transition $C^{12} \to Ni^{56}$ ($q = 9.10^{17}$ erg/g). Note that the subsequent neutronization is a secondary endothermal process and cannot lead to a reduction in the detonation velocity.[2]

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Qualitative isotropic cosmology with cosmological constant and with allowance for dissipation

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New types of evolution that arise in isotropic cosmological Friedmann models with cosmological constant when allowance is made for bulk viscosity are decribed. A family of solutions is obtained for a closed model without singularities of the metric and the energy density. The stability of static solutions in the presence of viscosity is investigated. The coefficient of viscosity is assumed to be a function of the energy density that has power-law asymptotic behaviors at small and large values of the argument.

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In the investigation of isotropic cosmological models with allowance for the λ term, attention has been mainly concentrated on the stability of static solutions and the description of the evolution of various types of solution. On the other hand, the equations that describe the evolution of models constitute a dynamical system and it is of interest to investigate the behavior of the integral curves on the phase plane. This makes it possible to give a perspicuous classification of the possible types of evolution.

For the case $\lambda = 0$, this was done in[1], in which one can find the necessary details that are omitted here.

The behavior of the solutions in the region of large values of the Hubble "constant" $H = \dot{R}/R$ or energy density are the same as in[1]. The most interesting effects occur near the static de Sitter and Einstein solutions.

We write the Friedmann metrics in the form

$$-ds^2 = -dt^2 + \frac{R^2(t)\,(dx^2+dy^2+dz^2)}{[1+{}^1/_4 k(x^2+y^2+z^2)]^2}, \tag{1}$$

where $k = +1$, -1 and 0 correspond to closed, open, and flat models, respectively. In an isotropic cosmological evolution, the shear (first) viscosity is not manifested and one need only consider the bulk viscosity, whose coefficient is ζ.

The energy-momentum tensor has the form

$$T_k^i = (\varepsilon + p')u^i u_k + p'\delta_k^i, \quad p' = p - \zeta u^k_{;k}. \tag{2}$$

For simplicity, we take the equation of state in the form $p = (\gamma - 1)\varepsilon$. A comoving frame can be chosen.

We introduce $H = (\ln R)^{\cdot} = \dot{R}/R$, the Hubble "constant". Then the Einstein equations $R_k^i + \lambda\delta_k^i = T_k^i - \frac{1}{2}\delta_k^i T$ and the hydrodynamic equations $T^i_{k;i} = 0$ reduce to the three equations

$$\dot{\varepsilon} = 3H(3\zeta H - w), \tag{3}$$

$$\dot{H} = {}^1/_3(\lambda + \varepsilon - 3H^2) + {}^1/_2(3\zeta H - w), \tag{4}$$

$$\lambda + \varepsilon - 3H^2 = 3kR^{-2}, \tag{5}$$

In the variables (H, ε), the system of equations (3)–(5) does not depend on k. The parabola $\lambda + \varepsilon - 3H^2 = 0$ separates the open and closed models: The integral curves of the closed model lie within it; those of the open model, outside it.

As in[1], we assume that $\zeta(\varepsilon) = \alpha\varepsilon^{a_2}$, as $\varepsilon \to 0$ and $\zeta(\varepsilon) = \beta\varepsilon^{b_2}$, as $\varepsilon \to \infty$ ($a_2 \geqslant 1$, $b_2 < \frac{1}{2}$), since unphysical effects do not occur for such exponents.

The singular points of the system (3)–(5) lie on the parabola $\lambda + \varepsilon = 3H^2$ and on the straight line $H = 0$.

