Expression for relativistic amplitudes in terms of wave functions

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The conditions under which relativistic amplitudes can be expressed in terms of wave functions are investigated within the framework of an invariant diagram technique that appears in a field-theoretical treatment of a light front. The obtained amplitudes depend on a 4-vector ω that defines the surface of the wave front. A prescription is formulated for the determination of values of the 4-vector ω that minimize the diagram contribution which is not expressed in terms of the wave functions. This investigation is the equivalent of a study of the dependence of the amplitudes of the old perturbation theory in a system with infinite momentum on the direction of the infinite momentum.

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1. INTRODUCTION

The investigation of nuclei in the region of relativistic nucleon momenta ($q \sim m$) and of nuclear reactions with large momentum transfers made it necessary to develop a formalism of relativistic wave functions that are relativistic invariants, admit of a probabilistic interpretation, depend just as the nonrelativistic wave functions on three-dimensional vectors, and go over in the nonrelativistic limit into the ordinary nonrelativistic wave functions. Wave functions in a relativistic coordinate were considered space by Shapiro.[1] The problem of relativistic wave functions was discussed by him also in Ref. 2. In my earlier paper[3] I developed a wave-function formalism with the properties indicated above. In this formalism, the wave functions are non-single-time Fock components of a state vector in an invariant Schrödinger representation (ISR) on the surface of the light front $\omega x = 0$ (ω is a 4-vector that lies on the light cone: $\omega = (\omega_0, \boldsymbol{\omega})$, $\omega^2 = 0$, $\omega_0 > 0$).

To obtain information on the wave functions from experiment, it is necessary that the amplitudes of the process be expressed in terms of the wave function of a bound system. To express the amplitudes in terms of non-single-time wave functions a non-single-time computation formalism is necessary. Such a formalism, in the ISR, was developed by Kadyshevskiĭ[4] and formulated for the case of a light front in Ref. 3. It combines the advantages of the Feynman diagram technique (explicit relativistic invariance) and of the old perturbation theory (possibility of working with probabilistically interpretable wave functions). Questions of unitary and causality in field theory formulated on a light front were investigated in Ref. 5.

However, the problem of expressing relativistic amplitudes in terms of wave functions is made complicated by the fact that contribution to the amplitude of scattering by a bound system is made not only by the diagrams that are expressed in terms of wave functions of the coupled system, but also diagrams which are not expressed in terms of wave functions. The causes of this phenomenon, which does not occur in the nonrelativistic case, will be explained in detail in Sec. 2. We note only that is is connected with the fact that in the $|\mathrm{in}\rangle$ and $\langle\mathrm{out}|$ state vectors, which provide a complete description of the system of interacting particles, for example an electron and a deuteron, there can be present, owing to the virtual production of particles, Fock components which cannot be expressed in terms of the Fock components of the state vector of the deuteron. In addition, the amplitudes turn out to depend on the 4-vector ω which defines the surface of the light front, and a dependence on ω is possessed also by the relative contributions of the diagrams which are expressed and are not expressed in terms of wave functions.

We note that the same problems exist in the old perturbation theory in a system with infinite momentum (SIM). Thus, the analog of the dependence of the IRS diagrams on ω is the dependence of the diagrams of the old perturbation theory in the SIM on the direction of the infinite momentum.

To obtain unambiguous expressions for the amplitudes in terms of the wave functions, it is necessary to find the conditions under which the relative contribution of diagrams not expressed in terms of wave functions is minimal, and fix in some way the position of ω on the light cone relative to the 4-momenta of the particles that take part in the reaction.

The purpose of the present paper is to find a method of uniquely expressing the amplitudes of the processes in terms of wave functions. We obtain a prescription for finding the values of the 4-vector ω (which depend on the type of diagram) that minimize the contribution of the diagrams that are not expressed in terms of the wave functions, and consequently the amplitudes of the processes can be unambiguously expressed in terms of the wave functions of the bound system.

In Sec. 2 we recall the main properties of the non-single-time wave functions investigated in Ref. 3, and the ISR diagram technique. We explain how diagrams that are not expressed in terms of wave functions arise and why the amplitudes become dependent on the 4-vector ω, and also illustrate these properties of the diagrams with examples.

In Sec. 3, we show that the ISR diagram technique is an invariant formulation of the old perturbation theory

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with SIM (a change of variables makes the expressions for the amplitudes identical).

