is seen from the table that the radius R_0 of the transition is only slightly larger than the combined dimension of the electron orbits $2/\alpha^2+2/\beta^2$, i.e., the asymptotic theory determines the cross section at the limit of applicability of the theory. It is also seen that the Van-der-Waals contribution to the splitting[5] at $R=R_0$ is less than the exchange contribution. The temperature dependence of the cross section is weak, $\sigma\sim\ln^2T$, as seen from (37).

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Translated by J. G. Adashko

Strong fluctuations of electromagnetic waves in a random medium with finite longitudinal correlation radius of the inhomogeneities

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(Submitted 13 December 1977)
Zh. Eksp. Teor. Fiz. 75, 56–65 (July 1978)

The propagation of electromagnetic waves in a medium with random inhomogeneities of the refractive index is considered in the parabolic-equation approximation. The statistical wave intensity moments $\langle I^n\rangle$ of arbitrary order are expressed as Feynman continual integrals (in operator form). Expressions are obtained for the higher intensity moments with account taken of the finite longitudinal correlation radius of the refractive-index fluctuations, for both weak and intense intensity fluctuations. The limits of applicability of the Markov approximation, in which this correlation radius is assumed equal to zero, are obtained in the course of the calculation of the intensity moments $\langle I^n\rangle$.

PACS numbers: 42.20.–y

I. INTRODUCTION

The passage of electromagnetic waves in a randomly refracting medium with inhomogeneities of a scale that is large compared with the wavelength is accompanied in a number of cases by concentration of the scattered radiation in a narrow angle interval around the initial propagation direction; this leads to a fast growth of the field intensity fluctuations, followed by their saturation at a certain level. This effect can arise when radio waves propagate through the ionosphere or through interplanetary or interstellar plasma, or when light passes through a turbulent atomosphere.[1,2]

A theoretical description of strong intensity fluctuations is based on methods that go beyond the scope of perturbation theory, and have been first reported in Refs. 3–6. These methods yield equations, in closed form, for the statistical moments of the field, suitable also in the region of strong fluctuations. The approach developed in one of the papers[4] is based on a model in which the longitudinal correlation radius of the permittivity ε can be neglected in comparison with all the remaining longitudinal scales; it is also based on the assumption that the fluctuations of ε have a Gaussian probability distribution. Such a model makes it possible to describe the wave-propagation process as a Markov random process, and has therefore been dubbed the Markov approximation.

The question of the applicability limits of the Markov approximation in the derivation of the mean field $\bar{u}$ and of the coherent-field function Γ_2 was considered in Refs. 7 and 8, while its use for the description of amplitude-phase fluctuations in the geometric approximation was dealt with in Ref. 9, where a successive-approximation method was developed in which the Markov approximation serves as the first step. This method has made it possible to estimate the limits of applicability of the Markov approximation for the first two moments of the field, but not for the higher moments, owing to the complexity of the resultant equations.

The method used in the present paper was proposed by Fradkin[10] in quantum field theory and yields an expression for an arbitrary moment of the field, in the form of a Feynman continual integral or in the corresponding operator form,[11] without resorting to the Markov-random-process approximation. This approach is used to derive corrections to the Markov solution for the intensity moments $\langle I^n\rangle$ for both weak and strong fluctuations. These corrections are the results of allowance for the fact that the longitudinal correlation radius of the refractive index of the medium is finite.

2. DERIVATION OF THE IMPROVED EXPRESSIONS FOR THE MOMENTS

The existing theory of both weak fluctuations (the method of smooth perturbations) and strong fluctuations in the propagation of waves in a randomly inhomogeneous medium, despite the difference between the methods, makes use in fact, in all cases, the following model for the permittivity ε. The mean value of the permittivity $\bar{\varepsilon}$ is assumed constant, and the relative fluctuations $\varepsilon=(\varepsilon-\bar{\varepsilon})/\bar{\varepsilon}$ are assumed to be a delta-correlated Gaussian random field along the wave-propagation direction (in our case, along the x axis) in such a way that its correlation function is assumed to be

$$\langle\tilde{\varepsilon}(x_1,\rho_1)\tilde{\varepsilon}(x_2,\rho_2)\rangle=B_\varepsilon(x_1-x_2,\rho_1-\rho_2)=\delta(x_1-x_2)A(\rho_1-\rho_2),$$

$$A(0)-A(\rho)=D(\rho)=2\pi\iint_{-\infty}^{\infty}\Phi_\varepsilon(0,\varkappa)(1-\cos\varkappa\rho)d^2\varkappa. \tag{2.1}$$

The angle brackets denote here averaging over all possible realizations of $\tilde{\varepsilon}$, while $\Phi_\varepsilon(\varkappa_x,\varkappa)$ is the three-dimensional spectral density of the fluctuations $\tilde{\varepsilon}$.