We consider first the case $\zeta = 0$. The system (3)–(5) can be integrated, and the equation of the integral phase curves is

$$\lambda + \varepsilon - 3H^2 = 3k(\varepsilon/\varepsilon_0)^{2/3\gamma}, \tag{6}$$

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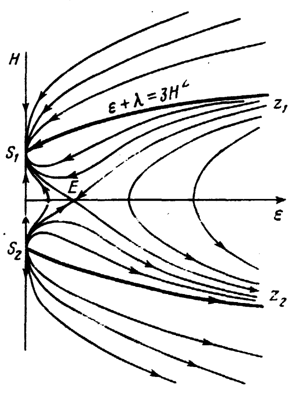

FIG. 1. Integral curves of isotropic models in the absence of viscosity. The parabola $\varepsilon+\lambda=3H^2$ divides the integral curves of the closed and open models and is itself the integral curve of the flat Friedmann model. The equation of the integral curves is $\lambda+\varepsilon-3H^2=3k(\varepsilon/\varepsilon_0)^{2/3\gamma}$, where ε_0 is an arbitrary constant. For $\gamma<4/3$, the integral curves in the limit $\varepsilon\to\infty$ move away from the parabola $\varepsilon+\lambda=3H^2$; for $\gamma=4/3$, they run parallel with it, and for $\gamma>4/3$ they begin to converge toward it. The points $Z_{1,2}$ correspond to $H=\pm\infty$, $\varepsilon=\infty$.

where ε_0 is an arbitrary constant. The phase plane is symmetric under the substitution $H\to-H$, $t\to-t$.

For finite ε and H, the system has three singular points. The point E corresponds to Einstein's static space: $H=0$, $\varepsilon=\lambda/(\frac{3}{2}\gamma-1)$, $R^{-2}=\gamma\lambda/(3\gamma-2)$. This singular point is a saddle and the slope of the separatrix is

$$\left.\frac{dH}{d\varepsilon}\right|_E=\pm\frac{{}^3/_2\gamma-1}{3(\gamma\lambda)^{1/2}},\quad \varepsilon=\varepsilon_E+C_1\exp[-(\gamma\lambda)^{1/2}t]+C_2\exp[(\gamma\lambda)^{1/2}t],$$
$$H=\frac{{}^3/_2\gamma-1}{3(\gamma\lambda)^{1/2}}C_1\exp[-(\gamma\lambda)^{1/2}t]-\frac{{}^3/_2\gamma-1}{3(\gamma\lambda)^{1/2}}C_2\exp[(\gamma\lambda)^{1/2}t]. \tag{7}$$

The points S_1 and S_2 correspond to the de Sitter spaces: $\varepsilon=0$, $H_0=\pm(\lambda/3)^{1/2}$. Near the singular points S_1 and S_2, the metric has the form

$$ds^2=dt^2-\exp(2H_0t)(dx^2+dy^2+dz^2)$$
$$=dt^2-\exp(2H_0t)(dr^2+r^2\sin^2\theta\,d\varphi^2+r^2d\theta^2). \tag{8}$$

The transformation found by Lemaître and Robertson:

$$r=\frac{\bar r\exp(-\bar t/R_0)}{(1-\bar r^2/R_0^2)^{1/2}},\quad t=\bar t+R_0\ln(1-\bar r^2/R_0^2)^{1/2},\quad R_0=1/H_0=\pm(3/\lambda)^{1/2} \tag{9}$$

reduces this metric to the usual de Sitter static interval:

$$ds^2=\left(1-\frac{\bar r^2}{R_0^2}\right)dt^2-\frac{d\bar r^2}{1-\bar r^2/R_0^2}-\bar r^2\,d\theta^2-\bar r^2\sin^2\theta\,d\varphi^2. \tag{10}$$

The equations for $\varepsilon=0$ can be readily integrated and give

$$H=\left(\frac{\lambda}{3}\right)^{1/2}\operatorname{th}\left(\frac{\lambda}{3}\right)^{1/2}t,\quad R=\left(\frac{\lambda}{3}\right)^{-1/2}\operatorname{ch}\left(\frac{\lambda}{3}\right)^{1/2}t \tag{11}$$

for $3H^2<\lambda$, $k=1$, and

$$H=\left(\frac{\lambda}{3}\right)^{1/2}\operatorname{cth}\left(\frac{\lambda}{3}\right)^{1/2}t,\quad R=\left(\frac{\lambda}{3}\right)^{-1/2}\left|\operatorname{sh}\left(\frac{\lambda}{3}\right)^{1/2}t\right| \tag{12}$$

for $3H^2>\lambda$, $k=-1$. As $t\to0$, we have $H\to\pm\infty$, $R\sim t$ and space is indeed Galilean, i.e., the points $H=\pm\infty$, $\varepsilon=0$ correspond to the same flat spacetime. The different behavior of the curves in the neighborhood of the points S_1 and S_2 corresponding to the same metric means that the de Sitter space is stable with respect to the introduction into it of low-density expanding matter but it is unstable with respect to the introduction into it of contracting matter of arbitrarily low density.

