Corrections to the hydrodynamics of liquids

A. F. Andreev

Institute of Physics Problems, USSR Academy of Sciences
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The principal correction terms of the linear equations of the hydrodynamics of liquids are obtained . The principal mechanism stems from long-wave thermal fluctuations. The low-frequency dispersion of sound is calculated.

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The hydrodynamics equations are obtained by expanding the equations of motion in the gradients of the velocity and of the thermodynamic quantities up to terms of second order in the spatial derivatives. In this approximation, the form of the equations, as is well known,[1] follows uniquely from the general conservation laws alone, and is therefore the same for all gases and liquids. Distinctions appear only in the thermodynamic functions and in the values of the kinetic coefficients.

The situation is different, as will be shown below, on going to the next-order approximation. There exist, generally speaking two entirely different types of corrections to hydrodynamics. On the one hand we have the usual "gas-kinetic" corrections obtained by Burnett[2] (see also Ref. 3) on the basis of the Boltzmann equation. If we confine ourselves to linearized equations, then the Navier-Stokes equation acquires in the Burnett approximation an additional term proportional to the third spatial derivative of the temperature. The order of magnitude of this term is $k^3nl^2\delta T$, where k is the wave vector or some other reciprocal of a characteristic length, n is the number of particles per unit volume, l is the mean free path of the particles, and δT is the characteristic temperature difference. On the other hand, in the present paper are calculated the fluctuation corrections due to the presence of long-wave thermal fluctuations, particularly acoustic fluctuations. Since sound absorption is proportional to the square of the frequency, acoustic fluctuations with sufficiently low frequency have an arbitrarily large mean free path. This is the physical reason why the fluctuation mechanism is always the basic one at sufficiently small gradients. In fact, the fluctuation correction to the Navier-Stokes equation, as will be shown below, is of the order of $k^{5/2}\times l^{-3/2}\delta T$, i.e., at sufficiently low k it greatly exceeds the gas-kinetic correction. It is important, however, that with increasing k the gas-kinetic corrections becomes the basic one when the condition $kl \gg (na^3)$[4] is satisfied, where $a\sim(nl)^{-1/2}$ is the particle dimension. Therefore for gases ($na^3 \ll 1$) there exists a wide wavefector region in which expansion with respect to the gradients is meaningful ($kl \ll 1$), but the fluctuation corrections are small. For liquids on the other hand $na^3\sim 1$ and the fluctuation corrections are always the principal ones. It is of interest to note that in this case the correction terms contain no new parameters whatever and are completely expressed in terms of the thermodynamic functions and kinetic coefficients that enter in the hydrodynamic equations themselves.

It must be emphasized that there are many phenomena that do not occur in the hydrodynamic approximation and are therefore due entirely just to the corrections that must be made to the hydrodynamics. For example, in hydrodynamics there is no thermomechanical effect, i.e., no onset of motion under the influence of a temperature gradient at constant pressure. It is clear from the foregoing that such phenomena in gases (at not very small k) and in liquids should differ qualitatively from one another. In gases they are described by the local Burnett equations (see Refs. 4 and 5). In the case of liquids, inasmuch as the fluctuation corrections depend on k in nonanalytic fashion, the equations are essentially nonlocal. Some of these nonlocal effects in liquids were considered earlier.[6-8]

1. We start with the hydrodynamic equations of an ideal liquid, expressed in the form of the conservation laws for the mass, momentum, and energy:

$$\dot\rho+\operatorname{div}\rho\mathbf{v}=0,$$

$$\frac{\partial}{\partial t}(\rho v_i)+\frac{\partial}{\partial x_k}(\rho v_i v_k+p\delta_{ik})=0,$$

$$\frac{\partial E}{\partial t}+\operatorname{div}\left\{\rho\mathbf{v}\left(w+\frac{v^2}{2}\right)\right\}=0, \tag{1}$$

where ρ is the density and $\mathbf{v}$ the velocity of the liquid, p is the pressure, E is the energy per unit volume, and w the thermal energy per unit mass.