We consider the corrections to the Markov solution for intensity moments of arbitrary order, necessitated by the allowance for the finite longitudinal correlation radius of the fluctuations $\tilde{\varepsilon}$. We assume, as before, that $\tilde{\varepsilon}(x,\rho)$ is a Gaussian random field, but take account of the fact that its correlation function ($B_\varepsilon(x_1-x_2, \rho_1-\rho_2)$, while a sharp function of the argument x_1-x_2, is not a delta function.

We consider the statistical moment of the field of the wave u:

$$\left\langle\prod_{k=1}^{n}u(x,\rho_{2k-1})u^*(x,\rho_{2k})\right\rangle=\Gamma_{2n}(x,\rho_1,\ldots,\rho_{2n}), \tag{2.2}$$

which is called the coherence function of $2n$-th order. If we describe the wave propagation with the aid of a stochastic parabolic equation then, using the methods of Ref. 10, we can write its solution in operator form[11]:

$$u(x,\rho)=\exp\left\{\frac{i}{2k}\int_0^x\frac{\delta^2}{\delta\tau^2(\xi)}\right\}u_0\left(\rho+\int_0^x\tau(\xi)d\xi\right)$$
$$\times\exp\left\{\frac{ik}{2}\int_0^x dx'\tilde{\varepsilon}\left(x',\rho+\int_{x'}^x\tau(\xi)d\xi\right)\right\}\Big|_{\tau=0}. \tag{2.3}$$

Substituting now (2.3) in (2.2) and averaging over the Gaussian fluctuations $\tilde{\varepsilon}$, we get

$$\Gamma_{2n}(x,\{\rho_i\})=\hat{L}\Gamma_{2n}^{(0)}\left(\left\{\rho+\int_0^x d\xi\,\tau(\xi)\right\}\right)$$
$$\times\exp\left\{-\frac{k^2}{8}\int_0^x\int_0^x dx'dx''\sum_{j,l=1}^{2n}(-1)^{j+l}\langle\tilde{\varepsilon}_j'\tilde{\varepsilon}_l''\rangle\right\}, \tag{2.4}$$

where

$$\hat{L}=\exp\left\{\frac{i}{2k}\int_0^x d\xi\sum_{j=1}^{2n}\frac{\delta^2}{\delta\tau_j^2(\xi)}(-1)^{j+1}\right\},$$

$$\langle\tilde{\varepsilon}_j'\tilde{\varepsilon}_l''\rangle=B_\varepsilon\left(x'-x'',\rho_j-\rho_l+\int_{x'}^x\tau_j(\xi)d\xi-\int_{x''}^x\tau_l(\xi)d\xi\right)$$

$$=\iint_{-\infty}^{\infty}d^2\varkappa\,F_\varepsilon(x'-x'',\varkappa)\exp(i\varkappa\mathbf{p}_{jl}),$$

$$\mathbf{p}_{jl}=\rho_j-\rho_l+\int_{x'}^x\tau_j(\xi)d\xi-\int_{x''}^x\tau_l(\xi)d\xi.$$

Here $F_\varepsilon(x'-x'',\varkappa)$ is the two-dimensional spectral density of the ε fluctuations.[12]

The expression for Γ_{2n} in the Markov approximation[13] can be obtained from (2.4) by substituting there $F_\varepsilon(x'-x'',\varkappa)$ in the form

$$F_\varepsilon(x'-x'',\varkappa)=2\pi\delta(x'-x'')\Phi_\varepsilon(0,\varkappa), \tag{2.5}$$

which corresponds to the approximation (2.1):

$$\Gamma_{2n}^{M}(x,\{\rho_i\})=\hat{L}\Gamma_{2n}^{(0)}\left(\left\{\rho+\int_0^x\tau(\xi)d\xi\right\}\right)$$
$$\times\exp\left\{\frac{k^2}{8}\sum_{j,l=1}^{2n}(-1)^{j+l}\int_0^x dx'D\left(\rho_j-\rho_l+\int_{x'}^x d\xi[\tau_j(\xi)-\tau_l(\xi)]\right)\right\}\Big|_{\tau=0}. \tag{2.6}$$

