u}$ perpendicular to $\mathbf{k}$. Thus, longitudinal and transverse sound may appear against a background of magnetic fluctuations. An example of steady-state magnetic fluctuations in the absence of velocity fluctuations may, in principle, be imagined. However, such a situation is not typical in applications. Therefore, the treatment in the present section is purely illustrative, demonstrating the elastic properties of random magnetic inhomogeneities.

§3. INSTABILITY OF LARGE-SCALE FLUCTUATIONS

Fluctuations of the velocity field v occur in Eq. (10) in the form of an additional term curl $[\mathbf{v}\times\mathbf{H}]$ and in Eq. (11) they occur as correlations of the $\langle H_iH_jv_f\rangle$ type. In view of the resultant problem of closure, we shall apply simple physical hypotheses relating to the nature of the interaction between the velocity $\mathbf{v}$ and the field $\mathbf{H}$. We must bear in mind that a steady-state turbulence is considered. This means that all deviations of $\hat{T}$ from its steady-state value disappear in a short relaxation time $\tau=l/v$, where l is the coordinate length and

$v = \langle v^2\rangle^{1/2}$. In the preceding section we have ignored $\langle v^2\rangle$. We shall now consider the opposite equidistribution case:

$$\tfrac{1}{2}\rho\langle v^2\rangle = \langle H^2\rangle/8\pi. \quad (15)$$

The amplitude of turbulent pulsations v depends strongly on the turbulence sources. If we assume that these sources maintain $\langle v^2\rangle$ at the steady-state level, we find that

$$\langle H^2\rangle = a\rho, \quad a = 4\pi\langle v^2\rangle. \quad (16)$$

Any more general law, such as $\langle H^2\rangle \propto \rho^\alpha$, $\alpha > 0$, does not affect qualitatively further considerations. Therefore, for simplicity, we shall adopt Eq. (16).

We shall consider perturbations against the background of an isotropic distribution of magnetic fluctuations: $T_{ij}^0 = A\delta_{ij}$. Since the macroprocess time is considerably greater than the relaxation time τ, we shall ignore the anisotropy of $\hat{T}$. Using Eq. (16), we obtain the following linearized equation for large-scale fluctuations of the velocity $\mathbf{u}$:

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{8\pi}\frac{a}{3}\nabla\rho, \quad (17)$$

from which the existence of longitudinal sound of velocity $(a/3\cdot 8\pi)^{1/3} = v_A/6^{1/2}$ follows in a natural manner. Let us assume that now $\hat{T}$ is given by Eq. (5), where $B \gg A$. Then, instead of Eq. (17), we have

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\frac{a}{8\pi}\nabla\rho' + \frac{a}{4\pi}\lambda(\lambda\nabla)\rho', \quad \frac{\partial\rho'}{\partial t} + \rho_0 \operatorname{div}\mathbf{u} = 0. \quad (18)$$

We shall select a coordinate system in which $\lambda = \{0, 0, 1\}$, $\mathbf{k} = \{0, k_2, k_3\}$. The dispersion equation is then

$$\omega^2 = \frac{a}{8\pi\rho_0}(k_2^2 - k_3^2), \quad u_1 = 0, \quad u_2 = bk_2, \quad u_3 = -bk_3. \quad (19)$$

Hence, it is clear that for $k_2^2 > k_3^2$ the second sound has both longitudinal and transverse components. If $k_2^2 < k_3^2$, the absolute instability sets in ($\operatorname{Re}\omega = 0$, $\operatorname{Im}\omega > 0$). The nature of the instability is related to the action of a negative pressure which appears if the tensor $\hat{T}$ is anisotropic. In general, the tensor $\hat{T}$ has the form (5) and

$$\langle\sigma_{ij}\rangle = \frac{\langle H^2\rangle}{4\pi(3+\varepsilon)}\left(\frac{1+\varepsilon}{2}\delta_{ij} - \varepsilon\lambda_i\lambda_j\right). \quad (20)$$