The presence of thermal fluctuations gives rise to the appearance of small corrections δp, $\delta\mathbf{v}$, . . . to the hydrodynamic quantities; these corrections oscillate in space and in time. In what follows it is essential to ascertain the relation between the fluctuation wave vectors $\mathbf{1}$, which play the principal role, and the wave vector $\mathbf{k}$ of the hydrodynamic motion. Let, for the sake of argument, the hydrodynamic motion of interest to us be a sound wave. From the formulas that follow it will be seen that the main contribution to the correction terms are made by fluctuations whose damping time is of the order of the reciprocal of the frequency of the hydrodynamic motion. Since the damping time of any fluctuation in a liquid is inversely proportional to the square of the wave vector q, and the sound frequency is proportional to the first power of k, it can be assumed that $q \gg k$. We therefore average all the quantities over volumes whose linear dimensions are much less than $1/k$ but much larger than $1/q$. All the quantities that are linear in the fluctuations vanish after such an averaging:

$$\langle\delta p\rangle=\langle\delta\mathbf{v}\rangle=\ldots=0,$$

and the effect of interest to us appears only in second

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order in the fluctuation amplitude.

An arbitrary small perturbation in a liquid is a superposition of acoustic, entropy, and vortical waves. If we choose as the independent thermodynamic variables the pressure p and the entropy per unit mass σ, then the acoustic fluctuations correspond to oscillations of the pressure and of the longitudinal part $\delta\mathbf{v}_l$ of the velocity (curl $\delta\mathbf{v}_l = 0$), the entropy fluctuations correspond to oscillations of σ, and the vortical fluctuations correspond to oscillations of the transverse part $\delta\mathbf{v}_t$ of the velocity (div $\delta\mathbf{v}_t = 0$), with the remaining variables constant. Since the different types of fluctuation can be regarded as statistically independent, the mean values of some quadratic combinations vanish. For example,

$$\langle\delta p\delta\sigma\rangle = \langle\delta p\delta v_t\rangle = \langle\delta\sigma\delta v_{t,l}\rangle = \langle\delta v_t\delta v_l\rangle = 0.$$

Expanding the equations in (1) accurate to terms quadratic in the fluctuation amplitudes, and carrying out the indicated averaging, we get

$$\frac{\partial}{\partial t}\left\{\rho(p,\sigma)+\frac{1}{2}\left(\frac{\partial^2\rho}{\partial p^2}\right)_\sigma\langle\delta p^2\rangle+\frac{1}{2}\left(\frac{\partial^2\rho}{\partial\sigma^2}\right)_p\langle\delta\sigma^2\rangle\right\}$$
$$+\operatorname{div}\left\{\rho\mathbf{v}+\left(\frac{\partial\rho}{\partial p}\right)_\sigma\langle\delta p\delta\mathbf{v}_l\rangle\right\}=0,$$
$$\frac{\partial}{\partial t}\left\{\rho v_i+\left(\frac{\partial\rho}{\partial p}\right)_\sigma\langle\delta p\delta v_{li}\rangle\right\}+\frac{\partial}{\partial x_k}\{p\delta_{ik}+\rho\langle\delta v_{li}\delta v_{lk}\rangle$$
$$+\rho\langle\delta v_{ti}\delta v_{tk}\rangle\}=0,$$
$$\frac{\partial}{\partial t}\left\{E(p,\sigma)+\left[T\left(\frac{\partial\rho}{\partial\sigma}\right)_p+\frac{\rho}{2}\left(\frac{\partial T}{\partial\sigma}\right)_p\right]\langle\delta\sigma^2\rangle\right.$$
$$+\frac{1}{2}\frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_\sigma\langle\delta p^2\rangle+\frac{w}{2}\left[\left(\frac{\partial^2\rho}{\partial\sigma^2}\right)_p\langle\delta\sigma^2\rangle+\left(\frac{\partial^2\rho}{\partial p^2}\right)_\sigma\langle\delta p^2\rangle\right]$$
$$\left.+\frac{\rho}{2}[\langle\delta v_l^2\rangle+\langle\delta v_t^2\rangle]\right\}+\operatorname{div}\left\{w\left[\rho\mathbf{v}+\left(\frac{\partial\rho}{\partial p}\right)_\sigma\langle\delta p\delta\mathbf{v}_l\rangle\right]+\langle\delta p\delta\mathbf{v}_l\rangle\right\}=0, \quad (2)$$