We shall be interested in the statistical moments of the intensity of a plane wave of unit amplitude, propagating in a randomly inhomogeneous medium. The expression for $\langle I^n\rangle$ is obtained from (2.4) by putting in it $\Gamma_{2n}^{(0)}=1$ and $\{\rho_i\}=0$:

$$\langle I^n\rangle\equiv\Gamma_{2n}(x,0)=\hat{L}\exp\left\{-\frac{k^2}{8}\int_0^x\int_0^x dx'dx''\right.$$
$$\left.\times\sum_{j,l=1}^{2n}(-1)^{j+l}B_\varepsilon\left(x'-x'',\int_{x'}^x\tau_jd\xi-\int_{x''}^x\tau_ld\xi\right)\right\}\Big|_{\tau_\alpha=0}. \tag{2.7}$$

In the region of weak intensity fluctuations, the argument of the exponentials in (2.4) and (2.7) is small, and the exponential can be expanded in a series. Confining ourselves to the first two terms of the expansion and applying the operator in them, we obtain

$$\langle I^n\rangle=1+\frac{n(n-1)}{2}[\langle I^2\rangle-1]+\ldots, \tag{2.8}$$

where

$$\langle I^2\rangle-1=\beta^2$$
$$=\frac{k^2}{2}\int_0^x\int_0^x dx'dx''\iint_{-\infty}^{\infty}d^2\varkappa\,F_\varepsilon(x'-x'',\varkappa)\left[\cos\frac{\varkappa^2}{2k}(x'-x'')-\cos\frac{\varkappa^2}{2k}(x'+x'')\right]. \tag{2.9}$$

It should be noted that expression (2.8) for the n-th intensity moment of a wave in an extended randomly inhomogeneous medium takes a form similar to that of the expressions obtained in Refs. 14 and 15 for the intensity moments $\langle I^n\rangle$ behind a phase screen. If we substitute (2.5) in (2.9) and (2.8), we find immediately the expression that follows from the smooth-perturbation method for $\langle I^n\rangle$. Formula (2.8) shows that all the moments of the weak intensity fluctuations are determined by the second moment.

We consider now the region of strong intensity fluctuations, which is characterized by the fact that in this region the parameter β^2 calculated in first-order perturbation theory tends to infinity. It is then convenient to represent Γ_{2n} in the form

$$\Gamma_{2n}=\Gamma_{2n}^{\text{M}}+\Delta\Gamma_{2n}, \tag{2.10}$$

where $\Gamma_{2n}^{\mathbf{M}}$ is the coherence function of order $2n$, calculated in the Markov approximation, and $\Delta\Gamma_{2n}$ is the correction added to $\Gamma_{2n}^{\mathbf{M}}$ when account is taken of the finite longitudinal correlation radius of $\tilde{\varepsilon}$. On the other hand, we can write the identities

$$\Gamma_{2n}\equiv\hat{L}\Gamma_{2n}^{(0)}\exp\{-\psi_{\text{M}}\}\exp\{-\psi+\psi_{\text{M}}\}|_{\tau_\alpha=0},$$
$$\Gamma_{2n}^{\text{M}}\equiv\hat{L}\Gamma_{2n}^{(0)}\exp\{-\psi_{\text{M}}\}|_{\tau_\alpha=0}, \tag{2.11}$$

where ψ is the argument of the exponential in (2.4) and $\psi_{\mathbf{M}}$ is the same for (2.6). If the correction to the Markov approximation is assumed small, then the second exponential in (2.11) can be expanded in a series in which only the first two terms are retained

$$\Gamma_{2n}\approx\Gamma_{2n}^{\text{M}}+\hat{L}\,\Gamma_{2n}^{(0)}\exp\{-\psi_{\text{M}}\}[\psi_{\text{M}}-\psi]|_{\tau_\alpha=0}+\dots \tag{2.12}$$

