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Duality transformations for discrete Abelian models. Simple example of duality transformation for non-Abelian model

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A duality transformation is proposed for gauge and non-gauge Abelian A_N models (generalization of the Ising model, the field assumes N discrete values on a circle) to include two-, three-, and four-dimensional cubic lattice. Besides the known cases of self-duality of the Ising model (A_2 model) in two and four dimensions, an entire series of self-dual models is found (particularly the Z_3 and Z_4 self-dual models), and accordingly, the phase-transition points for them. A duality transformation is also proposed for the simple case of a discrete non-Abelian model (the symmetry group is the group of symmetry axes of the tetrahedron). The model turns out to be self-dual and accordingly the dual transformation makes it possible to find the phase-transition point.

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1. INTRODUCTION. DESCRIPTION OF MODELS

It is well known that the two-dimensional (2D) Ising model has a definite thermodynamic symmetry, as established by Kramers and Wannier.[1] The gist of this symmetry is that the partition function $Z(\beta)$, as a function of the reciprocal temperature $\beta = 1/T$, is invariant (apart from an inessential factor) to the transformation

$$\beta \to \beta^* = -{}^1/_2 \ln \operatorname{th} \beta, \tag{1}$$

which converts low temperatures into high ones and vice versa. The values of $Z(\beta)$ at the points β and β^* are connected by the relation

$$Z(\beta) = (\operatorname{sh} 2\beta)^{\Omega} Z(\beta^*), \tag{2}$$

where Ω is the number of lattice points. One of the possible proofs of (2) can be found, for example, in Isihara's book.[2] The qualitative form of (1) is shown in Fig. 1. From (2) it follows that the thermodynamic properties of the high-and low-temperature phases are symmetrical.

In the reciprocal-temperature scale, there is a preferred point β_c defined by the condition $\beta^* = \beta$. From (1) we have

$$\beta_c = {}^1/_2 \ln(\sqrt{2} + 1). \tag{3}$$

If we assume that there is only one phase-transition point, or, which is the same, that there are only two different phases (for the Ising model this is obvious: ordered and disordered phases), then it follows from symmetry considerations that β_c is a phase-transition point. Thus, the Kramers–Wannier (KW) symmetry[1] has made it possible to obtain the phase-transition point for the 2D Ising model before an exact solution has been found for the model of Ref. 3.

The partition-function transformation whereby $Z(\beta)$ is expressed in terms of $Z(\beta^*)$ is called a duality transformation. The property of the Ising model, that it goes over into itself under this transformation, is called self-duality. The initial derivation of (2) was based on comparison of Van der Warden graphs of the expansions of the partition functions of the initial and dual models.[1] The KW symmetry (self-duality) is actively used in investigations of the Ising model.[4,5] Kadanoff and Ceva[5] investigated the physical meaning of the duality transformation, introduced the concept of the disorder parameter, and established that the

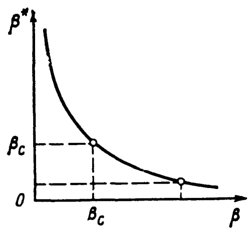

FIG. 1. Schematic plot of the function $\beta^* = -(1/2)\ln \tanh\beta$.

duality transformation of the Ising model is a transition from the order parameter to the disorder parameter.

In the study of phase transitions, it is of interest to generalize the Ising model. Potts[6] considered stochastic models in which, in contrast to the Ising model, the field variable φ (the order parameter) can assume N different values. Two types of different models were proposed, in which the field is specified at the lattice points and only nearest neighbors interact:

1) the field takes on N different values, and the neighbor interaction energy takes the form

$$u(\varphi_1,\varphi_2)=\begin{cases}0, & \varphi_1=\varphi_2\\ J, & \varphi_1\neq\varphi_2\end{cases};$$

2) the field assumes on a circle N discrete values $\varphi=2\pi n/N, n=0, 1, 2, \ldots, N-1$, and the interaction energy $u(\varphi_1, \varphi_2)$ depends on the angle between the points φ_1 and φ_2.

The first model is called[7] the Potts model proper, and the second the Potts vector model, since the field variable in this model can be taken to be a two-dimensional unit vector.

Using the KW method,[1] Potts found the phase-transition point for the model of the first type at any N.[6] For the vector model, the same method was used to obtain the phase-transition point for $N=3$ and 4. Mittag and Stephan[7] investigated duality transformations for the Potts model at any N in the transition-matrix formulation. It is interesting to note that, as shown by Suzuki,[8] the vector Potts model with $N=4$ breaks up into two independent Ising models and can thus be solved exactly.

It turns out that from the more general point of view the duality transformations constitute a Fourier transformation for the field variables. Using this approach, Berezinskii[9] was the first to effect the duality transformation for a planar XY model (the field takes on continuous values on a unit circle) (see also Ref. 10). In this paper we apply this approach in succession to various models. For Abelian models on a lattice, with a field that takes on values at discrete points on a circle, the duality transformation is very simple to perform, by a single method, for any N and for any potential of the neighbor interaction.

In field theory one considers gauge statistical models on a lattice; these models constitute a lattice formulation of the theory in Euclidean space-time. The first to introduce such a model was Wilson.[11] In Abelian lattice gauge theory, the field variables takes on values on a circle. It is of interest to study Abelian models with a field that takes on discrete values on a circle. Balian *et al.*[12] investigated the simplest particular case, in which a field specified on a cubic $4D$ lattice takes on only two values, ±1 (the Ising gauge model). This model, just as the usual Ising model on a quadratic $2D$ lattice, is self-dual and has the same phase-transition point.

As already noted, the duality transformations for

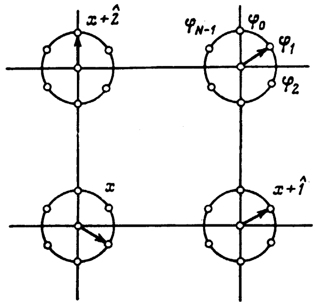

FIG. 2. The field φ specified at the points of a quadratic $2D$ lattice takes on N discrete values $\varphi_n=2\pi n/N$, $n=0,1,2,\ldots,N-1$ on the unit circle.

Abelian models are effected by a single method. For the Fourier transformation in the field variable, to which the dual transformation equivalent, an important role is played by the form of the space on which the field variable takes on its values. This is why the dual transformation is effected in the same way for both gauge and non-gauge Abelian models, regardless of the dimensionality of the lattice on which the field is specified.

We now define the Abelian models considered in this paper.

The $2DZ_N$ *model.* The field $\varphi(x)$ is specified at the points of the quadratic $2D$ lattice and takes on N discrete values on a circle: $\varphi=2\pi n/N$, $n=0, 1, \ldots, N-1$ (the Potts vector model), see Fig. 2. The Hamiltonian is

$$H=-\sum_{x,\alpha} V(\varphi(x)-\varphi(x+\hat{\alpha})).$$

Here x is a $2D$ vector and runs over the points of the $2D$ lattice; $\alpha=1, 2$; $\hat{\alpha}=\hat{1}, \hat{2}$ are the elementary lattice vectors; V is the potential of the neighbor interaction. The model has a global Abelian discrete symmetry group Z_N (group of simultaneous discrete rotations of the field variable over the entire lattice through angles that are multiples of $2\pi/N$).