We shall substitute Eq. (20) into Eq. (2) and use Eq. (16). We then obtain an equation of the (18) type but with different coefficients. For perturbations obeying $\sim \exp \times [i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, we have

$$\omega^2\mathbf{u} = v_A^2 \frac{1}{3+\varepsilon}\left(\frac{1+\varepsilon}{2}\mathbf{k} - \varepsilon(\mathbf{k}\lambda)\lambda\right)(\mathbf{k}\mathbf{u}). \quad (21)$$

For $\varepsilon = \infty$, we return to the system (18), An analysis shows that an instability appears for $\varepsilon > 1$, $\omega^2 \propto [k^2 + \varepsilon \times (k^2 - 2(\mathbf{k}\cdot\lambda)^2]$.

§4. MAGNETIC RESISTANCE

We shall now consider large-scale incompressible flow at a velocity $\mathbf{u}$, where $\operatorname{div}\mathbf{u} = 0$. We shall assume that magnetic fluctuations against the background of this flow are homogeneous: $\langle H^2\rangle$ is independent of the coordinates. We shall determine the reaction (a slight deviation) of $\hat{T}'$ from its steady-state value to the appearance of flow. In view of the incompressibility of flow, there are no density perturbations and, therefore, the principal effect described in § 3 is absent. In the present section we shall allow for finer effects, including a slight deviation of the tensor $\hat{T}$ from its steady-state value. The dynamics of $\hat{T}'$ is given by Eq. (12), where we have to allow additionally for the relaxation in a time τ. We shall use the "τ approximation":

$$\frac{\partial}{\partial t}T_{ij}' = T_{il}^0\partial_l u_j + T_{jl}^0\partial_l u_i - \frac{1}{\tau}T_{ij}'.$$

Assuming that all the macroprocesses occur in a time much longer than τ, we have

$$T_{ij}' = \tau(T_{il}^0\partial_l u_j + T_{jl}^0\partial_l u_i) \quad (22)$$

We shall now calculate the force acting on a plasma on the basis of Eq. (22). For an isotropic $\hat{T}^0$, we have

$$\mathbf{F} = \tfrac{1}{3}\rho v_A^2\tau\Delta\mathbf{u}. \quad (23)$$

According to Eq. (23), magnetic fluctuations have the effect of viscosity and the viscosity coefficient found subject to Eq. (17) is of the same order of magnitude as the hydrodynamic turbulence coefficient, which is $\rho v\, l/3$. In the presence of an anisotropic background [$\hat{T}^0$ is described by Eq. (5) and $B \gg A$] the viscosity is anisotropic:

$$\mathbf{F} = \rho v_A^2\tau(\lambda\nabla)^2\mathbf{u}, \quad (24)$$

where $(\lambda\nabla)^2 = \lambda_a\lambda_b\partial_a\partial_b$. The potential part of the force is not given above because the motion is assumed to be incompressible and the pressure compensates the potential force.

We shall now consider the situation when the tensor $\hat{T}^0$ is inhomogeneous. According to Eq. (11), we then have an additional perturbation $\hat{T}' \sim -u_f\partial_f\hat{T}_0$. Hence, it follows that the additional nonpotential force acting on the plasma is

$$F_i = -\frac{1}{4\pi}\tau\partial_j(u_f\partial_f\lambda_i\lambda_j B). \quad (25)$$

The isotropic part of the tensor $\hat{T}(A\delta_{ij})$ makes no contribution to Eq. (25). We shall select the coordinate system in which $\lambda = \{0, 0, 1\}$, $\mathbf{u} = \{0, 0, u(y)\}$ and we shall assume that B depends only on the coordinate z. The force (25) is parallel to the z axis:

$$F_z = -\frac{1}{4\pi}\tau u\frac{\partial^2 B}{\partial z^2}. \quad (26)$$

Consequently, an additional resistance appears for $\partial^2B/\partial z^2 > 0$. In the region of a maximum of B we have $\partial^2B/\partial z^2 < 0$ and the force (25) results in an instability (negative resistance). The viscous and magnetic resistance can be ignored if the length of the inhomogeneity $u(y)$ along the y axis is greater than the characteristic size of the inhomogeneity B along the z axis.