where we have neglected the terms that make no contribution to the linearized equations of interest to us.

The quantity

$$\bar\rho=\rho(p,\sigma)+\frac{1}{2}\left[\left(\frac{\partial^2\rho}{\partial p^2}\right)_\sigma\langle\delta p^2\rangle+\left(\frac{\partial^2\rho}{\partial\sigma^2}\right)_p\langle\delta\sigma^2\rangle\right] \quad (3)$$

in the first equation of (2) is the average renormalized density of the liquid. It is easy to determine analogously the average entropy $\bar S$ per unit volume:

$$\bar S=S(p,\sigma)+\left(\frac{\partial\rho}{\partial\sigma}\right)_p\langle\delta\sigma^2\rangle+\frac{\sigma}{2}\left[\left(\frac{\partial^2\rho}{\partial\sigma^2}\right)_p\langle\delta\sigma^2\rangle+\left(\frac{\partial^2\rho}{\partial p^2}\right)_\sigma\langle\delta p^2\rangle\right] \quad (4)$$

and the average velocity

$$\bar{\mathbf{v}}=\mathbf{v}+\frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_\sigma\langle\delta p\delta\mathbf{v}_l\rangle.$$

If we choose the renormalized quantities $\bar\rho$, $\bar S$, and $\bar{\mathbf{v}}$ as the new independent variables, then we can rewrite (2) in the form

$$\dot\rho+\rho\operatorname{div}\mathbf{v}=0,$$
$$\rho\dot v_i+\frac{\partial}{\partial x_k}\left\{p(\rho,S)+\frac{1}{c^3}\left(\frac{\partial c}{\partial\rho}\right)_\sigma\langle\delta p^2\rangle\right.$$
$$\left.-\frac{\rho^2c^2}{2}\frac{\partial}{\partial\sigma}\left(\frac{1}{\rho^2}\frac{\partial\rho}{\partial\sigma}\right)_p\langle\delta\sigma^2\rangle+\rho\langle\delta v_{li}\delta v_{lk}+\delta v_{ti}\delta v_{tk}\rangle\right\}=0,$$
$$\frac{\partial}{\partial t}\left\{E(\rho,S)+\frac{\rho T}{2c_p}\langle\delta\sigma^2\rangle+\frac{1}{2\rho c^2}\langle\delta p^2\rangle+\frac{\rho}{2}\langle\delta v_l^2+\delta v_t^2\rangle\right\}$$
$$+\operatorname{div}\{\rho w\mathbf{v}+\langle\delta p\delta\mathbf{v}_l\rangle\}=0. \quad (5)$$

where c is the speed of sound, c_p is the heat capacity per unit mass at constant pressure. Here and below we shall omit the bar over the letters ρ, S, $\mathbf{v}$, which will henceforth designate the renormalized quantities.