To the separatrices of the points S_1 and S_2 there correspond the eigenvalues $\nu_1=-2(\lambda/3)^{1/2}$ for $\varepsilon=0$, and $\nu_2=-3\gamma(\lambda/3)^{1/2}$ for $\lambda+\varepsilon=3H^2$ in the case S_1 and $-\nu_1$ and $-\nu_2$ in the case S_2, as must be for the reversible model.

For the integral curves we arrive at the classification given by Tolman.[2] In the region S_2ES_1, the integral curves begin at $R=+\infty$, $t=-\infty$, contract to a finite radius at $t=0$, and then expand to $R=\infty$ from $t=0$ to $t=\infty$. The integral curves in the region Z_1EZ_2 expand from $R=0$, $\varepsilon=\infty$, $H=\infty$ to $R_{\max}$, $H=0$, and then contract again to $R=0$ in a finite time.

Above the line Z_1ES_1, we have expansion from $R=0$ at $t=0$ to $R=\infty$ at $t=\infty$ (de Sitter space). Below the line S_2EZ_2 the opposite contraction occurs: from $R=\infty$ at $t=-\infty$ to $R=0$ at $t=0$.

We now turn to the case when viscosity is present. In contrast to the case $\lambda=0$, viscosity can also have a significant influence for small ε and H. The singular points are now determined by the equation

$$3\zeta H-w=0,\quad \varepsilon+\lambda-3H^2=0,$$

or

$$\varepsilon=\frac{\lambda}{{}^3/_2\gamma-1},\quad H=0. \tag{13}$$

If $a_2>1$, we obtain an even number of new singular points; as in the case $\lambda=0$, they alternate in a sequence of saddles and nodes. If $\zeta=\alpha\varepsilon$ as $\varepsilon\to0$ and $\alpha(3\lambda)^{1/2}>\gamma$, then there is an odd number of new singular points and they alternate in the sequence note-saddle-note.

We consider the first case ($a_2>1$ or $a_2=1$, i.e., $\zeta=\alpha$ for $\alpha<\gamma/(3\lambda)^{1/2}$). In this case, the qualitative picture of the integral curves within and near the region S_1S_2E is the same as for $\zeta=0$. For $a_2>1$, the eigenvalues of the points S_1 and S_2 are not changed; for $a_2=1$, to the separatrix $\varepsilon=0$ there corresponds $\nu_1=-2H_0$ and to the separatrix $\varepsilon+\lambda=3H^2$ there corresponds $\nu_2=3H_0(3\alpha H_0-\gamma)$. If $\gamma-3\alpha(\lambda/3)^{1/2}<\frac{2}{3}$, the curves squeeze up to the separatrix $\varepsilon+\lambda=3H^2$ at the point S_1. For the point E, the eigen-

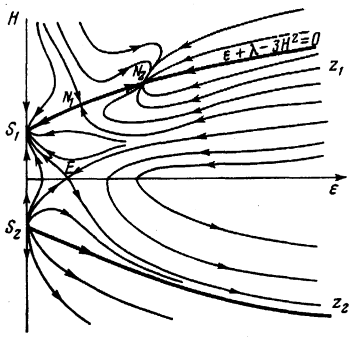

FIG. 2. Integral curves of isotropic models in the presence of viscosity. When $\varepsilon\to0$, we have $\zeta(\varepsilon)\ll\varepsilon$. Viscosity does not change the qualitative picture of the integral curves below the line $Z_1N_1S_1$. Above this line, the behavior of the integral curves is qualitatively the same as for $\lambda=0$.