We have considered so far only the Maxwell stress tensor. It follows from the general equation of motion

(2) that the turbulent stress tensor $\hat{V}=\langle\rho v_i v_j\rangle\approx\rho\langle v_i v_j\rangle$ may give a contribution comparable with $\langle\sigma_{ij}\rangle$. We shall consider briefly the contribution $\hat{V}$ primarily to the second sound. It is natural to assume that for isotropic $\hat{T}$ the tensor $\hat{V}$ is also isotropic. Using the hypothesis represented by Eqs. (15) and (16), we find that $\langle\rho v^2\rangle\approx\rho\langle v^2\rangle=\rho v_A^2$. Then, $\hat{V}$ compensates completely the LFS and instead of the longitudinal sound with the square of the velocity $(a/3\cdot 8\pi)^{1/2}$ [see Eq. (17)] we have the velocity described by $(a/8\pi)^{1/2}$. As pointed out in § 1, the factor $\frac{1}{3}$ appears because of the compensating effect of the LFS.

The problem of the role of $\hat{V}$ in the presence of anisotropy is more complex. Clearly, if we assume that $\hat{T}$ is anisotropic, we have to assume that $\hat{V}$ is also anisotropic. However, the relationship between these tensors is not known. We shall assume that $\hat{V}$ is given by $V_{ij}=A_1\delta_{ij}+B_1\lambda_i\lambda_j$. We shall introduce $B_1=\varepsilon_1 A_1$, where $-1\leqslant\varepsilon<\infty$; if we use Eq. (17), we find that $A_1=\rho_0\langle v^2\rangle/(3+\varepsilon_1)=v_A^2(3+\varepsilon_1)$. Now the equation for **u** differs from Eq. (21):

$$\omega^2\mathbf{u}=v_A{}^2(\mathbf{ku})\left[\left(\frac{1+\varepsilon}{2(3+\varepsilon)}+\frac{1}{3+\varepsilon_1}\right)\mathbf{k}-\left(\frac{\varepsilon}{3+\varepsilon}-\frac{\varepsilon_1}{3+\varepsilon_1}\right)(\mathbf{k}\boldsymbol{\lambda})\boldsymbol{\lambda}\right]. \qquad 27)$$

It follows from this expression that allowance for the contribution of $\hat{V}$ generally hinders the appearance of an instability. For example, if $\varepsilon=\varepsilon_1$, the perturbations are potential and stable. This is to be expected: for $\varepsilon=\varepsilon_1$, the tensors $\hat{T}/4\pi$ and $\hat{V}$ are identical and $\hat{V}$ compensates completely the part of the Maxwell tensor responsible for the LFS. We recall that the nonpotential force (and oscillations), negative pressure, and instability owe their origin to the LFS. The instability criterion is

$$\varepsilon>(7\varepsilon_1+9)/(1-\varepsilon_1), \quad \varepsilon_1<1$$

(if $\varepsilon_1=-1$, then $\varepsilon>1$). Since the relationship between $\hat{T}$ and $\hat{V}$ is not known, the values of ε and ε_1 should be deduced from the experimental data.

At the beginning of this section we have found the magnetic viscosity which appears because of the Maxwell tensor. The tensor $\hat{V}$ contributes the usual turbulence viscosity mentioned above; the expression for the force (23) is, in fact, compared with this viscosity.