Thus, all the results of the present paper pertain equally well also to the old perturbation theory with SIM. The dependence of the amplitudes on ω means that these amplitudes depend on additional invariant variables—scalar products of the 4-vector ω and the 4-momentum of the particles that take part in the reaction.

We introduce invariant variables that are connected with ω and obtain range of their variation. In Sec. 4 we investigate the singularities of the ISR amplitudes. The positions of some of these singularities turn out to depend on the values of the invariant variables that are connected with ω. In Sec. 5, investigating the character of the dependences of the amplitudes on the invariant variables connected with ω, and using information on the positions of the singularities that depend on the scalar products of ω with the 4-momenta, we obtain a prescription for finding for these invariants (which is equivalent to finding the 4-vector ω) values such that the contribution of the diagrams that are not expressed in terms of wave functions is minimal. Section 6 contains concluding remarks. In the Appendix we investigate the parametrization of the ISR amplitude off the energy shell.

2. RELATIVISTIC WAVE FUNCTIONS AND DIAGRAM TECHNIQUE

We recall the principal results of Ref. 3, which pertain to the relativistic wave functions. We carry out here a qualitative analysis of the properties of the wave functions.[1)]

Let $\psi(\mathbf{x}_1, \mathbf{x}_2, t)$ be a two-particle Fock component having the meaning of the probability amplitude of observing a particle in the reference frame A at the points $\mathbf{x}_1$ and $\mathbf{x}_2$ at the instant of time t. In the reference frame A', this wave function will not be single-time: it will describe particles at the points $\mathbf{x}_1'$ and $\mathbf{x}_2'$ at the instants of time t_1' and t_2'. The wave function $\psi'(\mathbf{x}_1', \mathbf{x}_2', t')$, which is single-time in the system A', does not coincide with $\psi(\mathbf{x}_1, \mathbf{x}_2, t)$, since the position of the particles changes in a time $\Delta t = t_2' - t_2'$. Therefore the single-time wave function is not a covariant quantity: there is no kinematic connection between ψ and ψ'. The connection between them is dynamic and contains the Hamiltonian of the system. A covariant quantity is the non-single-time wave function $\psi(x_1, x_2, \lambda)$, defined on an arbitrary space-like surface.

For simplicity we consider a plane surface $\lambda x = 0$, with $\lambda = (\lambda_0, \boldsymbol{\lambda})$, $\lambda^2 = 1$, $\lambda_0 > 0$. Such a wave function is transformed kinematically, inasmuch as it describes in the system A' particles at the same points of space-time as in the system A. The coordinates of these points and the position of the surface are different in the systems A and A'—they are connected by a Lorentz rotation. In the case of a nonrelativistic bound system, the position of the particles does not manage to change substantially within a time $\Delta t = t_2' - t_1'$. Therefore the non-single-time wave functions coincide in the nonrelativistic limit with the ordinary single-time wave functions.

In momentum space, the qualitative aspect of the situation reduces to the following. In nonrelativistic theory, the deuteron momentum p and the nucleon momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ are connected by the relation

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{p}. \tag{1}$$

In relativistic theory, Eq. (1) can hold only in a system where the wave function is single-time, inasmuch as in any other system we add up the momenta of particles 1 and 2 taken at different instants of time. Since the momentum of the deuteron is conserved, it is clear that for two instants of time t_1 and t_2 we have

$$\mathbf{k}_1(t_1) + \mathbf{k}_2(t_2) \neq \mathbf{p}(t_1) = \mathbf{p}(t_2).$$

In a system where $\mathbf{k}_1 = -\mathbf{k}_2 \equiv \mathbf{q}$, the deuteron momentum $\mathbf{p} \neq 0$ and in addition to the vector variable q (the relative relativistic momentum) there remains also a certain variable p. In other words, in a relativistic deuteron it is now possible to separate the variables of the "center of gravity," as can be done in relativistic theory. An exception is a system of noninteracting particles. In this case Eq. (1) is always valid, because the momenta $\mathbf{k}_i$ are invariant in time.