The correction $\Delta\Gamma_{2n}$ thus satisfies the expression

$$\Delta\Gamma_{2n}=\hat{L}\Gamma_{2n}^{(0)}\exp\{-\psi_{\text{M}}\}[\psi_{\text{M}}-\psi]|_{\tau_\alpha=0}. \tag{2.13}$$

Using the asymptotic method developed in Ref. 13, we can obtain from (2.13) the correction to the n-th intensity moment of a plane wave in the strong-fluctuation region:

$$\Delta\langle I^n\rangle\equiv\Delta\Gamma_{2n}(0)=\hat{L}\exp\{-\psi_{\text{M}}(0)\}[\psi_{\text{M}}(0)-\psi(0)]|_{\tau_\alpha=0}, \tag{2.14}$$

where

$$\hat{L}\exp\{-\psi_{\text{M}}(0)\}\psi(0)|_{\tau_\alpha=0}\equiv J_n$$
$$=\hat{L}n!\frac{k^2}{8}\sum_{j,l=1}^{2n}(-1)^{j+l}\int_0^x dx'\int_0^x dx''\,\text{B}_\varepsilon\left(x'-x'',\int_{x'}^x \tau_j d\xi-\int_{x''}^x \tau_l d\xi\right)$$
$$\times\exp\left\{-\frac{k^2}{4}\sum_{j=1}^{n}\int_0^x dy\,D\left(\int_y^x(\tau_{2j-1}-\tau_{2j})\,d\xi\right)\right\}\Big|_{\tau_\alpha=0}. \tag{2.15}$$

The second term of (2.14)

$$\hat{L}\exp\{-\psi_{\text{M}}(0)\}\psi_{\text{M}}(0)|_{\tau_\alpha=0}\equiv J_n^{\text{M}} \tag{2.16}$$

can be obtained from (2.15) by going in the limit to the δ-correlation function $\text{B}_\varepsilon(x_1-x_2)$.

If we use in the pre-exponential factor of (2.15) the spectral expansion of the functions B_ε, and introduce the new variables

$$\mathbf{r}_j=\tau_{2j-1}-\tau_{2j},\quad 2\mathbf{R}_j=\tau_{2j-1}+\tau_{2j},$$

then simple transformations convert this expression to

$$J_n=n!\frac{k^2}{4}\exp\left\{\frac{i}{k}\sum_{j=1}^{n}\int_0^1 d\xi\frac{\delta^2}{\delta\mathbf{r}_j\,\delta\mathbf{R}_j}\right\}\int_0^x\int_0^x dx'\,dx''\iint_{-\infty}^{\infty}d^2\varkappa\,F_\varepsilon(x'-x'',\varkappa)$$
$$\times\sum_{j,l=1}^{n}\exp\left[i\varkappa\left(\int_{x'}^x\mathbf{R}_j\,d\xi-\int_{x''}^x\mathbf{R}_l\,d\xi\right)\right]\left\{\cos\frac{\varkappa}{2}\left[\int_{x'}^x\mathbf{r}_j\,d\xi-\int_{x''}^x\mathbf{r}_l\,d\xi\right]\right.$$
$$\left.-\cos\frac{\varkappa}{2}\left[\int_{x'}^x\mathbf{r}_j\,d\xi+\int_{x''}^x\mathbf{r}_l\,d\xi\right]\right\}\exp\left\{-\frac{k^2}{4}\sum_{i=1}^{n}\int_0^x dy\,D\left(\int_y^x d\xi\,\mathbf{r}_i\right)\right\}\Big|_{\mathbf{R}_\alpha=\mathbf{r}_\alpha=0}. \tag{2.17}$$