§5. SPECIAL CASE: OSCILLATIONS IN A HOMOGENEOUS MAGNETIC FIELD

We shall consider a low-pressure plasma with $\beta=p8\pi/H_0^2\ll 1$, where H_0 is a homogeneous magnetic field. For a specific type of wave we can find directly the fluctuations of the magnetic field H and of the velocity **v** and thus find exactly the tensors $\hat{T}$ and $\hat{V}$. We shall assume the presence of the following oscillations against the background of a homogeneous field: in one case we shall postulate the presence of the Alfven waves and the other of fast magnetic sound. In calculating the tensor $\langle\sigma_{ij}\rangle$ we shall allow only for the fluctuations of H because the field $\mathbf{H}_0$ does not contribute to the force **F**. The general stress tensor is $\langle\hat{\sigma}+\hat{V}\rangle$. Since $H\ll H_0$, we can introduce the permittivity tensor $\hat{\varepsilon}$. Substituting $\hat{\varepsilon}$ in the expression for the stress tensor of Washimi and Karpman,[6] we shall demonstrate that the expression obtained is identical with $\langle\hat{\sigma}+\hat{V}\rangle$.

We shall first deal with the Alfven waves. Since $\mathbf{H}\propto[\mathbf{k}\times\mathbf{H}_0]$ and $\mathbf{v}\propto[\mathbf{k}\times\mathbf{H}_0]$, i.e., $\mathbf{v}\parallel\mathbf{H}$, it follows that $\hat{T}/4\pi=\hat{V}$ for a monochromatic wave and an isotropic ensemble of Alfven waves. Consequently, the tensor $\hat{V}$ compensates completely the part of the Maxwell tensor responsible for the LFS. The only term that remains is the potential force $\mathbf{F}=-\nabla\langle H^2\rangle/8\pi$, which generates the second sound. Therefore, for $\mathbf{u}\parallel\mathbf{H}_0$, we obtain longitudinal oscillations.

We shall now consider the fast magnetic sound. We shall assume the presence of a monochromatic wave. We shall expand $\mathbf{H}_0$ parallel to the z axis and assume that **k** is in the yz plane. The perturbations **v** and **H** lie in the yz plane (Fig. 1). Hence, it follows that

$$V_{ij}=B\nu_i\nu_j, \quad T_{ij}=B\lambda_i\lambda_j, \qquad (28)$$

where $\boldsymbol{\nu}$ is a unit vector along the y axis and $\boldsymbol{\lambda}$ is a unit vector in the direction of $[\mathbf{k}\times[\boldsymbol{\nu}\times\mathbf{H}_0]]$, defined by $\boldsymbol{\lambda}=\{0,\ -k_z, k_y\}/k$. We shall turn first of all to steady-state but homogeneous fluctuations, i.e., to the case when $\langle H^2\rangle$ is independent of time but depends on the coordinates. Since all the inhomogeneities are driven along the direction of **k**, a steady-state situation is possible if $\langle H^2\rangle$ appears only along directions perpendicular to **k**, i.e., if $B=\langle H^2\rangle$ is a function of $(\mathbf{r}\cdot\boldsymbol{\lambda})$ and x, where **r** is the radius vector. We shall write the component of the force F_z in the form

$$F_z=\frac{1}{8\pi}\frac{\partial B}{\partial z}=\frac{1}{8\pi}\frac{\partial}{\partial z}\langle H^2\rangle. \qquad (29)$$

It thus follows that a steady-state ensemble of the fast magnetic sound waves gives rise to negative pressure along $\mathbf{H}_0$.

The results of calculations of the other two components of the force are

$$F_x=0, \quad F_y=-\frac{1}{8\pi}\frac{\partial}{\partial y}\langle H^2\rangle.$$

If $\langle H^2\rangle$ depends only on $(\mathbf{r}\cdot\boldsymbol{\lambda})$, the component F_y is of no special interest because $\langle H^2\rangle$ is known to be much smaller than H_0^2 and it gives rise to very small fluctuations across the main field $\mathbf{H}_0$. However, if $\langle H^2\rangle$ depends also on x, then F_y causes the flow characterized by $\mathbf{u}=\{0, u(x), 0\}$. We can easily show that $\mathrm{curl}[\mathbf{u}\times\mathbf{H}_0]=0$ (the flow simply transposes the force lines) and, therefore, the magnetic field $\mathbf{H}_0$ does not offer a resistance to such flow.