We represent the fluctuations in the form of the expansions

$$\delta p(\mathbf{r})=V^{-1/2}\sum_{\mathbf{q}}p(\mathbf{q})e^{i\mathbf{qr}},\quad \delta\sigma(\mathbf{r})=V^{-1/2}\sum_{\mathbf{q}}\sigma(\mathbf{q})e^{i\mathbf{qr}},$$
$$\delta\mathbf{v}_t=V^{-1/2}\sum_{\mathbf{q}}\mathbf{l}_\alpha v_\alpha(\mathbf{q})e^{i\mathbf{qr}}$$

where V is the normalization volume, $\mathbf{l}_\alpha$ ($\alpha = 1, 2$) are mutually perpendicular unit vectors and lie in a plane perpendicular to the wave vector q and satisfy the condition $l_{\alpha i}l_{\alpha k} = \delta_{ik} - (q_iq_k/q^2)$, and introduce the distribution functions of the acoustic fluctuations

$$n(\mathbf{q})=|p(\mathbf{q})|^2/\rho c^3q,$$

of the entropy fluctuations

$$g(\mathbf{q})=|\sigma(\mathbf{q})|^2,$$

and the vortical fluctuations

$$f_{\alpha\beta}(\mathbf{q})=v_\alpha(\mathbf{q})v_\beta^*(\mathbf{q}).$$

The mean values in (5) can be expressed in terms of the distribution functions as follows:

$$\langle\delta p^2\rangle=\rho c^3\int d\tau\, qn(\mathbf{q}),\quad \langle\delta\sigma^2\rangle=\int d\tau\, g(\mathbf{q}),$$
$$\langle\delta p\delta\mathbf{v}_l\rangle=c^2\int d\tau\,\mathbf{q}n(\mathbf{q}),\quad \langle\delta v_{li}\delta v_{lk}\rangle=\frac{c}{\rho}\int d\tau\frac{q_iq_k}{q}n(\mathbf{q}),$$
$$\langle\delta v_{ti}\delta v_{tk}\rangle=\int d\tau\, l_{\alpha i}l_{\beta k}f_{\alpha\beta}(\mathbf{q}),$$

where $d\tau = d^3q/(2\pi)^3$. Substituting these formulas in (5), we obtain after simple transformations

$$\dot\rho+\rho\operatorname{div}\mathbf{v}=0,$$
$$\rho\dot v_i+\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_k}\left\{\delta_{ik}\rho\left(\frac{\partial c}{\partial\rho}\right)_\sigma\int d\tau\, qn(\mathbf{q})\right.$$
$$-\delta_{ik}\frac{\rho^2c^2}{2}\frac{\partial}{\partial\sigma}\left(\frac{1}{\rho^2}\frac{\partial\rho}{\partial\sigma}\right)_p\int d\tau\, g(\mathbf{q})+c\int d\tau\frac{q_iq_k}{q}n(\mathbf{q})$$
$$\left.+\rho\int d\tau\, l_{\alpha i}l_{\beta k}f_{\alpha\beta}(\mathbf{q})\right\}=0,$$
$$\rho T\dot\sigma+\frac{\rho T}{2c_p}\int d\tau\,\dot g(\mathbf{q})+\int d\tau\, cq\dot n(\mathbf{q})+\frac{\rho}{2}\int d\tau\dot f_{\alpha\alpha}(\mathbf{q})$$
$$+c^2\int d\tau(\mathbf{q}\nabla)n(\mathbf{q})=0, \quad (6)$$

where we have introduced the renormalized entropy $\sigma = S/\rho$ per unit mass. Equations (6), with only acoustic fluctuations taken into account, i.e., at $f_{\alpha\beta} = g = 0$, were obtained by the author earlier[6] by another method, by starting from the conservation laws.

