The thermodynamic description of hadronic matter

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The relationship between the thermodynamic potential of a system of strongly interacting particles and the two-particle, three-particle, etc. scattering characteristics of the system is obtained in the framework of a scheme describing the evolution of a quantum system not with time, as is usually the case, but with change in the magnitude of the coupling constant. The method proposed does not require solution of the Faddeev equations and does not lead to singularities of the forward-scattering type. Some very simple applications to the theory of hot nucleon matter are considered.

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1. INTRODUCTION

The thermodynamic characteristics of systems of strongly interacting particles[1] serve as an important element in the description of many systems in physics and related sciences: the excited states of an atomic nucleus, the systems that arise in the collision of high-energy particles, matter at catastrophic stages of the evolution of stars or in the early stages of the development of the Universe, etc.

A microscopic calculation of the thermodynamic parameters of hadronic matter on the basis of the modern theory of the strong interaction—quantum chromodynamics—is, as yet, possible only at extremely high pressures or temperatures, when the property of asymptotic freedom is manifested.[1] Under less extreme conditions, and, in particular, in the non-relativistic (with respect to the hadrons) region considered below, it is reasonable to use a semiphenomenological approach, relating the thermodynamics of the system to the particle-particle scattering parameters (cf., e.g., Ref. 2).

This approach takes its history from the work of Beth and Uhlenbeck[3] (see also Ref. 4, Sec. 77), in which the following expression for the second virial coefficient of a nonideal gas was found:

$$b_2 = \sum_n \exp(\beta|E_n|) + \sum_l (2l+1)\frac{1}{\pi}\int_0^\infty dk \exp\left(-\frac{\beta k^2}{2m^*}\right)\frac{d\delta_l(k)}{dk}. \quad (1.1)$$

Here and below, $\beta = 1/kT$, E_n is the energy of a bound state, m^* is the reduced mass, δ_l is the phase shift for orbital angular momentum l, and k is the momentum in the c.m. frame.

One of the lines of development of the approach under consideration led to the construction of the thermodynamics of dilute many-particle systems in which the average distance between particles is large compared with the scattering length (the gas approximation; cf., e.g., Refs. 5 and 6). The interaction process in such a system reduces to successive scatterings of particles by each other, such that in the interval between these events the wavefunction of a pair of particles has time to approach its asymptotic form. It is not surprising, therefore, that the macroscopic parameters of a dilute system are completely expressed in terms of the characteristics of the mutual scattering of a pair of particles.

To this class we can assign the Landau theory of the Fermi liquid, together with its applications to the theory of the nucleus (Migdal's theory[7]). Although the applicability of these theories is not restricted by the requirement that the system itself be dilute, they describe only weakly excited states of matter, when the gas of elementary excitations (quasi-particles) can be assumed to be dilute. In this case the answer is expressed in terms of the scattering parameters of the quasi-particles in the medium and not those of the the particles *in vacuo*. These parameters should be taken from other macroscopic characteristics of the system.

Another line of development of the semiphenomenological approach (see the paper by Dashen, Ma, and Bernstein, [8] and also the earlier papers by Berezin[9]) led to an expression for the virial coefficients in terms of the complete scattering matrix S, which connects an arbitrary number of particles and contains complete information on the energy spectrum of the system:

$$b_N \propto \mathrm{Im}\int dE\, e^{-\beta E}\, \mathrm{Sp}_N\left[S^{-1}\frac{dS}{dE}A\right]^c. \quad (1.2)$$

Here, E is the energy, A is the exchange operator (symmetrization or antisymmetrization of the wavefunction), the superscript c indicates that only connected scattering diagrams are taken into account, and the trace is taken over states with N particles.