The easiest to parametrize is the wave function $\omega x = 0$, which is single-time in a system with infinite momentum. In a system where $\mathbf{p} \to \infty$, it does not depend on $|\mathbf{p}|$, but depends on the variable $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$:

$$\psi = \psi(\mathbf{q}, \mathbf{n}). \tag{2}$$

By virtue of the invariance of the non-equal-time wave functions, the parametrization (2) is valid in any refererence frame. In addition, the fluctuations of the vacuum do not contribute to a wave function on the light front. It is precisely in this case that the wave function concept acquires a clear-cut meaning.

Changing over to four-dimensional notation, we note that the nonzero difference between the 4-momenta $k_1 + k_2$ and p (all the 4-momenta are on the mass shells) should be proportional to the 4-vector λ:

$$k_1 + k_2 - p = \lambda\tau, \tag{3}$$

where τ is a scalar parameter. Actually λ is the only 4-vector that identifies a wave function as non-single-time. In a system where the wave function is single-time we have $\boldsymbol{\lambda} = 0$ and $\lambda_0 = 1$, and we return to Eq. (1). In the case of a wave function on the light front, Eq. (3) becomes

$$k_1 + k_2 = p + \omega\tau \tag{4}$$

and the wave function depends on four 4-vectors $\psi = \psi(k_1, k_2, p, \omega\tau)$, which are connected by the conservation law (4). It is therefore convenient to express the wave function graphically in the form of the 4-point diagram of Fig. 1, and regard the 4-momentum $\omega\tau$ as the momentum of a ficticious particle, the spurion. The wave function, the "amplitude of the reaction," shown in Fig. 1 depends on the relative momentum q of the "final" particles and on the variable qn which is connected with the "scattering angle" (n is the direction

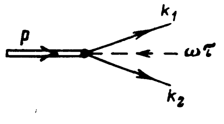

FIG. 1.

of the momentum of the "initial" spurion in the c.m.s. of the "reaction") in accordance with formula (2). In the nonrelativistic limit, the dependence on n vanishes. We emphasize that the introduction of the spurion does not mean that an unphysical particle is present in the state vector.

It is easy to express the variables $\mathbf{q}^2$ and qn in terms of the invariants

$$\mathbf{q}^2=\frac{s}{4}-m^2,\quad \frac{\mathbf{nq}}{\varepsilon(\mathbf{q})}=\frac{(u-t)\sqrt{s}}{2(s-M^2)},\tag{5}$$

where

$$s=(p+\omega\tau)^2=(k_1+k_2)^2,\quad t=(p-k_1)^2,\quad u=(p-k_2)^2.$$

It is possible also to introduce variables analogous to the variables $\mathbf{R}_\perp$ and x of the parton model in the SIM ($\mathbf{R}_\perp$ is the momentum projection perpendicular to the direction of motion of the SIM, and x is the ratio of the momentum of one of the particles to the total momentum of the system): $\psi=\psi(\mathbf{R}_\perp^2,x)$. The connection of $\mathbf{R}_\perp$ and x with q and n is the following[5]:

$$\mathbf{R}_\perp^2=\mathbf{q}^2-(\mathbf{qn})^2,\quad x={}^1/_2(1-\mathbf{nq}/\varepsilon(\mathbf{q})).$$

In these variables, the nonrelativistic wave function depends on the following combination of $\mathbf{R}_\perp$ and x:

$$\mathbf{q}^2=(\mathbf{R}_\perp^2+m^2)/4x(1-x)-m^2.$$

Thus, the relativistic wave function depends on an additional argument, which takes the form of a unit vector n. The dependence of the wave function on n is determined by the dynamics and describes the properties of a relativistic system, just as the dependence on the relative momentum q.

Consideration of the simplest dynamic models shows that the characteristic parameter that determines the dependence of the nuclear wave functions on the variable nq is the nucleon mass. An investigation of this dependence is a completely new aspect of the problem of nuclear wave functions and is of considerable interest.

The ISR diagram technique appears when the Schrödinger equation is solved for a state vector expressed in "oblique" time—along the direction of ω. The ISR diagrams can be obtained from the Feynman diagrams by "time ordering" of the vertices in all possible manners. Assuming that the "time axis" is directed in the figure from left to right then, if the vertices are numbered from right to left, they must be connected by a directed spurion line in increasing order of the numbers (a smaller instant of time corresponds to a larger number). The arrows on the particle lines are directed from left to right. In the vertices, just as in the wave functions, there are conservation laws for the 4-momenta, including also the momenta of the spurion. The spurion with momentum $\omega\tau$ corresponds to a propagator $1/2\pi(\tau_i-i0)$, and an internal particle line with momentum k_i corresponds to a propagator $\theta(\omega k_i)\delta(k_i^2-m_i^2)$ (for spinless particles). Diagrams with production of particles from vacuum, just as in the old perturbation theory in the SIM, make no contribution. The rules of the SIM diagram techniques are detailed in Refs. 4 and 3.