The $4DGZ_N$ *model.* The field $B(x)_\mu$, specified on the edges of a cubic $4D$ lattice (or the vector field specified at the lattice points) takes on N discrete values on a circle. The Hamiltonian is

$$H=-\sum_{x,\mu<\nu} V(f(x)_{\mu\nu}),$$

where

$$f(x)_{\mu\nu}=B(x)_\mu+B(x+\hat{\mu})_\nu-B(x+\hat{\nu})_\mu-B(x)_\nu=\partial_\mu B(x)_\nu-\partial_\nu B(x)_\mu$$

is the curl of the field over the $2D$ unit face $(x, \mu\nu)$; $=1, 2, 3, 4$ (see Fig. 3). The model has a local Abelian discrete symmetry group Z_N. We note that in the

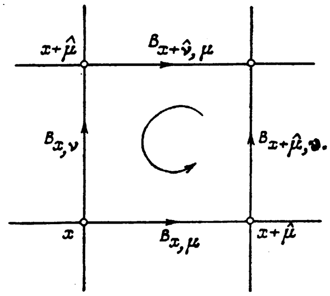

FIG. 3. Curl of the field $B(x)_\mu$ specified on the edge edges of the lattice, over the $2D$ elementary face $(x, \mu\nu)$.

gauge model with a lattice the energy is accumulated actively over the curls of the field on the $2D$ unit faces. For more details see Ref. 11.

The usual and gauge models are analogously defined for a cubic $3D$ lattice.

The $3DZ_N$ *model*. The Hamiltonian is

$$H=-\sum_{x,i} V(\varphi(x)-\varphi(x+\hat{i})).$$

The $3DGZ_N$ *model*. The Hamiltonian is

$$H=-\sum_{x,i<j} V(f(x)_{ij}).$$

Here x runs over the sites of the $3D$ lattice; $i=1, 2, 3$.

The generalizations of the Ising model were discussed above as interesting objects for the model study of the physics of phase transitions. The $2DZ_N$ models, however, can have also more concrete applications. For example they can serve as models of layered magnetic crystals with anisotropy in the layer Z_N.[13,10] It turns out that at sufficiently low temperatures the isotropic $2D$ system (the order parameter takes on continuous values on a circle) is unstable to weak anisotropy—see Ref. 14 and 10. Inasmuch anisotropy is always present in a real crystal, it follows that at low temperatures an isotropic system is transformed into the discrete model $2DZ_N$. For a layered magnetic crystal with sufficiently strong anisotropy the $2DZ_N$ model is a good one in a wide range of temperatures. Alexander[15] discusses also a realization of the $2DZ_3$ model in a $2D$ layer of atoms absorbed on the surface of a crystal with hexagonal lattice.

We turn now to the gauge models $4DGZ_N$. It is proposed in very recent papers[16,17] that the subgroup of the center of the $SU(N)$ group, i.e., of the Z_N group, plays an important role in the problem of quark confinement. It is proposed in Ref. 17 that the mechanism of quark confinement can be understood by studying gauge Z_N theories. The same paper introduces order and disorder operators that go over into each other under duality transformations. In the gauge theories the phase transition is with respect to charge, and the different phases are the phase with quark confinement and the phase in which the quarks are free. Knowledge of the phase-transition point in gauge theory provides us with a knowledge of the critical charge that separates the quark confinement and nonconfinement phases.

The significance of the duality transformations lies in the fact that they can reveal a deep internal symmetry of the investigated model, or equivalence, on the face of it, of different models. In those cases when the model is self-dual, the transformation makes it possible to find the phase-transition point.

Briefly speaking, the results of the transformations for the Abelian models discussed above consist in the fact that the models connected by duality transformations are

$$2DZ_N \leftrightarrow 2DZ_N$$

(we note that this still does not mean self-duality, since the initial and dual models have in general different interaction potentials V), and

$$3DZ_N \leftrightarrow 3DGZ_N, \quad 4DGZ_N \leftrightarrow 4DGZ_N.$$

Concrete examples of self-dual models have been found. In particular, the gauge models $4DGZ_3$ and $4DGZ_4$ are self-dual.

As already noted, from the formal point of view a duality transformation is a Fourier transformation for field variables. Of fundamental significance to the transformations is the type of the space on which the field assumes its value. The situation is simple in the Abelian case with a field that takes on values on a circle. For the non-Abelian model the Fourier transformation is of much more complicated form. For example, if the field takes on values on a sphere, then the harmonics for the Fourier expansion are spherical functions. In the last sections we make a first attempt at a duality transformation for a non-Abelian model using the following simple example:

The $2DT$ *model*. The field is specified at the points of a quadratic $2D$ lattice and assumes four discrete valus on a sphere at points v_1, v_2, v_3, and v_4, which the vertices of a tetrahedron inscribedin the sphere (see Fig. 4). The Hamiltonian of the model

$$H=-\sum_{x,\alpha} V(v(x), v(x+\hat{\alpha}))=-\sum_{x,\alpha} V(\mathbf{n}(x), \mathbf{n}(x+\hat{\alpha}))$$

has a symmetry corresponding to the group of symmetry axes of the tetrahedron T, i.e., it has a global non-Abelian discrete symmetry group. Using the terminology of group theory we can state that the field takes on values on a homogeneous space T/Z_3 (the T group factored in terms of the subgroup Z_3); the Hamiltonian is invariant to motions of this space. The model is self-dual, and the phase-transition point is obtained.

2. GENERAL SCHEME OF DUALITY TRANSFORMATIONS

It can be stated briefly that a duality transformation is a Fourier transformation for the field variables. What is being transformed is in fact the partition function of the field. We demonstrate first the transformation with a simple example of a $2D$ Abelian model (following Ref. 10).

Consider a field of unit vectors $\mathbf{n}(x)$ distributed over the points of a quadratic $2D$ lattice and capable of being rotated only in a plane. The rotating vector $\mathbf{n}(x)$ describes a circle, so that the field variables must be

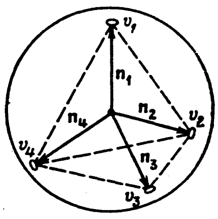

FIG. 4. The field n of the $2DT$ model takes on values at four discrete points on the sphere: at the vertices of a tetrahedron inscribed in the square.

taken to be the points of this circle, identified by the rotation angle $\varphi(x)$. The field Hamiltonian

$$H=-\sum_{x,\alpha}V(\mathbf{n}(x),\mathbf{n}(x+\hat{\alpha}))=-\sum_{x,\alpha}V(\varphi(x)-\varphi(x+\hat{\alpha}))$$

contains only the nearest-neighbor interaction (as is the case for all the models considered in this paper) and is invariant to uniform rotations of the vectors $\mathbf{n}(x)$ over the lattice.