We shall now consider oscillations against the background of a homogeneous ensemble of the fast magnetic sound waves. It is known that in the presence of an en-

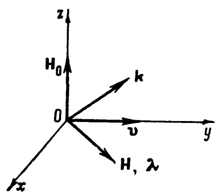

FIG. 1.

semble of the Alfven waves there are amplitude modulations of the second-sound type and in the presence of an ensemble of the fast magnetic sound waves there is a modulation instability for the motion along the field $\mathbf{H}_0$.[7] In this example we have obtained the same results by a different way. In fact, at the beginning of the section we have mentioned longitudinal sound in the presence of the Alfven waves. The appearance of a modulation instability in the field of the fast magnetic sound waves may be expected on the basis of Eq. (29). We shall now consider a density perturbation with the wave vector $\mathbf{q}\|\boldsymbol{\lambda}$ $(\boldsymbol{\lambda}\|\mathbf{H})$, as shown in Fig. 1. As pointed out by Al'tshul',[7] the energy $\langle H^2\rangle$ arises in a region of higher density: $\langle H^2\rangle \propto 1/v_A^0 \propto \rho_0^{1/2}$, $v_A^0 = H_0/(4\pi\rho_0)^{1/2}$, where ρ_0 is the density averaged over small-scale fluctuations. It then follows from Eq. (29) that a plasma is acted upon by a force due to a negative pressure and this gives rise to an instability.

A modulation instability in the potential motion (compression along $\mathbf{H}_0$) was considered by Al'tshul'[7] so that we shall turn to the incompressible motion. Let us assume that for $\mathbf{u}=\{0,0,u(y)\}$ all the quantities depend only on y. Then, the kinetic equation for the number of waves ("quasiparticles") is

$$\frac{\partial n_k}{\partial t}+v_A{}^0\frac{k_y}{k}\frac{\partial n_k}{\partial y}-\frac{\partial u}{\partial y}k_z\frac{\partial n_k}{\partial k_y}=0. \tag{30}$$

The Doppler frequency shift is allowed above: $\omega = v_A^0 k + uk_z$. In the linear approximation for perturbations of the $\sim\exp[i(qy-\Omega t)]$ type, we have

$$n_k=\frac{quk_z\,\partial n_k{}^0/\partial k_y}{v_A{}^0 qk_y/k-\Omega}. \tag{31}$$

The force acting on a plasma is then

$$F_i=-\partial_j\langle\sigma_{ij}\rangle=\frac{1}{4\pi}\partial_j\lambda_i\lambda_j B,\quad F_z=\frac{1}{4\pi}\frac{\partial}{\partial y}\lambda_y\lambda_z B.$$

The vector $\boldsymbol{\lambda}$ exhibits a weak dependence on the coordinates: according to Eq. (30), the wave vector and, therefore, $\boldsymbol{\lambda}$ vary from one quasiparticle to another. Therefore, it is necessary to calculate the quantity

$$\int \lambda_y\lambda_z n_k\omega_k d\mathbf{k}=-\int\frac{k_yk_z}{k^2}n_k\omega_k d\mathbf{k}. \tag{32}$$

We have allowed here for the fact that $n_k\omega_k$ is the energy density of quasiparticles. Substituting Eq. (31) into Eq. (32), and then Eq. (32) into a linearized equation of motion, we obtain the dispersion equation

$$\Omega\rho_0=-q^2\int\frac{k_yk_z^2v_A{}^0}{k}\frac{\partial n_k{}^0/\partial k_y}{\Omega-v_A{}^0k_yq/k}d\mathbf{k}. \tag{33}$$