Applying the operators, we ultimately get

$$J_n=-n!\frac{k^2}{4}\int_0^x dx'\int_0^x dx''\iint d^2\varkappa\,F_\varepsilon(x'-x'',\varkappa)\sum_{s,l=1}^{n}\left[\cos\frac{\varkappa^2}{2k}(x''-x')\right.$$
$$\left.-\cos\frac{\varkappa^2}{2k}(2x-x'-x''-2(x-\max(x',x''))\delta_{sl})\right]$$
$$\times\exp\left\{-\frac{k^2}{4}\sum_{i=1}^{n}\int_0^x dy\,D\left(\frac{\varkappa}{k}\int_y^x d\xi[\delta_{is}\theta(\xi-x')-\delta_{il}\theta(\xi-x'')]\right)\right\}, \tag{2.18}$$

where δ_{ls} is the Kronecker symbol and $\theta(\xi-x')$ is the step function. Summation over s and l yields

$$J_n=n!n(n-1)\frac{k^2}{4}\int_0^x dx'\int_0^x dx''\iint_{-\infty}^{\infty}d^2\varkappa\,F_\varepsilon(x'-x'',\varkappa)$$
$$\times\left\{\cos\frac{\varkappa^2}{2k}(2x-x'-x'')-\cos\frac{\varkappa^2}{2k}(x'-x'')\right\}$$
$$\times\exp\left\{-\frac{k^2}{4}\int_0^x dy\left[D\left(\frac{\varkappa}{k}(x-\max(x'',y))\right)+D\left(\frac{\varkappa}{k}(x-\max(x',y))\right)\right]\right\}. \tag{2.19}$$

Substituting here $F_\varepsilon(x'-x'',\varkappa)$ from (2.5) and integrating with respect to x'' we obtain for $J_n^{\mathbf{M}}$ the expression

$$J_n^{\text{M}}=n!n(n-1)\frac{\pi k^2}{2}\int_0^x dx'\iint d^2\varkappa\,\Phi_\varepsilon(0,\varkappa)\left\{\cos\frac{\varkappa^2}{k}(x-x')-1\right\}$$
$$\times\exp\left\{-\frac{k^2}{2}\int_0^x dy\,D\left(\frac{\varkappa}{k}(x-\max(x',y))\right)\right\}. \tag{2.20}$$

If we now denote by J_2 and $J_2^{\mathbf{M}}$ the expressions (2.19) and (2.20) with $n=2$, then it is seen that

$$\Delta\langle I^n\rangle=n!\frac{n(n-1)}{4}[J_2^{\text{M}}-J_2]=\frac{n!n(n-1)}{4}\Delta\langle I^2\rangle. \tag{2.21}$$

Thus, recognizing that in the region of strong fluctuations the Markov approximation yields[13]

$$\langle I_{\text{M}}^n\rangle=n![1+n(n-1)(\langle I_{\text{M}}^2\rangle-2)/4+\dots], \tag{2.22}$$

we obtain for the corrected moment $\langle I^n\rangle$ an analogous expression

$$\langle I^n\rangle=n![1+n(n-1)(\langle I^2\rangle-2)/4+\dots], \tag{2.23}$$

where $\langle I^2\rangle$, with allowance for (2.21), takes the form

$$\langle I^2\rangle=\langle I_{\text{M}}^2\rangle+\Delta\langle I^2\rangle. \tag{2.24}$$

Consequently, just as in the case of weak fluctuations, the n-th moment of the intensity is expressed in terms of the second moment. It is then easy to conclude that the limits of applicability of the Markov approximation, for both weak and strong fluctuations, are the same for all intensity moments. These limits will be determined in the following sections for both fluctuation regions.

3. LIMITS OF APPLICABILITY OF THE MARKOV APPROXIMATION FOR WEAK INTENSITY FLUCTUATIONS

In the region of weak fluctuations, the expression for the dispersion of the intensity fluctuations, with account taken of the finite correlation radius of ε, is given by formula (2.9). If we change in this formula from the variables x' and x'' to their sum and difference

$$\eta'=(x'+x'')/2,\quad \xi'=x'-x'',$$

we obtain

$$\beta^2=2\pi k^2\int_0^x d\xi'\int_{\xi'/2}^{x-\xi'/2}d\eta\int_0^\infty d\varkappa\,\varkappa F_\varepsilon(\xi',\varkappa)\left[\cos\frac{\varkappa^2\xi'}{2k}-\cos\frac{\varkappa^2\eta}{2k}\right].\quad (3.1)$$