We carry out in the partition function

$$Z=\Big(\prod_x\int_0^{2\pi}\frac{d\varphi(x)}{2\pi}\Big)\prod_{x,\alpha}\exp\{V(\varphi(x)-\varphi(x+\hat{\alpha}))\} \tag{4}$$

a Fourier transformation in the variables $\varphi(x)$. The field φ takes on values on a circle, so that functions of φ can be expanded in Fourier series in elementary harmonics of the circle

$\exp\{iS\varphi\},\quad S=0,\ \pm1,\ \pm2,\ldots$

In particular, each Gibbs factor $\exp\{V(\varphi_1-\varphi_2)\}$ can be represented in the form

$$\exp\{V(\varphi(x)-\varphi(x+\hat{\alpha}))\}$$
$$=\sum_{S(x,x+\hat{\alpha})=-\infty}^{+\infty}\exp\{\tilde{V}(S(x,x+\hat{\alpha}))+iS(x,x+\hat{\alpha})(\varphi(x)-\varphi(x+\hat{\alpha}))\}. \tag{5}$$

Here

$$\exp\{\tilde{V}(S)\}=\frac{1}{2\pi}\int_0^{2\pi}d\varphi\exp\{V(\varphi)-iS\varphi\}.$$

We substitute the expansion (5) in (4) and integrate over the field φ. The result is the partition function for the field $S(x,\ x+\hat{\alpha})$, specified on the edges constraints of the square lattice and taking on integer values:

$$Z=\sum_{\{S\}}\prod_{x,\alpha}\delta(S(x-\hat{1},x)-S(x,x+\hat{1})+S(x-\hat{2},x)-S(x,x+\hat{2}))$$
$$\times\exp\{\tilde{V}(S(x,x+\hat{\alpha}))\}. \tag{6}$$

The constraints

$S(x-\hat{1},\ x)-S(x,\ x+\hat{1})+S(x-\hat{2},\ x)-S(x,\ x+\hat{2})=0$

appeared in the summation over the field φ and have the meaning of the vanishing of the divergence of the field S. They can be easily resolved by representing the field on the edges in the form of the curl of a new field specified at the points of the dual lattice:

$$S(x,\ x+\hat{\alpha})=\varepsilon_{\alpha^*\beta^*}\partial_{\beta^*}s(x^*)=\varepsilon_{\alpha^*\beta^*}(s(x^*+\hat{\beta}^*)-s(x^*)). \tag{7}$$

Here $\varepsilon_{\alpha\beta}$ is an antisymmetrical tensor. The definition of the dual lattice is given in Fig. 5. To simplify the formulas, we shall omit the asterisks of the subscripts pertaining to the dual lattice. It must therefore only be remembered that the subscripts of the initial field pertain in the initial lattice and those of the dual field to the dual lattice.

Substituting (7) in (6), we obtain the partition function for the dual field $s(x)$, specified in the points of the dual lattice and taking on integer values on a straight line:

$$Z=\sum_{\{s\}}\prod_{x,\alpha}\exp\{\tilde{V}(s(x+\hat{\alpha})-s(x))\}. \tag{8}$$

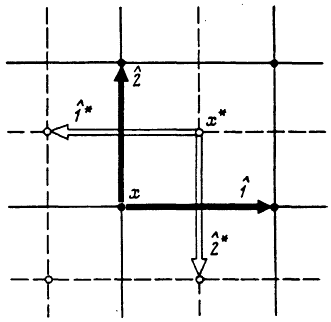

FIG. 5. Definition of the dual lattice and of its elementary vectors.

The transition from (4) to (8) is an example of the duality transformation. From a more general point of view, if the calculation of the partition function is taken to mean the evaluation of a functional integral over the field, then the duality transformation is a generalization of the operation of going over to Fourier variables under the integral sign to include the functional case.

The actual transformation is, of course, different in each concrete case. For Abelian models with a field that takes on values on a circle (as e.g., in the example above), the transformation is relatively easy to perform. The field-variable functions are expanded in the harmonics of the circle $\exp\{iS\varphi\}$ and the series have the usual simple form. It is also easy to integrate with respect to the initial field φ after the expansion. The resultant constraints are simple δ functions and can be easily resolved. The situation is much more complicated in non-Abelian models. For example, if the field takes on values on a sphere unit vector $\mathbf{n}$ rotating in three dimensions), then the Fourier expansion is now in the harmonics of the sphere, i.e., in the spherical functions $Y_{lm}(\mathbf{n})$. The dual variables are the fields are in this case the indices that label the spherical functions.

For example, if in the example considered above $\mathbf{n}(x)$ rotates in three dimensions, then we must expand the Gibbs factor in Legendre polynomials, and these in turns must be expanded in spherical functions:

$$\exp\{V(\mathbf{n}_1,\mathbf{n}_2)\}=\sum_{l=0}^{\infty}c_lP_l(\mathbf{n}_1,\mathbf{n}_2)$$
$$=\sum_{l=0}^{\infty}c_l\frac{4\pi}{2l+1}\sum_{m=-l}^{+l}Y_{lm}^*(\mathbf{n}_1)Y_{lm}(\mathbf{n}_2)$$
$$=\sum_{lm}\exp\{\tilde{V}(l)\}Y_{lm}^*(\mathbf{n}_1)Y_{lm}(\mathbf{n}_2).$$

Substituting this expansion in the partition function

$$Z=\Big(\prod_x\int\frac{d\mathbf{n}(x)}{4\pi}\Big)\prod_{x,\alpha}\exp\{V(\mathbf{n}(x),\mathbf{n}(x+\hat{\alpha}))\}$$

and integrating over the initial field $\mathbf{n}$, we obtain the partition function for the field of the indices $(l,\ m)$, specified on the constraints of the square lattice:

$$Z=\sum_{\{lm\}}\prod_{x,\alpha}\begin{pmatrix}l(x-\hat{1},x) & l(x,x+\hat{1}) & l(x-\hat{2},x) & l(x,x+\hat{2})\\ m(x-\hat{1},x) & m(x,x+\hat{1}) & m(x-\hat{2},x) & m(x,x+\hat{2})\end{pmatrix}$$
$$\times\exp\{\tilde{V}(l(x,x+\hat{\alpha}))\}. \tag{9}$$

Integration over the field $\mathbf{n}$ on each site gives rise to a constraint that constitutes an integral of the product of four spherical functions (as against the simple δ functions in the Abelian case):

$$\begin{pmatrix} l(x-\hat{1},x) & l(x,x+\hat{1}) & l(x-\hat{2},x) & l(x,x+\hat{2}) \\ m(x-\hat{1},x) & m(x,x+\hat{1}) & m(x-\hat{2},x) & m(x,x+\hat{2}) \end{pmatrix}$$
$$=\int\frac{d\mathbf{n}(x)}{4\pi}Y^{*}_{lm(x-\hat{1},x)}(\mathbf{n})Y_{lm(x,x+\hat{1})}(\mathbf{n})Y^{*}_{lm(x-\hat{2},x)}(\mathbf{n})Y_{lm(x,x+\hat{2})}(\mathbf{n}).$$

Whether these constraints can be removed by a redefinition of the field variables is presently unclear, yet unless this can be done the transformation yields nothing, since the obtained partition function in terms of the variables (l,m) is of much more complicated form than in terms of the initial field variables **n**.