The last equation of (6) contains, under the sign of the derivative with respect to time, besides the entropy also a comination of distribution functions; this combination constitutes the "combinatorial" (see Ref. 6) entropy of the fluctuations. In what follows it will be convenient to carry out one more renormalization of the entropy, by including in it the combinatorial entropy. In addition, it is possible to replace in all the equations the distribution functions by their deviations δn, δg, $\delta f_{\alpha\beta}$ from the equilibrium values, since the equilibrium fluctuations can be incorporated in the definitions of the thermodynamic functions. As a result, the equations take on the form

$$\dot{\rho}+\rho\,\mathrm{div}\,\mathbf{v}=0,$$

$$\rho\dot{v}_i+\frac{\partial p}{\partial x_i}+\frac{\partial}{\partial x_i}\Big\{\rho\left(\frac{\partial c}{\partial\rho}\right)_\sigma\int d\tau\, q\delta n(\mathbf{q})-\frac{\rho^2c^2}{2}\frac{\partial}{\partial\sigma}\left(\frac{1}{\rho^2}\frac{\partial\rho}{\partial\sigma}\right)_p\int d\tau\,\delta g(\mathbf{q})-\left(\frac{\partial p}{\partial\sigma}\right)_\rho\Big[\frac{1}{2c_p}\int d\tau\delta g(\mathbf{q})+\frac{c}{\rho T}\int d\tau\, q\delta n(\mathbf{q})$$

$$+\frac{1}{2T}\int d\tau\,\delta f_{\alpha\alpha}(\mathbf{q})\Big]\Big\}+\frac{\partial}{\partial x_k}\left\{c\int d\tau\frac{q_iq_k}{q}\delta n(\mathbf{q})+\rho\int d\tau l_{\alpha i}l_{\beta k}\delta f_{\alpha\beta}(\mathbf{q})\right\}=0,$$

$$\rho T\dot{\sigma}+\mathrm{div}\left\{c^2\int d\tau\,\mathbf{q}\delta n(\mathbf{q})\right\}=0. \qquad (7)$$

The acoustic-fluctuation distribution function satisfies[6] the usual Boltzmann equation

$$\dot{n}+\frac{\partial n}{\partial\mathbf{r}}\frac{\partial H}{\partial\mathbf{q}}-\frac{\partial n}{\partial\mathbf{q}}\frac{\partial H}{\partial\mathbf{r}}+\frac{\gamma q^2}{\rho}\left(n-\frac{T}{cq}\right)=0,$$

where $H=cq+\mathbf{q}\cdot\mathbf{v}$, $\gamma=\frac{4}{3}\eta+\zeta+\varkappa(1/c_v-1/c_p)$, c_v is the specific heat per unit mass at constant volume, η and ζ are the first and second viscosity coefficients, and $\varkappa$ is the thermal conductivity coefficient. Putting $n=n_0+\delta n$, where $n_0=T/cq$ is the equilibrium distribution function, and linearizing the kinetic equation, we get

$$\delta\dot{n}+c\mathbf{n}\nabla\delta n+(\gamma q^2/\rho)\delta n=-\frac{T}{cq}\left\{\frac{c}{T}\mathbf{n}\nabla T+n_in_k\frac{\partial v_k}{\partial x_i}+\frac{\dot{T}}{T}-\frac{\dot{c}}{c}\right\}, \qquad (8)$$

where $\mathbf{n}=\mathbf{q}/q$.

The time derivatives in the right-hand side of (8) can be expressed in terms of the space derivatives of the velocity with the aid of the linearized equations of an ideal liquid:

$$\dot{T}=-\rho\left(\frac{\partial T}{\partial\rho}\right)_\sigma\mathrm{div}\,\mathbf{v},\quad \dot{c}=-\rho\left(\frac{\partial c}{\partial\rho}\right)_\sigma\mathrm{div}\,\mathbf{v}.$$

The nonequilibrium part of the acoustic distribution function is thus equal to

$$\delta n(\mathbf{q})=-\frac{\rho T}{cq}\frac{1}{\gamma q^2-i\rho(\omega-c\mathbf{n}\mathbf{k})}\left\{\frac{c}{T}\mathbf{n}\nabla T+n_in_k\frac{\partial v_k}{\partial x_i}+(\varphi-\psi)\,\mathrm{div}\,\mathbf{v}\right\}, \qquad (9)$$

where $\varphi=(\rho/c)(\partial c/\partial\rho)_\sigma$, $\psi=(\rho/T)(\partial T/\partial\rho)_\sigma$, and ω and $\mathbf{k}$ are the frequency and wave vector of the considered hydrodynamic motion.