The two-dimensional spectral density $F_\varepsilon(\xi',\varkappa)$ corresponding to the structure function

$$D_\varepsilon(r)=C_\varepsilon^2r^\mu\quad(0<\mu<2),$$

is of the form[12]

$$F_\varepsilon(\xi',\varkappa)=\frac{C_\varepsilon^2}{\pi^2}\sin\frac{\pi\mu}{2}\cdot 2^{\mu/2-1}\Gamma\left(1+\frac{\mu}{2}\right)\varkappa^{-\mu-2}(\varkappa\xi')^{1+\mu/2}K_{1+\mu/2}(\varkappa\xi').\quad (3.2)$$

If we substitute (3.2) in (3.1), then β^2 can be calculated, but after rather cumbersome computations. The same result, apart from numerical coefficients, can be obtained much faster by approximating the function $F_\varepsilon(\xi',x)$ by

$$F_\varepsilon(\xi',\varkappa)=\begin{cases}2^\mu\dfrac{C_\varepsilon^2}{\pi}\dfrac{\Gamma(1+\mu/2)}{\Gamma(-\mu/2)}\varkappa^{-2-\mu}, & \xi'\varkappa\leqslant 1,\\ 0, & \xi'\varkappa>1.\end{cases}\quad (3.3)$$

We now substitute (3.3) in (3.1) and introduce new dimensionless variables

$$\xi'=x\xi,\quad \eta'=x\eta,\quad \varkappa=\left(\frac{k}{x}\right)^{1/2}q.\quad (3.4)$$

We then get the expression

$$\beta^2=N\int_0^\infty dq\,q^{-\mu-1}R(\alpha,q),\quad (3.5)$$

where $N\sim C_\varepsilon^2k^{(3-\mu)/2}x^{(3+\mu)/2}\sim\beta_0^2$ is the variance of the intensity fluctuations, calculated in first-order perturbation theory, $\alpha=(kx)^{1/2}$ is a large parameter, and

$$R(\alpha,q)=\alpha\int_0^1 d\xi\,\theta\left(\frac{1}{q\alpha}-\xi\right)T(q,\xi),\quad (3.6)$$

where

$$T(q,\xi)=\int_{\xi/2}^{1-\xi/2}d\eta\left[\cos\frac{q^2\xi}{2}-\cos q^2\eta\right]$$

$$=(1-\xi)\cos\frac{q^2\xi}{2}-\frac{2}{q^2}\sin\frac{q^2(1-\xi)}{2}\cos\frac{q^2}{2}.\quad (3.7)$$

The condition $\varkappa\xi'\leqslant 1$ takes in terms of the dimensionless variables (3.4) the form $q^2\xi\leqslant 1/\alpha$, so that $q^2\xi/2$ satisfies the inequality

$$q^2\xi/2\leqslant q/2\alpha.$$

Since the maximum value of q that can be of interest to us is of the order of $q_m\sim l_0^{-1}(x/k)^{1/2}$, where l_0 is the internal turbulence scale, we have

$$q_m/\alpha\sim\lambda/l_0\ll 1,$$

and consequently $q^2\xi/2\ll 1$ in our case and we can expand in (3.7) the sine and the cosine, which contain $q^2\xi/2$:

$$T(q,\xi)\approx 1-\frac{1}{q^2}\sin q^2-\xi\sin^2\frac{q^2}{2}+\ldots.\quad (3.8)$$

Substituting (3.8) in (3.6) and integrating, we get

$$R(\alpha,q)=\alpha\min\left(1,\frac{1}{q\alpha}\right)\left[1-\frac{1}{q^2}\sin q^2\right]-\frac{\alpha}{2}\left[\min\left(1,\frac{1}{q\alpha}\right)\right]^2\sin^2\frac{q^2}{2}.$$

Substituting in turn $R(\alpha,q)$ in (3.5), we have

$$\beta^2\sim N\alpha\int_0^{1/\alpha}dq\,q^{-\mu-1}\left[1-\frac{1}{q^2}\sin q^2-\frac{1}{2}\sin^2\frac{q^2}{2}\right]+$$

$$+N\int_{1/\alpha}dq\,q^{-\mu-2}\left[1-\frac{1}{q^2}\sin q^2\right]-\frac{N}{2\alpha}\int_{1/\alpha}dq\,q^{-\mu-3}\sin^2q^2/2.\quad (3.9)$$