Even for $N = 3$, serious difficulties stand in the way of the practical utilization of formula (1.2); these are connected with the fact that the scattering matrix is obtained in this case as the outcome of a cumbersome numerical integration of the Faddeev equations. Moreover, for three and more particles, special purely kinematic singularities corresponding to the vanishing of the energy denominator in the propagator appear in the corresponding "forward"-scattering amplitude that enters in the right-hand side of (1.2).[8] For example, even in the simplest diagram, in Fig. 1, in which a blob denotes the pair scattering amplitude, there appears the denominator

$$E(\mathbf{k}_1) + E(\mathbf{k}_2) - E(\mathbf{k}_1') - E(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_1'), \quad (1.3)$$

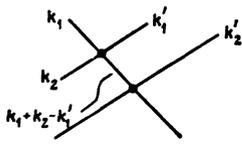which vanishes for forward scattering $(\mathbf{k}_{1,2} = \mathbf{k}_{1,2}')$. Al-

FIG. 1.

though this difficulty has a fictitious character and all such singularities should cancel each other in the sum, a simple and effective prescription for removing them has not yet been proposed (see Ref. 10).

To overcome these difficulties we propose in this article to use a special quantum-mechanical approach, based on a description of the evolution of the system not with time, as is usually the case, but with change of the magnitude of the coupling constant (see Ref. 11). The essence of this approach is briefly described in Sec. 2, in which subsequently needed information pertaining to the two-nucleon problem is also given. In Sec. 3 the connection between the virial coefficients and the scattering characteristics is considered. Section 4 contains an investigation of the singularity of the forward-scattering type. Finally, in Sec. 5 some very simple applications involving a hot many-nucleon system are given.

The advantages of the proposed method in application to the problem of constructing a semiphenomenological thermodynamics are as follows. First of all, in the framework of this method a virial coefficient is expressed in the form of a rapidly convergent series, each term of which depends comparatively simply on the pair-scattering characteristics. In the end the virial coefficient can be expressed in the form of an explicit analytic function of the pair-scattering phase shift, the energy of the two-particle bound state, etc. Furthermore, the singular terms of the virial coefficient can be summed in closed form, and, as will be shown, their sum is exactly equal to zero. Therefore, such terms can be disregarded from the outset. Details pertaining to applications of the method of evolution in the coupling constant to the problem of three and more bodies can be found in a paper by the authors[12] (see also Ref. 13).

2. EVOLUTION IN THE COUPLING CONSTANT

Referring the reader to previous papers by the authors[11,12] for the details, in this section we give the basic relations of the method proposed. The coupling constant g is introduced by the relation

$$H = H_0 + gV, \tag{2.1}$$

where H and H_0 are the full and free Hamiltonians, and gV is the interaction Hamiltonian. The evolution equations should connect the real state of the system with its state when the interaction is switched off; the latter state corresponds to $t \to -\infty$ in the usual time-dependent technique, to $\beta = 0$ in the Matsubara technique in quantum statistics (see Refs. 5 and 6), and to $g = 0$ in the scheme being proposed. The matter reduces to replacing the usual evolution equations for the wavefunction Ψ and an operator O

$$i\frac{\partial \Psi}{\partial t} = H\Psi, \quad \frac{\partial O}{\partial t} = i[H, O]$$

by equations describing the evolution with change of the quantity g:

$$i\frac{\partial \Psi}{\partial g} = \Gamma\Psi, \quad \frac{\partial O}{\partial g} = i[\Gamma, O],$$

where Γ is the operator of translations in the coupling constant. The equations cited, together with (2.1), give

$$\Gamma(t) = \int_{-\infty}^{t} dt' V(t'),$$

where the time dependence of the operator corresponds to the Heisenberg picture.

From this, in the energy representation, there follow equations for the energy (the Hellmann-Feynman formula)

$$\frac{\partial E_n}{\partial g} = V_{nn}, \tag{2.2}$$

for the wavefunction (the index + corresponds to an outgoing wave)

$$\frac{\partial |n\rangle_+}{\partial g} = \sum_s \frac{V_{sn}|s\rangle_+}{E_n - E_s + i\delta} \tag{2.3}$$

and analogously for an incoming wave (index −), and, finally, for the operator:

$$\frac{\partial O_{mn}}{\partial g} = \sum_s \left\{ \frac{V_{ms}O_{sn}}{E_m - E_s - i\delta} + \frac{O_{ms}V_{sn}}{E_n - E_s + i\delta} \right\}. \tag{2.4}$$

Using (2.3) we obtain an equation for the scattering matrix $S_{mn} = {}_-\langle m|n\rangle_+$, which is nonzero only for $E_m = E_n$:

$$\frac{\partial S_{mn}}{\partial g} = -2\pi i \sum_s S_{ms} V_{sn} \delta(E_s - E_n). \tag{2.5}$$