A kinetic equation for the entropy distribution function $g(\mathbf{g})$ was derived in the Appendix of the paper by Meierovich and the author[8] from the general theory of hydrodynamic fluctuations.[9] If we are interested in the linearized equations this kinetic equation can be written in the form

$$\dot{g}(\mathbf{q})+2\chi q^2\left(g-\frac{c_p}{\rho}\right)=0,$$

where $\chi=\varkappa/\rho c_p$ is the thermal-conductivity coefficient. The equilibrium function is equal to c_p/ρ. From this we get, by the same method as above, the nonequilibrium part of the entropy distribution function:

$$\delta g(\mathbf{q})=\frac{\rho}{2\chi q^2-i\omega}\left(\frac{\partial}{\partial\rho}\frac{c_p}{\rho}\right)_\sigma\mathrm{div}\,\mathbf{v}. \qquad (10)$$

The distribution function of the vortical fluctuations satisfy the equation

$$\dot{f}_{\alpha\beta}+\frac{2\eta q^2}{\rho}\left(f_{\alpha\beta}-\delta_{\alpha\beta}\frac{T}{\rho}\right)=0,$$

from which we get the nonequilibrium part $\delta f_{\alpha\beta}$:

$$\delta f_{\alpha\beta}(\mathbf{q})=\delta_{\alpha\beta}\frac{\rho^2}{2\eta q^2-i\omega\rho}\left(\frac{\partial}{\partial\rho}\frac{T}{\rho}\right)_\sigma\mathrm{div}\,\mathbf{v}. \qquad (11)$$

Substitution of (9) in, say, the last equation of (7), produces under the divergence sign, in particular, an integral of the form

$$I_i(\omega,\mathbf{k})=\int d\tau\frac{n_in_k}{\gamma q^2-i\rho(\omega-c\mathbf{n}\mathbf{k})}\frac{\partial T}{\partial x_k}.$$

This integral diverges at large q, while the difference $I_i(\omega,\mathbf{k})-I_i(0,0)$ is finite. The quantity $I_i(0,0)$ determines the contribution of the acoustic fluctuations to the heat flux in the presence of a temperature gradient that is constant in space and in time, i.e., the contribution to the static thermal-conductivity coefficient. It is clear that the regularization of the diverging integral should consist of a renormalization of the thermal-conductivity coefficient and of subtraction of its value at $\mathbf{k}=\omega=0$ from the integral. It is easily seen that all the diverging integrals obtained by substituting Eqs. (9)–(11) in the equations of (7) can be regularized in similar fashion by renormalizing the static kinetic coefficients η, ζ, and $\varkappa$.

As a result we obtain the following final equations:

$$\dot{\rho}+\rho\,\mathrm{div}\,\mathbf{v}=0,$$

$$\rho\dot{v}_i+\frac{\partial p}{\partial x_i}-\eta\Delta v_i-\left(\zeta+\frac{\eta}{3}\right)\frac{\partial}{\partial x_i}\mathrm{div}\,\mathbf{v}=\frac{\partial}{\partial x_k}\left[\delta\eta_{iklm}(\omega,\mathbf{k})\frac{\partial v_l}{\partial x_m}+\alpha_{ikl}(\omega,\mathbf{k})\frac{\partial T}{\partial x_l}\right],$$

$$\rho T\dot{\sigma}-\varkappa\Delta T=\frac{\partial}{\partial x_i}\left[\delta\varkappa_{ik}(\omega,\mathbf{k})\frac{\partial T}{\partial x_k}+\beta_{ikl}(\omega,\mathbf{k})\frac{\partial v_l}{\partial x_k}\right], \qquad (12)$$

which are written in a form that makes clear the contributions of the fluctuations to the momentum and heat fluxes.