For a monochromatic wave with $n_k^0 \propto \delta(k_x)\delta(k_x-k_y^0)\times\delta(k_z-k_z^0)$, we obtain

$$(v_A{}^0k_y{}^0q-\Omega k)^2=q^2(k_z{}^0)^2\left[1-\frac{(k_y{}^0)^2}{k^2}\right]v_A{}^2. \tag{34}$$

It follows from Eq. (34) that the system acquires a new transverse oscillation branch for which $\mathbf{u}\|\mathbf{H}_0$. The oscillation frequency is $\Omega \approx v_A^0 k_y^0 q/k$ because $v_A \ll v_A^0$. Nevertheless, in spite of the smallness of the correction $\sim v_A/v_A^0$, we cannot substitute $v_A=0$ in Eq. (34) because the denominator of Eq. (33) then vanishes and the discussion loses its meaning.

In the example considered, the denominator of Eq. (33) does not vanish. If the distribution of n_k^0 is such as to include the vicinity of a singularity, a resonance takes place when the phase velocity of the modulation perturbation Ω/q becomes equal to the quasiparticle velocity (the group velocity along the y axis). In this case the integral (33) can be calculated by writing the denominator in the casual from $(\Omega - v_A^0 k_y q/k + i\varepsilon)$.

We shall now consider the situation when the maximum of the distribution $n_k^0(k_y)$ lies in the region k_y^0, $k_y^0 v_a^0 \ll k_z^0 v_A$. The real part of the frequency is now $\Omega_1 = \pm q v_A k_z^0/k$. We shall use n_k^0 in the form $n_k^0 = A(k_y)\delta(k_x)\times\delta(k_z-k_z^0)$. We shall write the imaginary part of the integral (33) as

$$i\pi\frac{q}{|q|}\frac{|k_z|^3}{v_A{}^0}\Omega\frac{\partial A}{\partial k_y}\bigg|_{k_y=\Omega|k_z|/v_A{}^0q} \tag{35}$$

and the expression for Ω (assuming $|k_z|\approx k$) as

$$\Omega=\Omega_1\left(1+\frac{1}{2}\frac{i\pi}{\rho_0}\frac{q}{|q|}\frac{k^3}{v_A{}^0}\frac{\partial A}{\partial k_y}\right), \tag{36}$$

where the derivative $\partial A/\partial k_y$ is taken at the point $k_y = \Omega|k_z|/v_A^0 q$. A simple estimate of the dimensionless correction to the frequency (36) gives

$$\Omega=\Omega_1\left(1+\frac{1}{2}i\pi\frac{k^2}{k_y{}^{02}}\frac{v_A{}^2}{v_A{}^{02}}\eta\right),$$

where $\eta = (\partial A/\partial k_y)/A_0/k_y^0)$ and $\partial A/\partial k_y$ is taken in the "tail" of the distribution $A(k_y)$ at the point $k_y = \Omega|k_z|/qv_A^0$, because $\Omega k_z/v_A^0 q = v_A k/v_A^0$ is considerably greater than k_y^0; here, A_0 is the characteristic value of the function $A(k_y)$. Hence, it is clear that $\eta \ll 1$. It is this smallness of η that ensures that the correction to the frequency is small. By analogy with the streaming instability, the modulation instability appears, in accordance with Eq. (36), in the presence of a hump in the function $A(k_y)$ and is associated with the transfer of momentum from a quasiparticle to a modulation wave of frequency given in Eq. (34).

DISCUSSION

1. The term modulation instability is used many times above (and also by Al'tshul'[7]) although this instability is not related directly to dispersion or, in particular, to the Lighthill criterion.[8] Nevertheless, the use of this term is justified because, in the final analysis, the amplitude is modulated on a scale much greater than the wavelength (the correlation length in § § 2–4) and at frequencies much lower than the frequencies of the fundamental oscillations.

2. In the Introduction we have mentioned the problem of the magnetic pressure because of the presence of magnetic inhomogeneities in stellas convective shells. Equally important is allowance for this pressure in studies of the solar wind. In fact, at relatively short distances from the sun (less than 30 solar radii) the magnetic pressure is greater than the plasma pressure. The quantity $\langle H^2\rangle$ decreases with the distance and, consequently, the plasma is acted upon by the radial force facilitating the wind.