In the case of elastic scattering of two particles (elementary or composite), there follows from this an equation for the phase shift:

$$\frac{\partial \delta_l(k)}{\partial g} = -\frac{m^* k}{2\pi} V_{kk,l}. \tag{2.6}$$

Finally, substituting the operator V in place of O in (2.4), we obtain closed equations for the matrix elements V_{mn} that determine the evolution of all the physical quantities:

$$\frac{\partial V_{mn}}{\partial g} = \sum_s V_{ms} V_{sn} \left\{ \frac{1}{E_m - E_s - i\delta} + \frac{1}{E_n - E_s + i\delta} \right\}. \tag{2.7}$$

This equation plays a central role in the method described.

We shall begin with an an application of the above equations to a two-body problem; specifically, we have in mind the neutron-proton system in the triplet state. In this case it is convenient to use the "separable" Yamaguchi potential, putting, for $g = 0$,

$$V_{kk'} = -\frac{2\pi\gamma^3}{m^*} [(k^2 + \gamma^2)(k'^2 + \gamma^2)]^{-1}. \tag{2.8}$$

In this expression, which is in the form of a product of two functions depending on k and k', respectively, the inverse range of the forces is $\gamma = 1.44 F^{-1}$, and the coupling constant is normalized in such a way that the

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value $g=1$ corresponds to the threshold for formation of the deuteron. The potential (2.8) acts only in the s state, and the index 1 is henceforth omitted.

The separated form (2.8) makes it possible to seek the solution of Eq. (2.7) in separated form too. This enables us to solve the problem completely in explicit analytic form.[2] We give the expressions for the deuteron energy:

$$-\frac{\varkappa^2}{2m^*} = -\frac{\gamma^2}{2m^*}(g^{1/2}-1)^2, \qquad (2.9)$$

where the inverse scattering length $\varkappa = 0.23\,\mathrm{F}^{-1}$, for the phase shift

$$\delta_{np}(k) = \pi - \operatorname{arctg}\frac{k}{\varkappa} + \operatorname{arctg}\frac{k}{2\gamma+\varkappa} - 2\operatorname{arctg}\frac{k}{\gamma} \qquad (2.10)$$

and for the interaction potential

$$V_{\mathbf{k}\mathbf{k}'} = v_{\mathbf{k}\mathbf{k}'} = -\frac{2\pi}{m^*}\left[\frac{1}{k}\frac{\partial\delta_{np}(k)}{\partial g}\frac{1}{k'}\frac{\partial\delta_{np}(k')}{\partial g}\right]^{1/2}\exp[i(\delta_{np}(k')-\delta_{np}(k))].$$
$$(2.11)$$

We turn now to the problem of three or more bodies, which is characterized by the presence of a larger number of particles than can be directly connected by a pair interaction. Here it is necessary, first of all, to separate out the unconnected diagrams corresponding to free flight of one or more particles. With this purpose we put

$$V_{mn} = V_{mn}{}^c + U_{mn},$$

where V^c is the connected part of the matrix element and U is the unconnected part, which is a combination of a matrix element for less than N particles and the δ-functions of the momentum-conservation law for freely moving particles. Substitution of the latter equality into (2.7) gives an equation for the connected part (the part of interest to us in what follows) of the matrix element of the potential:

$$\frac{\partial V_{mn}{}^c}{\partial g} = \sum_s [(U_{ms}U_{sn})^c + U_{ms}V_{sn}{}^c + V_{ms}{}^c U_{sn} + V_{ms}{}^c V_{sn}{}^c] \qquad (2.12)$$
$$\times \left[\frac{1}{E_m - E_s - i\delta} + \frac{1}{E_n - E_s + i\delta}\right].$$

Here, in the free term, we have omitted the unconnected part of the combination UU, which vanishes owing to the presence of the derivative $\partial U/\partial g$ in the left-hand side.