Calculating the integrals in (3.9), we get

$$\beta^2\sim N(1-1/\alpha-1/\alpha^{2-\mu}+\ldots).\quad (3.10)$$

The first term in (3.10) coincides with β_0^2, i.e., it is the Markov approximation for β^2. Comparing the remaining terms of the expansions, we reach the conclusion that at $\beta^2\ll 1$ the Markov approximation can be used to calculate all the intensity moment, provided that

$$\alpha\gg 1,\quad\text{or}\quad (kx)^{1/2}\gg 1.\quad (3.11)$$

4. LIMITS OF APPLICABILITY OF THE MARKOV APPROXIMATION IN THE REGION OF STRONG FLUCTUATIONS

It follows from (2.21) that the correction to the Markov expression for $\langle I^2\rangle$ is of the form $\Delta\langle I^2\rangle=J_2^M-J_2$. We consider first the value of obtained from (2.19) at $n=2$. We make in J_2 the same change of variables as in the preceding section:

$$J_2=N\int_0^1 d\xi\int_{\xi/2}^{1-\xi/2}d\eta\int_0^\infty dq\,q\tilde F_\varepsilon(q,\xi)\left[\cos q^2\xi/2-\cos q^2\eta\right]\exp\{-pN\varphi(\xi,\eta)q^{1+\mu}\}.\quad (4.1)$$

Here $\tilde F_\varepsilon(q,\xi)$ is a function that can be obtained from (3.3) by going to the dimensionless variables ξ and q while p is a numerical coefficient of order of unity and

$$\varphi(\xi,\eta)=\left|\eta+\frac{\xi}{2}\right|^{1+\mu}\left[1-\frac{1+\mu}{2+\mu}\left(\eta+\frac{\xi}{2}\right)\right]$$

$$+\left|\eta-\frac{\xi}{2}\right|^{1+\mu}\left[1-\frac{1+\mu}{2+\mu}\left(\eta-\frac{\xi}{2}\right)\right].\quad (4.2)$$

We represent J_2 in the form

$$J_2=N\int_0^\infty dq\,q^{-\mu-1}R(N,\alpha,q),\quad (4.3)$$

where

$$R(N,\alpha,q)=\alpha\int_0^1 d\xi\,\theta\left(\frac{1}{q\alpha}-\xi\right)T(N,q,\xi),\quad (4.4)$$

$$T(N,q,\xi)=\int_{\xi/2}^{1-\xi/2}d\eta\left[\cos(a^2\xi/2)-\cos q^2\eta\right]\exp\{-pN\varphi(\xi,\eta)q^{1+\mu}\}.\quad (4.5)$$

In the strong-fluctuation region, N is a large parameter. On the other hand, $\alpha=(kx)^{1/2}$ is also a large parameter. At the same time, we shall regard N/α as a small quantity, as is usually the case under real experimental conditions. We shall make use of this circumstance in the calculation of J_2.

At $0\leqslant\xi\leqslant 1$ we can carry out the following expansion:

$$T(N, q, \xi) \approx T(N, q, 0) + T_\xi'(N, q, 0)\xi + \dots \qquad (4.6)$$

which leads subsequently to expansion in powers of the parameter $N\alpha^{-(1+\mu)}$. We calculate the expansion coefficients $T(N, q, \xi)$ and substitute them in (4.6):

$$T(N, q, \xi) \approx \int_0^1 d\eta[1-\cos q^2\eta]\exp\left\{-pN\left[\eta^{1+\mu}\left(1-\frac{1+\mu}{2+\mu}\eta\right)\right]q^{1+\mu}\right\} - \xi(1-\cos q^2)\exp\left(-\frac{pN}{2+\mu}q^{1+\mu}\right) + \dots \qquad (4.7)$$

Substituting (4.7) first in (4.4) and then in (4.3), calculating the integrals, and using then their asymptotic form in terms of the small parameter $N\alpha^{-(1+\mu)}$, we get

$$J_2 \sim N^{(2\mu-2)/(1+\mu)}\left(1+\frac{N^{1/(1+\mu)}}{\alpha}\right) - N\alpha^{\mu-3}\left[c_1 + c_2\frac{N}{\alpha^{1+\mu}} + c_3\left(\frac{N}{\alpha^{1+\mu}}\right)^2 + \dots\right]. \qquad (4.8)$$