This example of a model with a field that takes on values on a sphere is given here to demonstrate the difficulties that arise in duality transformations for non-Abelian models. In Sec. 5 we consider a simple example of a non-Abelian model in which this difficulty can be overcome. Even in this case the duality transformation is quite useful: a thermodynamic symmetry is revealed—the analog of the KW symmetry for the Ising model.

To conclude the section, we describe the general scheme for duality transformations. Let the field $v(x)$ take on values on a homogeneous space O with a group of motions G. (For the mathematical theory of harmonic analysis on homogeneous spaces see, e.g., Ref. 18). For example, for a planar **n**-field a homogeneous space is a unit circle with a motion group $U(1)$ or $SO(2)$; for a three-dimensional **n** field, O is a unit sphere with motion group $SO(3)$.

Let the field Hamiltonian

$$H=-\sum_{x,\alpha}V(v(x),v(x+\hat{\alpha})),\quad v\in O$$

(we have in mind a lattice model with a nearest-neighbor approximation) be invariant to the motions of the homogeneous space O, specified by elements of the group G. In the partition function of the field

$$Z=\sum_{\{v\}}\prod_{x,\alpha}\exp\{V(v(x),v(x+\hat{\alpha}))\}$$

the Gibbs weight $\exp\{V(v_1,\ v_2)\}$, being a function on the space O, can be expanded in elementary harmonics, i.e., in the orthonormal basis of a linear space (designated L) of functions specified on a homogeneous space O.

According to group theory, the complete set of harmonics is constructed in the following manner. Let T_g be a representation of the group G in linear space L. It breaks up into irreducible representations $T_g^{(\delta)}$ and, correspondingly, the space of the representation L breaks up into subspaces of irreducible representations $L^{(\delta)}$. The functions of the bases of the subspace $\{L^{(\delta)}\}$ comprise the required set of harmonics for the functions on the homogeneous space O.

If we have a complete set of harmonics constructed in this manner, we can expand the Gibbs factors $\exp\{V(v_1,v_2)\}$ and sum over the initial field v. The result is a partition function for the sum of the indices that label the functions of the bases of the irreducible representations. The field of the indices is specified on the edges of the lattice. Summation over the initial field v at each lattice site yields a constraint for the values of the field of the indices on the edges adjacent to the given point. These constraints express the conservation law connected with the invariance of the Hamiltonian to the motions of the homogeneous space O. We are left with the usually unsimple task of attempting to resolve the constraints by redefining the field.

3. DUALITY TRANSFORMATIONS FOR ABELIAN MODELS

In this section we carry out a duality transformation for the discrete Abelian models $2DZ_N$, $3DZ_N$, and $4DGZ_N$ described in Sec. 1.

The $2DZ_N$ model. The duality transformation for this model is a trivial generalization, to include the discrete case, of the transformation for a planar **n**-field[9,10] described in Sec. 2. In the partition function

$$Z=\frac{1}{N^{\Omega}}\sum_{\{\varphi\}}\prod_{x,\alpha}\exp\{V(\varphi(x)-\varphi(x+\hat{\alpha}))\},\tag{10}$$

where Ω is the number of lattice points, we expand each Gibbs factor $\exp\{V(\varphi(x)-\varphi(x+\hat{\alpha}))\}$ in a Fourier series:

$$\exp\{V(\varphi(x)-\varphi(x+\hat{\alpha}))\}$$
$$=\sum_{S(x,x+\hat{\alpha})=0}^{N-1}\exp\{\tilde{V}(S(x,x+\hat{\alpha}))+iS(x,x+\hat{\alpha})(\varphi(x)-\varphi(x+\hat{\alpha}))\}.\tag{11}$$

We recall that in the Z_N mode the field φ takes on discrete values on the circle $\varphi=2\pi n/N$, $n=0, 1, 2, \ldots, N-1$, and therefore the Fourier series for functions of φ are finite syms. The inverse Fourier transform is

$$\exp\{\tilde{V}(S)\}=\frac{1}{N}\sum_{n=0}^{N-1}\exp\left\{V\left(\frac{2\pi}{N}n\right)-iS\frac{2\pi}{N}n\right\}.\tag{12}$$

It is obvious that $\exp\{\tilde{V}(S)\}$ is a periodic function of the whole-number argument S with a period equal to N. It can therefore be assumed that S takes on discrete N values on the circle: $S=0, 1, 2, \ldots, N-1$. Substituting (11) in (10) and summing over the field φ, we get

$$Z=\sum_{\{S\}}\prod_{x,\alpha}\delta(S(x-\hat{1},x)-S(x,x+\hat{1})+S(x-\hat{2},x)-S(x,x+\hat{2}))$$
$$\times\exp\{\tilde{V}(S(x,x+\alpha))\}.\tag{13}$$

The constraints $S(x-\hat{1},\ x)-S(x,\ x+\hat{1})+S(x-\hat{2},\ x)-S(x,\ x+\hat{2})=0$, which have the meaning of the vanishing of the divergence of the field S, can be easily obtained by representing the field S in the form of the curl of the dual field

$$S(x,\ x+\hat{\alpha})=\varepsilon_{\alpha\beta}\partial_\beta s(x)=\varepsilon_{\alpha\beta}(s(x+\hat{\beta})-s(x))\tag{14}$$

(see Sec. 2). Substituting (14) in (13), we obtain the partition function for the dual fields s specified at the points of the dual lattice and having N discrete values on the circle ($s=0, 1, 2, \ldots, N-1$):

$$Z=\sum_{\{s\}}\prod_{x,\alpha}\exp\{\tilde{V}(s(x)-s(x+\hat{\alpha}))\}. \tag{15}$$

Thus, the dual of the $2DZ_N$ model is the same model $2DZ_N$, but with an interaction potential $\tilde{V}$ instead of V.

The $3DZ_N$ *model.* Carrying out the expansion (11) in the parition function

$$Z=\frac{1}{N^\Omega}\sum_{\{\varphi\}}\prod_{x,i}\exp\{V(\varphi(x)-\varphi(x+\hat{i}))\}$$

and summing over the field φ, we get

$$Z=\sum_{\{S\}}\prod_{x,i}\delta\left(\sum_{k=1}^{3}[S(x-\hat{k},x)-S(x,x+\hat{k})]\right)\exp\{\tilde{V}(S(x,x+\hat{i}))\}. \tag{16}$$

The constraint

$$\sum_{k=1}^{3}(S(x-\hat{k},x)-S(x,x+\hat{k}))=0$$

is again resolved by substituting the field S in the form of the curl of a new field

$$S(x,x+\hat{i})=\varepsilon_{ijk}\partial_j s(x)_k=\varepsilon_{ijk}(s(x+\hat{j})_k-s(x)_k). \tag{17}$$

Here ε_{ijk} is an antisymmetrical tensor and $s(x)_k$ is a dual field specified on the edges of the dual lattice. The definition of the dual lattice for the case of 3D system (analogous to the 2D-lattice case) and of the field $s(x)_k$ having the same vector spatial index as the fields on the lattice edges is shown in Fig. 6 (see also Fig. 3).