In the case of the three-body problem,

$$U_{mn} = \sum_{i=1}^3 \delta_{m_i n_i} v_{m_i n_i}, \quad (U_{ms}U_{sn})^c = \sum_{i\neq j} \delta_{m_i s_i} \delta_{s_j n_j} v_{m_i s_i} v_{s_j n_j}, \quad (2.13)$$

where i labels the freely moving particle (and, simultaneously, the interacting pair). Henceforth we use the abbreviated forms

$$U = \sum_i v_i, \qquad (UU)^c = \sum_{i\neq j} v_i v_j.$$

Of course, Eq. (2.12) cannot be solved exactly. However, a series of successive iterations of it, starting from the free term, is found to be rapidly convergent, and even the zeroth iteration is in fair agreement with experiment in the problem of neutron–deuteron scattering.[12] This is not surprising, since, unlike in the usual Born series (and the series from successive

iterations of the Faddeev equations), the conditions of unitarity and causality of the scattering matrix are fulfilled exactly at each stage of the successive approximations. The unitarity condition is connected with the conservation of the property of hermiticity of the matrix V_{mn} [cf. (2.5)], or, in another language, with the expansion of the phase shift rather than of the scattering amplitude (cf. (2.6)); the causality condition stems from the rules for going round the pole in the energy denominator of (2.12).

On the other hand, the usual considerations of "pology" are in force: the higher the order of iteration, the further into the complex energy plane are the singularities of the matrix element V_{mn} located, and the smaller are the corresponding residues and discontinuities across the cuts. For the three-body problem the zeroth iteration is depicted in Fig. 2a (a blob denotes the matrix element v), and the next iterations are depicted in Figs. 2b and 2c.

3. THE THERMODYNAMIC POTENTIAL AND SCATTERING MATRIX

The method expounded above makes it possible to find a simple connection between the thermodynamic potential Ω of the system and the scattering matrix. We shall start from the familiar expression (see Ref. 4, Sec. 35)

$$\Omega = -\frac{1}{\beta}\ln\left(\sum_N \xi^N Z_N\right), \qquad (3.1)$$

where N is the total number of particles, $\xi = e^{\beta\mu}$ is the activity, μ being the chemical potential, and

$$Z_N = \sum_n' \exp(-\beta E_{nN})$$

is the partition function of a system of N particles. The latter quantity, when expressed in diagrammatic language, contains both connected and unconnected diagrams. We can get rid of the latter[8,9] using the well known rule of Hubbard,[14] which has the symbolic form $a \to \exp(a^c)$. As applied to (3.1), this gives[8]

$$\Omega = \Omega_0 - \frac{1}{\beta}\sum_{N=2}^\infty \xi^N Z_N{}^c, \qquad (3.2)$$

where Ω_0 is the potential of an ideal gas ($N=1$). We note that the expression (3.2) pertains to a system of particles of one kind (or to a mixture whose components have the same values of N and μ); in the general case the quantity N is replaced by the set $N_1, N_2, \ldots$, and μN by the sum $\sum_i \mu_i N_i$.

Since the potential Ω is related directly to the pressure, the series in N in (3.2) has the meaning of the virial expansion for the pressure. Separating out the
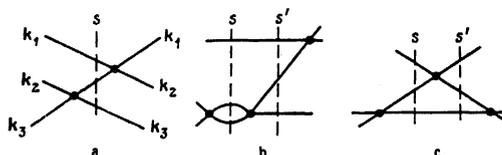

FIG. 2.

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center-of-mass motion, which leads to the factor

$$\sigma_N = (m_N/2\pi\beta)^{\frac{3}{2}},$$

where m_N is the total mass of the system, we finally obtain

$$\Omega = \Omega_0 - \frac{1}{\beta} \sum_{N=2}^{\infty} \sigma_N \xi^N b_N, \qquad (3.3)$$

where the virial coefficient[3)]

$$b_N = \mathrm{Sp}_N (e^{-\beta H})^c = \sum_n \exp(-\beta E_{nN})^c \qquad (3.4)$$

and the trace is taken over the wavefunctions in the c.m. frame.

It is important to emphasize that the analysis carried out does not take explicit account of exchange effects, which are, in fact, important at not very high temperatures. These effects lead to the result that correlations of an arbitrarily large number of particles make a contribution to the quantity Ω_0; correspondingly, the quantity b_N becomes dependent on the correlation of more than N particles. In this article we shall not take exchange effects explicitly into account, but refer the reader to Refs. 8 and 9 and to the next publication by the authors.