3. An instability in an anisotropic turbulent medium

(§ 3) reaches saturation only when the magnetic pressure becomes comparable with the plasma pressure. Let us assume that initially we have $\beta \ll 1$ (low-pressure plasma). According to Eq. (16) the value of $\langle H^2 \rangle$ increases proportionally to ρ, whereas $p \propto \rho^\gamma$ in the case of adiabatic compression (γ is the specific heat ratio). Since $\gamma > 1$, the plasma pressure finally prevents compression. Consequently, a turbulent medium acquires regions with a much higher local pressure.

4. The nonpotential forces acting on a plasma appear because of the anisotropy (§ 1) or are manifested as a negative resistance [§ 4, Eq. (27)]. Large-scale flow may be produced in a plasma under the action of these forces. By way of example, we shall mention the excitation of shear motion in the solar wind because of the inhomogeneity of $\langle H^2 \rangle$ and a possible anisotropy of the magnetic inhomogeneities.

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Magnetic moments of iron atoms in the fcc lattice of a transition *d*-metal

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The atomic magnetic moment m_{Fe} was investigated in the fcc modification of iron (the γ phase) and in its alloys with transition d-metals. The behavior of m_{Fe} was investigated as a function of the integral V of the transition between the localized levels of neighboring atoms at a fixed value of the mixing constant of the s and d states. In the assumed model, the magnetic state of the atom is determined only by the influence of its nearest neighbors. It is shown that the atomic magnetic moment of γ-iron decreases with increasing V. At $V \approx 0.35$ eV, m_{Fe} vanishes, but in the same region of V the value of m_{Fe} for the bcc modification of iron changes insignificantly. The function $m_{Fe}(V)$ for γ-Fe is considered with allowance for the transition of the conduction electrons to localized levels as V increases. It is shown that m_{Fe} of impurity iron atoms in d-metals with fcc lattices depends on the quantity E_{Fe}^- and E_M^-, where E_{Fe}^- is the energy of the d level of the iron atom at $V=0$ with a spin direction corresponding to the less occupied part of this level, and E_M^- is the analogous quantity for the matrix atoms. If the condition $E_{Fe}^- - E_M^- > 4V$ is satisfied, then m_{Fe} of the impurity iron atoms depends little on V. This conclusion explains why impurity iron atoms in many d-metals with fcc lattice have large and approximately equal values of m_{Fe}, equal to (2.5–3) μ_B, whereas γ-Fe has a small atomic magnetic moment, (0.5–0.7) μ_B. The dependence of the mean value $\bar{m}_{Fe}$ on the iron concentration in alloys with d-metals having an fcc lattice is considered. In accord with the experimental data, the obtained relation $\bar{m}_{Fe}(c)$ tends rapidly to zero at a certain critical iron concentration c_{cr}.

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INTRODUCTION

The magnetic properties of iron in the face-centered cubic (fcc) modification (γ phase) differ strongly from the properties of its usual body-centered cubic (bcc) modifications (α phase). Iron in the γ modification is antiferromagnetic. Its atomic magnetic moment is $(0.7-0.5)\mu_B$,[1,2] much higher than the atomic magnetic moment $2.2\mu_B$ of α iron. In the fcc lattice of transition d-metals, however, impurity iron atoms have relatively large spin magnetic moments that vary over a small range, approximately from $2.5\mu_B$ to $3\mu_B$ for different matrices. Magnetic moments of this order are possessed by alloys based on Ag[3]; Au[4,5]; Co[6]; Cu[7]; Ni[8,5]; Pd[9]; Pt.[10] These alloys are either ferromagnets or paramagnets.

The magnetic properties of the iron atoms vary strongly in alloys based on γ iron. In the fcc alloys FeCrNi (stainless steel) and FeMn the magnetic mo-

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