We see that in the case of the 3D lattice we encounter a new important singularity—the dual field is of the vector type. The intermediate (in the transformations) field $S(x, x+\hat{i})$ specified on the edge $(x, x+\hat{i})$ of the initial lattice is expressed in this case in terms of the curl of the field over the 2D face of the dual lattice perpendicular to the edge of the initial lattice—see Fig. 6.

Substituting (17) in (16) we get

$$Z=\frac{1}{N^\Omega}\sum_{\{s_i\}}\prod_{x,i<j}\exp\{\tilde{V}(g(x)_{ij})\}. \tag{18}$$

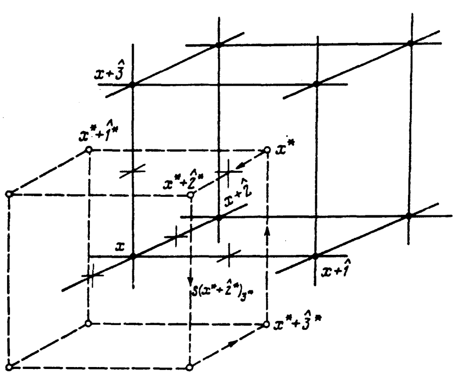

FIG. 6. Changeover from a field specified at the points (the energy is specified on the edges) of the initial 3D lattice to a dual field specified on the edges (the energy is specified on the 2D faces) of a dual 3D lattice, i.e., $4DZ_N \to 3DGZ_N$.

Here $g(x)_{ij}=\partial_i s(x)_j-\partial_j s(x)_i$ is the curl of the vector field s_i over the 2D face (x, ij) (see Fig. 3, as well as Fig. 3). The factor $1/N^\Omega$ in (18) cancels out the gauge degree of freedom of the field s_i. The model dual to $3DZ_N$ is therefore the gage model $3DGZ_N$ (a definition of a gauge model on a lattice[11] is given in Sec. 1).

The $4DGZ_N$ *model.* The partition function of the model is

$$Z=\frac{1}{N^{4\Omega}}\sum_{\{B_\mu\}}\prod_{x,\mu<\nu}\exp\{V(f(x)_{\mu\nu})\}. \tag{19}$$

The curl of the field $f(x)_{\mu\nu}=\partial_\mu B(x)_\nu-\partial_\nu B(x)_\mu$ takes on the same discrete N values $2\pi n/N$ on the circle. We can therefore use the expansion

$$\exp\{V(f(x)_{\mu\nu})\}=\sum_{S(x)_{\mu\nu}=0}^{N-1}\exp\{\tilde{V}(S(x)_{\mu\nu})+iS(x)_{\mu\nu}f(x)_{\mu\nu}\}. \tag{20}$$

Substituting (20) in (19) and summing over the field B_μ, we get

$$Z=\sum_{\{S\}}\prod_{x,\mu<\nu}\delta\left(\sum_{\lambda=1}^{4}(S_{x,\lambda\nu}-S_{x-\hat{\lambda},\lambda\nu})\right)\exp\{\tilde{V}(S(x)_{\mu\nu})\}. \tag{21}$$

We next represent $S(x)_{\mu\nu}$, which is specified on the 2D face $(x, \mu\nu)$ of the initial lattice, in the form of the curl of the dual field over the 2D face of the dual lattice perpendicular to the 2D face of the initial lattice (it is difficult to draw a figure for this case):

$$S(x)_{\mu\nu}=\varepsilon_{\mu\nu\lambda\tau}\partial_\lambda s(x)_\tau. \tag{22}$$

Substituting (22) in (21) we get ultimately

$$Z=\frac{1}{N^\Omega}\sum_{\{s_\mu\}}\prod_{x,\mu<\nu}\exp\{\tilde{V}(g(x)_{\mu\nu})\}. \tag{23}$$

Here $g(x)_{\mu\nu}=\partial_\mu s(x)_\nu-\partial_\nu z(x)_\mu$ is the curl of the dual field over the $(x, \mu\nu)$ face. The factor $1/N^\Omega$ cancels the gauge degree of freedom of the field s_μ.

We note in conclusion that continuous Abelian fields on a circle are obtained as a particular case as $N\to\infty$, by making in the formulas the substitution

$$\frac{1}{N}\sum_{\varphi}\to\int_0^{2\pi}\frac{d\varphi}{2\pi}$$

etc. The circle on which the dual field s takes on the values 0, 1, 2, ..., $N-1$ is then transformed into a circle of infinite radius, i.e., into a straight line, and we obtain the ordinary Fourier series.

4. CONCRETE EXAMPLES OF SELF-DUAL ABELIAN MODELS

In the preceding section it was shown that the models $2DZ_N$ and $4DGZ_N$ become their own duals if the interaction potential V is replaced by $\tilde{V}$. For these models to be self-dual it is necessary that the excitation energies of the initial and dual models are the same. We introduce the natural definitions

$$V(0)-V(\varphi)=\beta u(\varphi),\quad \tilde{V}(0)-\tilde{V}(s)=\beta\tilde{u}(\varphi_s),$$

where $\varphi_s=2\pi s/N$, β is the reciprocal temperature, and $u(\varphi)$ is the energy of the excitations. Self-duality means that

$$\tilde{V}(0)-\tilde{V}(s)\infty V(0)-V(\varphi_s). \qquad (24)$$

If this condition is satisfied, then $\tilde{\beta}$ can be chosen such that

$$\tilde{u}(\varphi)=u(\varphi). \qquad (25)$$

In this case

$$\tilde{V}(0)-\tilde{V}(s)=\frac{\beta^*}{\beta}(V(0)-V(\varphi_s)), \qquad (26)$$

where β^* is the temperature of the dual model and is defined by the condition (25).

We emphasize that the self-duality condition is (24). The potential $V(s)$ is determined by the Fourier transformation (12) and is obvious that the proportionality in (24) is possible only in particular cases. For the models Z_2 and Z_3 the condition (24) is always satisfied, since there is only one value of the excitation energies. For the models Z_N with $N>3$ it is necessary to make a special choice of the interaction potential in order to satisfy (24). It turns out that the model Z_4 is self-dual with the natural interaction

$$V(\varphi)=\beta\cos\varphi, \quad u(\varphi)=1-\cos\varphi. \qquad (27)$$

We present concrete results for the models Z_2, Z_3, and Z_4 with interaction (27).