The subject of the following analysis will be the virial coefficients b_N, which differ from zero only because of the interaction between the particles. Therefore, at extremely high temperatures, when the non-relativistic system of particles is transformed into an ideal gas, the following condition holds, for $\beta = 0$:

$$b_N = 0 \qquad (3.5)$$

[cf. (3.4)]. It is convenient to subject the contributions to the expression (3.4) from the discrete and continuous spectra to this condition separately. The former contribution, corresponding to N-particle bound states with energy E_{nN}, has the form

$$b_N^{discr} = \sum_n [\exp(-\beta E_{nN}) - 1]. \qquad (3.6)$$

As regards the continuous spectrum, it is conveniently described using the method expounded in the preceding section. Differentiating (3.4) with respect to g and using (2.2), we have

$$\frac{\partial b_N^{cont}}{\partial g} = -\beta \sum_n \exp(-\beta E_{nN}) V_{nn}^c.$$

Here, in the argument of the exponential, the quantity E_{nN} can be regarded as simply the sum of the kinetic energy and the binding energies of the composite complexes (the energy correction due to scattering is inversely proportional to the normalization volume and tends to zero in the limit). Taking the condition (3.5) into account, we obtain

$$b_N^{cont} = -\beta \int_0^g dg \sum_n \exp(-\beta E_{nN}) V_{nn}^c. \qquad (3.7)$$

Thus, to determine the virial coefficients it is necessary to know not only the energies of the bound states (these can be taken directly from experiment) but also the diagonal matrix elements V_{nn}^c of the connected part

of the potential. The latter can be found from Eq. (2.12), using its lowest iterations. Then the answer will be expressed in an explicit analytic form in terms of the phase shift in the scattering of a pair of particles [cf. (2.11)]. This is the first advantage of the proposed approach.

For the second virial coefficient the formula (2.6) gives directly

$$b_2^{cont} = \sum_l (2l+1) \frac{\beta}{\pi} \int_0^{\infty} dE \, e^{-\beta E} \delta_l(E),$$

which, when (3.6) is taken into account, returns us to the Beth-Uhlenbeck formula (1.1). It is necessary only to take into account Levinson's theorem

$$\delta_l(0) - \delta_l(\infty) = N_l \pi,$$

where N_l is the number of bound states with angular momentum l.

Despite the fact that the proposed method, unlike the method of Dashen, Ma, and Bernstein, makes it possible to avoid introducing a many-particle scattering matrix, for the sake of completeness we give the analog of the formula (1.2), which differs from the latter by replacement of the differentiation with respect to the energy by differentiation with respect to g. With is purpose we use Eq. (2.5), rewritten in the form

$$\left(S^{-1} \frac{dS}{dg}\right)_{mn} = -2\pi i V_{mn} \delta(E_m - E_n),$$

and assume the quantity E_m to be free parameter. In the absence of bound states (e.g., for sufficiently high temperatures), we obtain from (3.7)

$$b_N = \frac{\beta}{\pi} \int_0^{\infty} dE \, e^{-\beta E} \mathrm{Sp}_N \delta(E), \qquad (3.8)$$

where we have introduced the generalized phase shift

$$\delta = \frac{(\ln S)^c}{2i}.$$

In the presence of bound states, when speaking of a system of N particles we must distinguish the different combinations of elementary and composite particles. We shall label each of them by the index α, denoting the corresponding energy by $E_k + E_\alpha$, where E_k is the kinetic energy and E_α is the binding energy of the composite particles. In the end we obtain

$$b_{N\alpha}^{cont} = \frac{\beta}{2\pi i} \int_0^g dg \int_0^{\infty} dE_k \exp[-\beta(E_k + E_\alpha)] \mathrm{Sp}_{N\alpha} \left[S^{-1} \frac{\partial S}{\partial g}\right], \qquad (3.9)$$

where S is the many-channel scattering matrix describing all possible transitions $\alpha \to \alpha'$. The formula (3.9) expresses the virial coefficient not only in terms of the phase shifts but also in terms of the inelasticity coefficients corresponding to transitions to other channels.