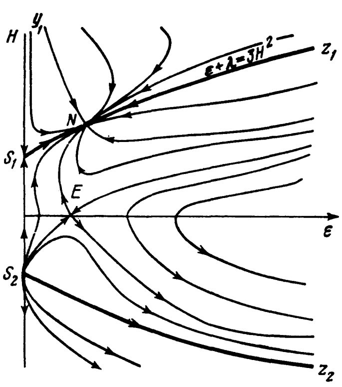

FIG. 3. The special case when $\zeta=\alpha\varepsilon$ and $\alpha>\gamma/(3\lambda)^{1/2}$. The point S_1 becomes unstable and the integral curves from S_1 now lead to N. On the segment S_1Z_1 of the parabola there is an odd number of singular points (in Fig. 3, one).

values are

$$\nu_{1,2}={}^3/_4\zeta(\varepsilon_E)\pm[{}^9/_{16}\zeta^2(\varepsilon_E)+\gamma\lambda]^{1/2},$$
$$\varepsilon_E=\lambda/({}^3/_2\gamma-1). \tag{14}$$

Near E, the equation of the integral curves is

$$H=C_1\nu_1\exp(\nu_1 t)+C_2\nu_2\exp(\nu_2 t),$$
$$\varepsilon=\varepsilon_E-3\gamma C_1\varepsilon_E\exp(\nu_1 t)-3\gamma C_2\varepsilon_E\exp(\nu_2 t). \tag{15}$$

The points N_1 and N_2 lie on the parabola $\lambda+\varepsilon=3H^2$. Their position is determined by the equation

$$w(\varepsilon)=\gamma\varepsilon=\zeta(\varepsilon)[3(\varepsilon+\lambda)]^{1/2}. \tag{16}$$

The cosmological term does not have any appreciable influence on them. The eigenvalues of these singular points are

$$\nu_1=3(\varepsilon_N+\lambda)\left[\frac{d}{d\varepsilon}\left(\zeta(\varepsilon)-\frac{w(\varepsilon)}{(3(\varepsilon+\lambda))^{1/2}}\right)\right]_N, \tag{17}$$

$$\nu_2=-2H_N, \tag{18}$$

where ε_N and H_N are the values of ε and H at the corresponding singular point (N_1 or N_2). The eigenvalue ν_1 corresponds to the separatrix $\varepsilon+\lambda=3H^2$. In the region of large ε or H, the integral curves behave in exactly the same way as for $\lambda=0$.

In the special case when $a_2=1$, i.e., $\zeta=\alpha\varepsilon$ for $\alpha>\gamma/(3\lambda)^{1/2}$, the viscosity is also dominant for small ε in the neighborhood of ε the points S_1 and S_2. The point S_1 becomes a saddle. The eigenvalues of the points S_1 and S_2 are

$$\nu_1=-2H_0,\quad \nu_2=3H_0(3\alpha H_0-\gamma),\quad H_0=(\lambda/3)^{1/2} \tag{19}$$

for S_1, and

$$H_0=-(\lambda/3)^{1/2} \tag{20}$$

for S_2.

The dependence $R(t)$ for integral curves from the region S_1S_2EN corresponds to contraction from $R=\infty$, $t=-\infty$ to $R=R_{\min}$ and expansion to $R=\infty$ for $t=+\infty$. These integral curves are analogous to the integral curves in the region S_1S_2E in Fig. 2, since the metric at the point N:

$$ds^2=dt^2-\exp(2H_N t)(dx^2+dy^2+dz^2) \tag{21}$$

reduces to the de Sitter metric. Thus, if there is nonzero cosmological term, then for all physically reasonable dependences of the viscosity coefficient ζ on ε (including $\zeta=0$) there exists a family of integral curves on which the metric and energy density do not have singularities.

In Fig. 3 there is also the special trajectory S_1N, which corresponds to expansion from $R=0$, $\varepsilon=0$ (de Sitter space) to $R=\infty$, $\varepsilon=\varepsilon=\varepsilon_N$. The physical meaning of this integral curve is not clear.

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