The $2DZ_2$ *model*. This model is cited here only to demonstrate the method with a well known example. In this case

$$\exp\{\tilde{V}(s)\}=\frac{1}{2}\sum_{n=0,1}\exp\left\{\beta\cos\left(\frac{2\pi}{2}n\right)-i\frac{2\pi}{2}ns\right\},$$

$$\exp\{\tilde{V}(0)\}=\text{ch}\,\beta, \quad \exp\{\tilde{V}(1)\}=\text{sh}\,\beta. \qquad (28)$$

From (28), (27), and (26) we have

$$\beta^*=-{}^1/_2\ln\text{th}\,\beta. \qquad (29)$$

The phase-transition point β_c is determined from the condition $\beta^*=\beta$:

$$\beta_c={}^1/_2\ln(\sqrt{2}+1). \qquad (30)$$

From (10) and (15) we easily obtain

$$Z(\beta)=(\text{sh}\,2\beta)^{\Omega}Z(\beta^*). \qquad (31)$$

The $2DZ_3$ *model*. In analogy with the preceding, we have

$$\exp\{\tilde{V}(0)\}={}^1/_3(e^{\beta}+2e^{-\beta/2}), \quad \exp\{\tilde{V}(1)\}={}^1/_3(e^{\beta}-e^{-\beta/2}).$$

Next,

$$\beta^*=-\frac{2}{3}\ln\left(\frac{1-e^{-3\beta/2}}{1+2e^{-3\beta/2}}\right), \qquad (32)$$

$$\beta_c={}^2/_3\ln(\sqrt{3}+1) \qquad (33)$$

or

$$\beta_c=\frac{1}{J}\ln(\sqrt{3}+1) \text{ at } u(\varphi)=\begin{cases}0, & \varphi=0.\\ J, & \varphi=\pm 2\pi/3\end{cases} \qquad (34)$$

From (10) and (15) we get

$$Z(\beta)=[{}^1/_3(e^{\beta}+2e^{-\beta/2})^{2/3}(e^{\beta}-e^{-\beta/2})^{4/3}]^{\Omega}Z(\beta^*). \qquad (35)$$

The $2DZ_4$ *model*. Here

$$\exp\{\tilde{V}(0)\}=\text{ch}^2(\beta/2), \quad \exp\{\tilde{V}(1)\}=\exp\{\tilde{V}(3)\}=\text{sh}(\beta/2)\text{ch}(\beta/2),$$
$$\exp\{\tilde{V}(2)\}=\text{sh}^2(\beta/2).$$

For this case we have

$$\beta^*=-\ln\text{th}(\beta/2), \quad \beta_c=\ln(\sqrt{2}+1), \qquad (36)$$

$$Z(\beta)=(\text{sh}\,\beta)^{2\Omega}Z(\beta^*). \qquad (37)$$

For gauge $4D$ models everything is exactly similar, since the curls $f(x)_{\mu\nu}$ and $g(x)_{\mu\nu}$ of the initial and dual models take on the same values as the "two-dimensional curls" $\varphi(x)-\varphi(x+\hat{\alpha})$ and $s(x)-s(x+\hat{\alpha})$ of the initial and dual $2D$ models (all that changes is the degree of the factor of $Z(\beta^*)$, $\Omega\to 3\Omega$).

We write down the results in compact form:

The $4DGZ_2$ *model*:

$$\beta^*=-{}^1/_2\ln\text{th}\,\beta, \quad \beta_c={}^1/_2\ln(\sqrt{2}+1),$$

$$Z(\beta)=(\text{sh}\,2\beta)^{3\Omega}Z(\beta^*). \qquad (38)$$

The $4DGZ_3$ *model*:

$$\beta^*=-\frac{2}{3}\ln\left(\frac{1-e^{-3\beta/2}}{1+2e^{-3\beta/2}}\right), \quad \beta_c=\frac{2}{3}\ln(\sqrt{3}+1),$$

$$Z(\beta)=[{}^1/_3(e^{\beta}+2e^{-\beta/2})^{2/3}(e^{\beta}-e^{-\beta/2})^{4/3}]^{3\Omega}Z(\beta^*). \qquad (39)$$

The $4DGZ_2$ *model*:

$$\beta^*=-\ln\text{th}(\beta/2), \quad \beta_c=\ln(\sqrt{2}+1),$$

$$Z(\beta)=(\text{sh}\,\beta)^{6\Omega}Z(\beta^*). \qquad (40)$$

We present also an example of a $2DZ_N$ model (a $4DGZ_N$ model similarly defined) that is self-dual for any N, but in which the parameter β does not have the meaning of the reciprocal temperature. Namely, if in a model with a Gibbs weight of the constraint

$$\exp(V(\varphi_1-\varphi_2))=\sum_{m=-\infty}^{\pm\infty}\exp\{-{}^1/_2\beta(\varphi_1-\varphi_2-2\pi m)^2\} \qquad (41)$$

(the Berezinskiĭ model, see Ref. 9) the field φ assumes N discrete values $\varphi_n=2\pi n/N$ on a circle, then the model is self-dual. The Gibbs weight of the constraint of a dual model is given by

$$\exp\{\tilde{V}(\varphi_1^*-\varphi_2^*)\}=\sqrt{\frac{N}{2\pi\beta}}\sum_{m=-\infty}^{+\infty}\exp\left\{-\frac{\beta^*(\varphi_1^*-\varphi_2^*-2\pi m)^2}{2}\right\}, \qquad (42)$$

where $\varphi^*=2\pi s/N$ is the dual field, which also takes on N discrete values on the unit circle:

$$\beta^*=\left(\frac{N}{2\pi}\right)^2\frac{1}{\beta}. \qquad (43)$$

The symmetry point relative to the duality transformation is

$$\beta_c=N/2\pi. \qquad (44)$$

At $\beta\gg 1$ it is possible to regard β as a reciprocal temperature to the extent that the following expansion is valid:

$$\exp\{\beta(\cos\varphi-1)\}\approx\sum_{m=-\infty}^{+\infty}\exp\{-{}^1/_2\beta(\varphi-2\pi m)^2\}.$$

We note that at any N the model (41) has only one special point β_c (44). As $N\to\infty$, however, the model (41) goes over into the Berzinskiĭ isotropic model[9] (the field φ takes on continuous values on the circle) in which there is a singularity at $\beta_{c1} \sim 1$ (Ref. 9, and also 10, 14, and 19). In our case there is no such point for any N, and $\beta_c \to \infty$ ($T_c \to 0$) as $N \to \infty$. The following explanation can be offered for this situation.

For any finite N in a discrete model there is one phase-transition point (44). At $\beta > \beta_c$ there exists an ordered phase:

$$G(x-x') = \langle \exp\{i(\varphi(x)-\varphi(x'))\}\rangle \underset{|x-x'|\to\infty}{\longrightarrow} \text{const} \neq 0$$

and at $\beta < \beta_c$ we have a disordered phase:

$$G(x-x') \to 0, \quad |x-x'| \to \infty.$$

The ordered phase vanishes in the limit as $N \to \infty$ and $T_c \to 0$. In this case, however, a new singularity arises at the point $\beta_{c1} \sim 1$, corresponding to a phase transition in the isotropic model. Namely, a singularity is present only in the limit as $N \to \infty$, and does not exist for any finite N.