4. SINGULARITIES OF THE FORWARD-SCATTERING TYPE

The kinematic singularities in the scattering of three or more particles lead to the result that, in iterations of Eq. (2.12), zero energy denominators, i.e., singularities of the type $1/i\delta$, appear in the expression for

V_{Nn}^c. Below, for the example of a three-particle problem, we separate out the singular terms explicitly and show that their sum is exactly equal to zero. Owing to the reality of the quantity V_{mn}^c, the matter reduces to elucidating the question of whether the coefficient of the $1/i\delta$ singularity is real.

Using formulas (2.12) and (2.13), we shall analyze the structure of the iterative series from the standpoint of interest to us. The zeroth iteration (Fig. 2a) gives

$$V_{mn}{}^c = \sum_{i \neq j} \sum_s \int \frac{dg\, v_{ms}{}^i v_{sn}{}^j}{E_m - E_s - i\delta} + \text{h.c.}$$

For $m = n$ the kinematics leads to the appearance of a singularity because of the equality $E_m = E_n = E_s$. However, the corresponding coefficient is real and the singularity does in fact disappear (cf. Fig. 1).

However, starting from the next iteration it is found that the property of reality of the coefficient of $1/i\delta$ is lost and the singularities no longer disappear in the framework of an iteration of a given order. Thus, for the first iteration, the diagrams of Fig. 2b in which $E_m = E_n = E_s$, and Fig. 2c in which $E_s = E_{s'}$, are dangerous. They give the singular part of V_{nn}^c in the form

$$\frac{2}{i\delta} \sum_{i \neq j} \int_0^t dg' \int_0^{t'} dg'' v_{ns}{}^i(g') v_{sn}{}^i(g'')\,[v_{ns}{}^j(g'') - v_{sl}{}^j(g'')]\, P\frac{1}{E_n - E_s} + \text{c.c.}$$

A straightforward analysis shows that the dangerous diagrams of higher order can be described by Fig. 3, in which rescattering of particles in the initial or final states has been added to the diagrams of Fig. 2. Then singularities do not appear in the term quadratic in V^c in (2.12).

We can separate out the singular part of V_{mn}^c explicitly in the following way. We write V_{mn}^c in the form

$$V_{mn}{}^c = W_{mn} + (A_{mn} + \text{h.c.}), \tag{4.1}$$

having subjected the quantity $A = \sum_{i \neq j} A^{ij}$ to the following equation:

$$\frac{\partial A_{mn}^{ij}}{\partial g} = \sum_s \left\{ \frac{v_{ms}{}^i v_{sn}{}^j}{E_m - E_s - i\delta} + A_{ms}^{ij} v_{sn}{}^j \left(\frac{1}{E_m - E_s - i\delta} + \frac{1}{E_n - E_s + i\delta} \right) \right\}. \tag{4.2}$$

This equation describes the rescattering effects: iteration of it leads directly to the series corresponding to the diagrams of Fig. 3a. Owing to the separability of the quantity v_{mn}, it is possible, using (2.7), to find the explicit solution of Eq. (4.2):

$$A_{mn}^{ij} = v_{mn}^j \int \frac{dg}{v_{mn}^j} \sum_s \frac{v_{ms}^i v_{sn}^j}{E_m - E_s - i\delta}. \tag{4.3}$$

The important point is that, for $m = n$, the coefficient of $1/i\delta$ in (4.3), equal to
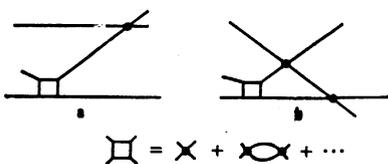
$$-v_{sn}^j \int dg\, v_{ns}^i,$$

FIG. 3.

is real. This leads to the disappearance of the singularities for $m = n$ in the expression in the brackets in (4.1).

It remained to convince ourselves that the singularities corresponding to the diagrams of Fig. 3b are also absent; this corresponds to a finite W in (4.1). These diagrams correspond to the expression

$$\sum_{i \neq j} \sum_s v_{ms}^i A_{sn}^{ji} + \text{h.c.}$$

which arises when (4.1) is substituted into (2.12). Using the explicit form of (4.3) it is easy to convince oneself that together these terms cancel the singularities appearing in their structure.