The tensors $\delta\eta_{iklm}$, $\delta\varkappa_{ik}$, α_{ikl}, and β_{ikl} are defined by the formulas

$$\delta\eta_{iklm}(\omega,\mathbf{k})=\frac{T}{6\pi}\left(\frac{\rho}{2\gamma}\right)^{3/2}(i-1)\,\omega^{1/2}\Big\{\frac{k_ik_kk_lk_m}{k^4}(3F_1-15F_2+35F_3)$$

$$+\frac{1}{k^2}(\delta_{ik}k_lk_m+\delta_{lm}k_ik_k)\left[-F_1+3F_2-5F_3+(\varphi-\psi)(-4F_1+6F_2)\right]$$

$$+\frac{1}{k^2}(\delta_{il}k_kk_m+\delta_{im}k_kk_l+\delta_{kl}k_ik_m+\delta_{km}k_ik_l)(-F_1+3F_2-5F_3)$$

$$+(\delta_{il}\delta_{km}+\delta_{im}\delta_{kl})(F_1-F_2+F_3)+\delta_{ik}\delta_{lm}\Big[F_1-F_2+F_3+4(\varphi-\psi)(2F_1-F_2)$$

$$+8(\varphi-\psi)^2F_1+\frac{3}{8}(1-\psi)\left(\frac{2}{3}-\psi\right)\left(\frac{2\gamma}{\eta}\right)^{1/2}$$

$$+\frac{3}{16}\frac{c^2}{Tc_p}\left[\rho c_p\frac{\partial}{\partial\sigma}\left(\frac{1}{\rho^2}\frac{\partial\rho}{\partial\sigma}\right)_p-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial\sigma}\right)_p\right]\left(\frac{2\gamma c_p}{\varkappa}\right)^{1/2}\rho^2\left(\frac{\partial}{\partial\rho}\frac{c_p}{\rho}\right)_\sigma\Big]\Big\},$$

$$\delta\varkappa_{ik}(\omega,\mathbf{k})=c^2\left(\frac{\rho}{2\gamma}\right)^{1/2}\frac{i-1}{3\pi}\omega^{1/2}\left[\delta_{ik}(2F_1-F_2)+\frac{k_ik_k}{k^2}(-2F_1+3F_2)\right],$$

$$\alpha_{ikl}=\frac{1}{T}\beta_{lik}=c\left(\frac{\rho}{2\gamma}\right)^{1/2}\frac{i-1}{6\pi}\omega^{1/2}\Big\{\frac{k_ik_kk_l}{k^3}(5F_5-3F_4)$$

$$+\frac{1}{k}(\delta_{il}k_k+\delta_{kl}k_i)(F_4-F_5)+\frac{1}{k}\delta_{ik}k_l[F_4-F_5+(\varphi-\psi)2F_4]\Big\},$$

$$F_1=\frac{1}{8}\frac{\omega}{ck}f_3,\quad F_2=\frac{1}{4}\left(\frac{\omega}{ck}\right)^3\left(f_3-\frac{6}{5}f_5+\frac{3}{7}f_7\right),$$

$$F_3=\frac{1}{8}\left(\frac{\omega}{ck}\right)^5\left(f_3-\frac{12}{5}f_5+\frac{18}{7}f_7-\frac{4}{3}f_9+\frac{3}{11}f_{11}\right),$$

$$F_4=\frac{1}{2}\left(\frac{\omega}{ck}\right)^2\left(f_3-\frac{3}{5}f_5\right),$$

$$F_5=\frac{1}{2}\left(\frac{\omega}{ck}\right)^4\left(f_3-\frac{9}{5}f_5+\frac{9}{7}f_7-\frac{1}{3}f_9\right),$$

$$f_n=(1+ck/\omega)^{n/2}-(1-ck/\omega)^{n/2}.$$

A part of the corrections to hydrodynamic equations is thus equivalent to the appearance of dispersion (temporal and spatial) of the kinetic coefficients. In addition, the expressions for the heat flux acquire terms with velocity gradients, and correspondingly terms with temperature gradients appear in the momentum-flux tensor. The value given above for the ratio of the tensors α_{ikl} and β_{ikl} agrees with the principle of symmetry of the kinetic coefficients.