We emphasize that we are dealing here with a discrete Z_N model to which a physical model with sufficiently strong anisotropy corresponds. If the Z_N isotropy is weak, it is possible to reconstruct the $SO(2)$ symmetry, and this leads to new singularities (see Refs. 10 and 14). We recall that in an isotropic model we have $G(x) \to 0$ as $|x| \to \infty$ for any β, but at $\beta > \beta_{c1}$ the correlator decreases in power-law fashion (Berezinskiĭ phase), and at $\beta < \beta_{c1}$ the decrease is exponential (see Refs. 9, 10, 14, 19).

5. SIMPLE EXAMPLE OF DUALITY TRANSFORMATION FOR NON-ABELIAN MODEL

The $2DT$ model was described in Sec. 1. We carry out here a duality transformation for this model, using the general scheme described in Sec. 2.

The representation of the group T in a linear space of functions specified on a homogeneous space T/Z_3 (see Fig. 4) breaks up into a identity transformation with a basis function

$$f_{00}(v) = (1/2,\ 1/2,\ 1/2,\ 1/2) \tag{45}$$

(it is implied that $f_{00}(v_1) = f_{00}(v_2) = f_{00}(v_3) = f_{00}(v_4) = 1/2$) and a three-dimensional irreducible representation whose basis functions are chosen to be

$$f_{11}(v) = (1/2,\ 1/2,\ 1/2,\ 1/2), \quad f_{12}(v) = (1/2,\ -1/2,\ 1/2,\ -1/2),$$
$$f_{13}(v) = (1/2,\ -1/2,\ -1/2,\ 1/2). \tag{45'}$$

The complete set of harmonics for the functions on T/Z_3 is

$$\{f_{lm}(v)\};\ l=0,\ m=0; \quad l=1,\ m=1,\ 2,\ 3.$$

In the partition function of the field

$$Z = \frac{1}{4^\Omega} \sum_{\{v\}} \prod_{x,\alpha} \exp\{V(v(x), v(x+\hat{\alpha}))\} \tag{46}$$

we expand the Gibbs weight of each constraint:

$$\exp\{V(v_1, v_2)\} = C_0 P_0(v_1, v_2) + C_1 P_1(v_1, v_2). \tag{47}$$

Here

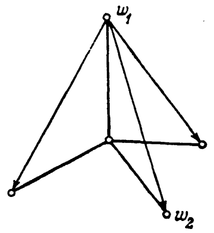

FIG. 7. Differences of the dual field w, which take on four values; this corresponds to $l=0$ at $w_1 = w_2$ and $l=1$ and $m=1, 2, 3$ at $w_1 \neq w_2$.

$$P_0(v_1, v_2) = 4 f_{00}(v_1) f_{00}(v_2) = 1,$$

$$P_1(v_1, v_2) = \frac{4}{3} \sum_{m=1}^{3} f_{1m}(v_1) f_{1m}(v_2) = \begin{cases} 1, & v_1 = v_2 \\ -1/3, & v_1 \neq v_2 \end{cases}.$$

The expansion in P_0 and P_1 and the expansions of P_0 and P_1 in products of the functions $f_{lm}(v)$ is the analog of the expansion in Legendre polynomials and spherical functions. Next,

$$\exp\{V(v_1, v_2)\} = \exp\{\tilde{V}(0)\} f_{00}(v_1) f_{00}(v_2) + \exp\{\tilde{V}(1)\} \sum_{m=1}^{3} f_{1m}(v_1) f_{1m}(v_2); \tag{47'}$$

where

$$\exp\{\tilde{V}(0)\} = 4C_0, \quad \exp\{\tilde{V}(1)\} = 4/3\, C_1.$$

Since the model has symmetry with respect to motions of the space T/Z_3, it follows that $V(v_1, v_2)$ depends only on the distance between the point v_1 and v_2 (or on the absolute value of the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$ corresponding to the points v_1 and v_2). Therefore $V(v_1, v_2)$ takes on two values (for v_1 and ν_2 on T/Z_3):

$$V(v_1, v_2) = \begin{cases} V(0), & v_1 = v_2 \\ V(1), & v_1 \neq v_2 \end{cases}.$$

We substitute the expansion (47′) in (46) and sum over the field v:

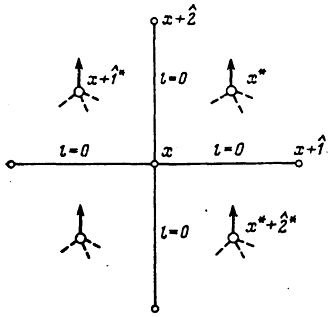

FIG. 8. Configuration corresponding to the coefficient $\begin{pmatrix} 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{pmatrix}$.

The statistical weight of the picture and of the coefficient is equal to unity. We note that one of the vectors, e.g., $w(x+\hat{1}^*)$ must be fixed, since we cannot turn the configuration of the vicinity of a given lattice point completely independently of the neighbors.

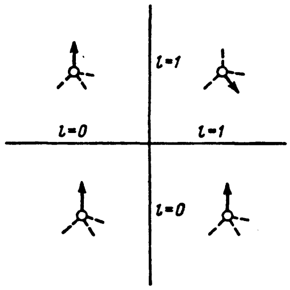

FIG. 9. Configuration corresponding to the coefficient

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & m & m \end{pmatrix}$$

The statistical weight of the picture and of the coefficient is 3×4.

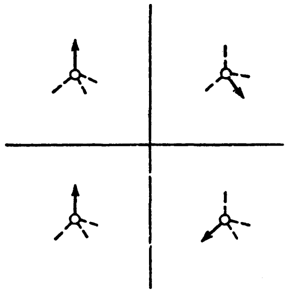

FIG. 11. Configuration corresponding to

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & m_1 & m_2 & m_3 \end{pmatrix}, \quad m_1 \neq m_2 \neq m_3.$$

Statistical weight 3×2×4.

$$Z = \frac{1}{4^\Omega} \sum_{\{v\}} \prod_{x,\alpha} \sum_{lm(x,x+\alpha)} \exp\{\tilde{V}(l(x, x+\hat{\alpha}))\}$$
$$\times f_{lm(x,x+\hat{\alpha})}(v(x)) f_{lm(x,x+\hat{\alpha})}(v(x+\hat{\alpha}))$$
$$= \frac{1}{4^\Omega} \sum_{\{lm\}} \prod_{x,\alpha} \begin{pmatrix} l(x-\hat{1},x) & l(x,x+\hat{1}) & l(x-\hat{2},x) & l(x,x+\hat{2}) \\ m(x-\hat{1},x) & m(x,x+\hat{1}) & m(x-\hat{2},x) & m(x,x+\hat{2}) \end{pmatrix}.$$
$$\times \exp\{\tilde{V}(l(x,x+\hat{\alpha}))\}. \tag{48}$$

Here

$$\begin{pmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix} = \sum_v f_{lm_1}(v) f_{lm_2}(v) f_{lm_3}(v) f_{lm_4}(v). \tag{49}$$

It will be shown below that the limitations imposed on the field of the indices (lm) by the coefficients (49) can be interpreted as the requirement that the divergence vanish for the field (lm), i.e., the values of (lm) can be interpreted as corresponding to the values of the spatial vectors of the dual field. This field is specified at the points of the dual lattice (see Fig. 5) and takes on values on the same T/Z_3 tetrad as the initial field v. We designate the dual field by $w(x)$.