To obtain J_2^M from J_2 we must let α go to infinity in J_2. We then get $J_2^M \sim N^{(2\mu-2)/(1+\mu)}$. Thus,

$$\Delta\langle I^2\rangle \sim -\frac{N^{(2\mu-1)/(1+\mu)}}{\alpha} + N\alpha^{\mu-3}\left[c_1 + c_2\frac{N}{\alpha^{1+\mu}} + c_3\left(\frac{N}{\alpha^{1+\mu}}\right)^2 + \dots\right]. \qquad (4.9)$$

Comparing $\Delta\langle I^2\rangle$ with $\langle I_M^2\rangle = 2 + cN^{(2\mu-2)/(1+\mu)}$, we obtain the condition

$$N^{1/(1+\mu)} \ll \alpha. \qquad (4.10)$$

The condition (4.10) can be written in the form

$$\left[C_\varepsilon^2 k^3 x\left(\frac{x}{k}\right)^{(1+\mu)/2}\right]^{1/(1+\mu)} \ll (kx)^{1/2}. \qquad (4.11)$$

If it is recognized that the combination $(C_\varepsilon^2 k^2 x)^{-1/(1+\mu)}$ is proportional to the coherence radius ρ_{coh}, then (4.11) goes over into

$$\rho_{coh} \gg \lambda. \qquad (4.12)$$

This condition can be recast in still another form: $r_0 \ll x$, where $r_0 = x/k\rho_{coh}$ is the larger of the correlation scales of the intensity fluctuations at $N \gg 1$.

5. CONCLUSION

The results show that the statistical intensity moments $\langle I^n\rangle$ calculated with account taken of the finite longitudinal correlation radius of the fluctuations $\tilde{\varepsilon}$, are expressed in terms of $\langle I^2\rangle$ just as in the Markov approximation, but $\langle I^2\rangle$ should now taken to mean the quantity corrected for the finite radius of the correlation of $\tilde{\varepsilon}$. In other words, the Markov approximation does not change the form of the probability distribution of the intensity fluctuation, and changes only its parameter $\langle I^2\rangle$.

In the weak-intensity-fluctuation region we find that the Markov approximation is valid when $(kx)^{1/2} \gg 1$, which is equivalent to the inequality

$$\lambda \ll (\lambda x)^{1/2} \ll x. \qquad (5.1)$$

In the strong-fluctuation region we have the condition $N^{1/(1+\mu)} \ll (kx)^{1/2}$, which can be recast in the form

$$\lambda \ll \rho_{coh} \ll r_0 \ll x. \qquad (5.2)$$

The inequalities (5.1) and (5.2) have a simple physical meaning.

First, it is easily understood why the limits of applicability of the Markov approximation are different for weak and strong intensity fluctuations. So long as the smallest of all the longitudinal scales in the problem of wave propagation in a medium with random inhomogeneities is the correlation radius of $\tilde{\varepsilon}$ (its role is played by the dimension of the Fresnel zone), the Markov approximation remains valid. On going into the region of strong fluctuations, a new scale $\Delta x \sim \rho_{coh}(kx)^{1/2}$ appears and decreases gradually, so that if the parameter N is large enough the scale Δx can become smaller than the correlation radius of $\tilde{\varepsilon}$. When this situation arises, the Markov approximation can no longer be used.

Second, the inequalities (5.1) and (5.2) can be regarded as lower and upper bounds of the scale of the correlation function of the intensities $B_I(x, \rho_1 - \rho_2) = \langle I(x, \rho_1) I(x, \rho_2)\rangle - \langle I(x, \rho_1)\rangle\langle I(x, \rho_2)\rangle$. Actually, $(\lambda x)^{1/2}$ is the only characteristic scale of the function B_I at $N \ll 1$. The quantities ρ_{coh} and r_0 are respectively the smaller and larger scales of the function B_I at $N \gg 1$. Thus, the Markov approximation is valid only when arbitrary characteristic scales that appear when the wave propagates remain small compared with the length of the route. It is interesting to note that conditions similar to (5.2) arise also in the analysis of the limits of the Markov approximation for the coherence function of the wave Γ_2 in Ref. 8.

The author thanks V. I. Tatarskiĭ and V. I. Klyatskin for useful discussions.

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Translated by J. G. Adashko