3. The obtained equations can be used to calculate the low-frequency sound dispersion in liquids. From (12) we can readily determine the correction terms for the phase velocity of the sound $c(\omega)$ and for its damping $\Gamma(\omega)$. Without dwelling on the simple calculations, we present the final result:

$$c(\omega)=c-\frac{T}{12\pi\rho c}\left(\frac{\rho\omega}{\gamma}\right)^{3/2}\Phi,$$

$$\Gamma(\omega)=\frac{\gamma\omega^2}{2\rho c^2}\left\{1-\frac{\rho T}{6\pi\gamma^2}\left(\frac{\rho\omega}{\gamma}\right)^{1/2}\Phi\right\},$$

where Φ is a dimensionless quantity equal to

$$\Phi=\frac{211}{1155}+\frac{22}{35}(\varphi-\psi)+(\varphi-\psi)^2+\frac{11}{35}\frac{c^2}{T}\frac{c_p-c_v}{c_pc_v}$$
$$+\frac{3}{8}(1-\psi)\left(\frac{2}{3}-\psi\right)\left(\frac{\gamma}{\eta}\right)^{3/2}-\frac{2}{5\rho c_v}\left(\frac{\partial p}{\partial T}\right)_\rho\left[\frac{13}{21}+(\varphi-\psi)\right]$$
$$+\frac{3}{16}\frac{\rho^2c^2}{Tc_p}\left[\rho c_p\frac{\partial}{\partial\sigma}\left(\frac{1}{\rho^2}\frac{\partial\rho}{\partial\sigma}\right)_p-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial\sigma}\right)_p\right]\left(\frac{\gamma c_p}{\varkappa}\right)^{3/2}\left(\frac{\partial}{\partial\rho}\frac{c_p}{\rho}\right)_\sigma.$$

The relative corrections to the speed of sound and to the damping are thus proportional respectively to $\omega^{3/2}$ and $\omega^{1/2}$.

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Critical phenomena in cholesteric liquid crystals

S. A. Brazovskiĭ and V. M. Filev

L. D. Landau Theoretical Physics Institute, USSR Academy of Sciences
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Phase transitions in cholesteric liquid crystals are considered. A phase diagram is derived which makes it possible to explain the existence of intermediate phases in a narrow region between the uniform isotropic (UI) phase and the spiral phase. The critical phenomena are investigated in the light of experiments on the supercooling of the UI phase.

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1. INTRODUCTION

The critical properties of cholesteric liquid crystals (CLC) in phase transitions from the uniform isotropic (UI) phase to the spiral phase have a number of important differences from the critical properties of other systems. A number of experimental[1-4] and theoretical[1,5-7] papers have been devoted to the study of the phase transitions in CLC, but some pertinent problems are still far from being completely solved. In particular, the natural supercooling of the UI phase observed in Ref. 4 and the anomalies in the temperature dependence of the precritical scattering of light require deeper investigation. As will be shown below, these anomalies agree qualitatively with the predictions made in Refs. 7 and 8, and with a more complex experimental investigation it ought to be possible to pose the question of the quantitative comparison of the theoretical and experimental results.

The theory developed in Refs. 7 and 8 predicts a discontinuous transition to the spiral phase, occurring in a region of substantial manifestation of critical anomalies due to the effect of critical fluctuations. The alternative is the formation of a planar lattice of spirals, with a triangular structure.[9] In the experiment of Ref. 4 a discontinuity was observed only in the transition from

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