First of all, if the dual field takes values on T/Z_3, then its differences $w_2 - w_1$ at a fixed, say w_1 actually take on four values corresponding to (see Fig. 7)

$$l=0 \text{ при } w_2=w_1; \quad l=1,\ m=1,\ 2,\ 3 \text{ at } w_2 \neq w_1.$$

$\tilde{V}(l(x, x+\hat{\alpha}))$ in (48) depends only on the modulus of the difference, as it should.

We now establish the correspondence between the (lm) combinations picked out by the coefficients (49) in the partition function (48), and the combinations stemming from the argument that (lm) is equivalent to the differences of the field w. It is easy to verify, by using the definition of the functions $f_{lm}(v)$ (45), the following of the coefficients (49) differ from zero:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{4},$$
$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & m_1 & m_2 \end{pmatrix} = \frac{1}{4}\delta_{m_1 m_2},$$
$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & m_1 & m_2 & m_3 \end{pmatrix} = \frac{1}{4} \quad \text{at } m \neq m_2 \neq m_3,$$
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix} = \frac{1}{4} \text{ at any pairwise equality of } m_1, m_2, m_3, m_4.$$

Figures 8–12 show pictures that interpret the different cases when the coefficients (49) do not vanish as states of dual field w. We see that the correspondence is exact, i.e., each nonzero coefficient (49) is set in correspondence with a geometric configuration of the dual function, and that the coefficient and the geometric configuration have equal statistical weights.

The partition function (48) can then be rewritten in the form

$$Z = \frac{1}{4^{2\Omega}} \sum_{\{w\}} \prod_{x,\alpha} \exp\{\tilde{V}(w(x), w(x+\hat{\alpha}))\}. \tag{50}$$

Since w, just as v, takes on values on T/Z_3, we have

$$\tilde{V}(w_1, w_2) = \begin{cases} \tilde{V}(0), & w_1 = w_2 \\ \tilde{V}(1), & w_1 \neq w_2 \end{cases}$$

and consequently there is only one excitation energy. Therefore the $2DT$ model with arbitrary V is self-dual.

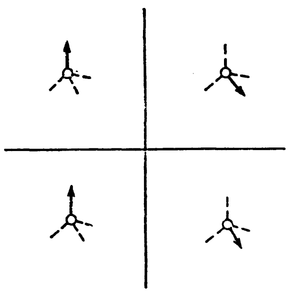

FIG. 10. Configuration corresponding to

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & m & 0 & m \end{pmatrix}.$$

The statistical weight is 4×2 (we recall that $w(x^* + \hat{1}^*)$ is fiex). The cases of Figs. 8 and 9 can be combined into one.

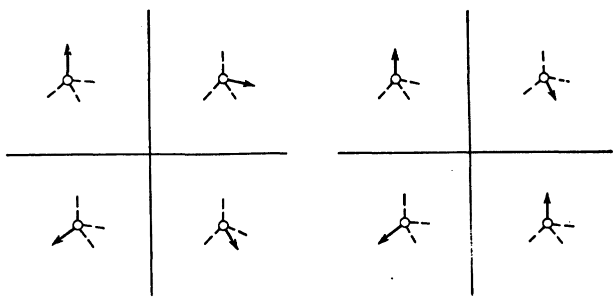

FIG. 12. Configuration corresponding to

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}; m_1, m_2, m_3, m_4$$

are pairwise equal. The statistical weight of the picutre equals 3×3×2+3×1×3=21. The statistical weight of the coefficient is 3×2×3+3=21.

Let $V(0)=0$ and $V(1)=-\beta J$; we then have from (47)

$$1=C_0+C_1,\quad e^{-\beta J}=C_0-{}^1/_3C_1;$$
$$\exp\{\tilde V(0)\}=4C_0=1+3e^{-\beta J},\quad \exp\{\tilde V(1)\}={}^4/_3C_1=1-e^{-\beta J}.$$

Using these expressions we get from (26)

$$\beta^*=-\frac{1}{J}\ln\left(\frac{1-e^{-\beta J}}{1+3e^{-\beta J}}\right). \tag{51}$$

The critical temperature is

$$\beta_c=J^{-1}\ln 3. \tag{52}$$

From (46) and (50) we easily obtain

$$Z(\beta)=[{}^1/_2(1+3e^{-\beta J})]^{2\Omega}Z(\beta^*). \tag{53}$$

We note in conclusion that tne models $2DZ_3$ and $2DT$ are equivalent to Potts models of the first type (see Sec. 1) with $N=3$ and 4. For these models the phase transition point is known from other transformations of the partition function (see Refs. 6 and 20) and coincides with the results (34) and (52) obtained here as a result of the self-duality of these models.

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Additional localized degrees of freedom in spin glasses

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It is shown that besides the three propagating "acoustic" modes predicted by Halperin and Saslow (Phys. Rev. B **16**, 2154, 1977) there exist in spin glasses also localized zero-gap degrees of freedom connected with a system of uniformly distributed disclinations. In this connection, the Poisson-bracket method is used to derive nonlinear equations of motion, which generalize the linearized version presented in a preceding article by the authors [J. Phys. (Paris) **39**, 693, 1978]. The cited version of spin-glass description is furthermore extended to other systems and it is shown that it is in fact a variant of the renormalization-group method of Kadanoff and Wilson.

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1. INTRODUCTION

We have recently[1] constructed a microscopic spin-glass theory wherein the spin glass is represented as consisting of balls of disclinations in a spin system, which are entangled in a complicated and disorderly manner (something recalling a dish of spaghetti). We have arrived at this picture by starting from the microscopic "frustration" concept[1)] developed to apply to spin glasses by Toulouse[2] and Villain.[3] We have shown that at the microscopic level such a magnet can be described by specifying at each point of space x a coordinate frame rigidly connected to a disclination ball located at this point, as well as a continuously distributed macroscopic disclination density. The orientation of the coordinate frame secured at the point x_i is specified in natural fashion by its rotation angle $\varphi(x_j)$ in the "isotopic" space of the spin directions.[2)]

To describe the disclinations we introduce (see Ref. 1), in analogy with dislocation theory (see, e.g., Ref. 4), the quantity $\mathbf{b}_i$. If the macroscopic disclination density is zero, then $\mathbf{b}_i=\partial\varphi/\partial x_i$, so that the disclination density is (in the linear approximation)

$$\rho_i=e_{ikl}\partial\mathbf{b}_l/\partial x_k, \tag{1}$$

where e_{ikl} is a unit antisymmetrical tensor. In analogy with the theory of plastic flow in elasticity (see, e.g., Ref. 4) we introduce also the disclination flux

$$\mathbf{j}_i=\frac{\partial\mathbf{b}_i}{\partial t}-\frac{\partial}{\partial x_i}\frac{\partial\varphi}{\partial t}. \tag{2}$$

A characteristic feature of the microscopic spin-glass theory based on the "frustration" concept is the use of local discrete invariance (LDI) of the exchange

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