When considering terms of the iterative series for Eq. (2.12) one may also encounter expression containing singularities of the type

$$[(E_m - E_n)^2 + \delta^2]^{-1},$$

which also lead to singularities of the $1/\delta$ form, and such singularities give a zero result in the sum.

To summarize, in calculating the iterative series for Eq. (2.12) we can simply disregard the singular terms, discarding all the singularities proportional to $1/\delta$ that arise, without fear that any finite remainder will be lost in this procedure.

5. THE VIRIAL COEFFICIENTS OF A HOT NUCLEON GAS

In this concluding section we shall carry out an application of the method described above to the calculation of the first few virial coefficients of a hot nucleon gas. The examples cited are principally of an illustrative character. They should show that there exists a comparatively simple way of describing the effects of the interaction of three and more particles in the thermodynamics of a many-nucleon system.

We shall confine ourselves below to treating the region of temperatures for which

$$\beta \gamma^2/m \gg 1, \qquad \beta \varkappa^2/m < 1. \tag{5.1}$$

This means that the temperature is not so high that it would be necessary to take relativistic effects into account. On the other hand, it is not so low that exchange effects are appreciable. Numerically, this corresponds to a temperature of the order of tens of MeV.

In using the method described above we shall confine ourselves to taking the zeroth iteration of Eq. (2.12) into account. As shown in a previous paper by the authors,[12] this is sufficient at low energies (temperatures). It will be all the more adequate at high temperatures, since the effective expansion parameter is the quantity βv, which decreases with increase of temperature. We start from the virial coefficient corresponding to neutron-proton scattering in the triplet state. Using (1.1) and (2.9), (2.10), we find

$$b_{2(np)} = \tfrac{1}{2}. \tag{5.2}$$

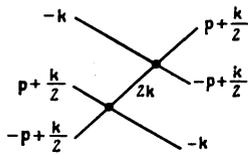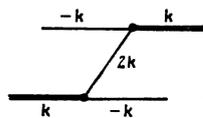For simplicity we assume that $\mu_p = \mu_n$.

FIG. 4.

FIG. 5.

Next, we estimate the contribution of the three-particle scattering (two neutrons and a proton, in the quartet state). According to (3.7) and Fig. 4,

$$b_{3(nnp)} = -\beta \int_0^\ell dg \int d^3k \, d^3p \exp\left[-\frac{\beta}{m}\left(\frac{3}{4}k^2 + p^2\right)\right] V_{nnp,nnp},$$

where

$$V_{nnp,nnp} = \frac{64\pi^2}{3kp} \int_0^\ell dg \, \frac{\partial \delta_{np}(p)}{\partial g} \, \frac{\partial \delta_{np}(3k/2)}{\partial g} \bigg/ \left(p^2 - \frac{9}{4}k^2\right).$$

Hence,

$$b_{3(nnp)} = \text{const}, \tag{5.3}$$

where the numerical value of the constant is of order unity.

Finally, we consider the contribution of the scattering of neutrons by deuterons in the quartet state. Using Fig. 5, we have

$$b_{3(nd)} = -\beta \int_\ell^\ell dg \int d^3k \exp\left[\frac{\beta}{m}\left(\varkappa^2 - \frac{3}{4}k^2\right)\right] V_{nd,nd},$$

where (cf. Ref. 12)

$$V_{nd,nd} = -\frac{16\pi}{3m} \frac{d\varkappa^2}{dg} \int_\ell^\infty \frac{dg}{\varkappa^2 + {}^9/_4 k^2} \frac{\partial \delta_{np}(3k/2)}{\partial g}.$$

Hence,

$$b_{3(nd)} = \frac{4}{27} \frac{\beta\varkappa^2}{m} \ln \frac{\beta\varkappa^2}{m}. \tag{5.4}$$

The small value of this quantity as compared with (5.3) is explained by the dissociation of the deuterons with increase of temperature.

[1] Below, the "strong interaction" has the same meaning as in the theory of elementary particles, although many of the results given below also apply to atomic-molecular systems with strong (in the literal sense of the word) interaction between the particles.

[2] The problem is solved even more simply by means of the Jost function.[12]

[3] The quantity σ_N is more often included in the expression for the virial coefficient, but we prefer the dimensionless quantity (3.4).

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