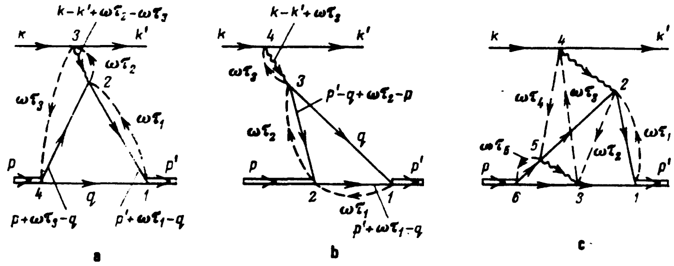

FIG. 2.

The vertex part Γ, which has the same diagram representation as the wave function (Fig. 1), is connected with the wave function by the formula[3]

$$\psi(k_1,k_2,p,\omega\tau)=\frac{1}{2\pi(s-M^2)}\Gamma(k_1,k_2,p,\omega\tau).\tag{6}$$

We illustrate the ISR diagram technique using elastic ed scattering as an example. Some of the diagrams that contribute to the ed-scattering amplitudes are shown in Fig. 2. The expression for the amplitudes of diagram 2a is

$$M=\int\Gamma_1\Gamma_4\theta(\omega(p-q))\delta((p+\omega\tau_3-q)^2-m^2)$$
$$\times\theta(\omega(p'-q))\delta((p'+\omega\tau_1-q)^2-m^2)\theta(\omega q)\delta(q^2-m^2)\theta(\omega(k-k'))$$
$$\times\delta((k-k'+\omega\tau_2-\omega\tau_3)^2)d^4q\frac{d\tau_1}{2\pi(\tau_1-i0)}\frac{d\tau_2}{2\pi(\tau_2-i0)}\frac{d\tau_3}{2\pi(\tau_3-i0)}.\tag{7}$$

Integrating with respect to $d\tau_i(i=1,2,3)$, and with respect to dq_0, and connecting with the wave function in accordance with formula (6), we obtain

$$M=\frac{1}{2\pi}\int\frac{\psi_1\theta(1-\omega q/\omega p)}{1-\omega q/\omega p}\frac{\psi_4\theta(1-\omega q/\omega p')}{1-\omega q/\omega p'}$$
$$\times\frac{\theta(\omega(p'-p))}{\omega(p'-p)[m^2-(p-q)^2]/\omega(p-q)-(p-p')^2-i0}\frac{d^3q}{2\varepsilon_q},\tag{8}$$

where the wave functions ψ_1 and ψ_2 depend on the 4-momenta in the vertices 1 and 4.

It is seen from (8) that the amplitude M is a function not only of the invariant $t=(p-p')^2$, but also of the scalar products ωp and $\omega p'$, with M dependent only on the ratio $y=\omega p'/\omega p$ of these scalar products. This becomes obvious if it is noted that expression (8) does not change when ω is multiplied by a number.

Diagrams 2b and 2c are not expressed in terms of the wave function of the deuteron. The reason for the appearance of these diagrams can be explained in the following manner. As already mentioned, a complete description of a system of a deuteron interacting with an electron is given by the state vectors $|\text{in}\rangle$ and $\langle\text{out}|$ of the continuous spectrum, and the S matrix is determined by their scalar product: $S=\langle\text{out}|\text{in}\rangle$. The diagrams identify in fact the virtual particles that are contained in the $|\text{in}\rangle$ and $\langle\text{out}|$ states. In the nonrelativistic theory, in the intermediate states, besides the incident particle, there are always present only those particles which are contained in the deuteron state vector. The latter appear as a result of the virtual decay of the deuteron, therefore in the nonrelativistic theory the amplitudes are always expressed in terms of the wave functions. In relativistic theory, since additional particles can be produced, the situation changes. Thus, in diagram 2b in the intermediate state between the "instants of time" 3 and 2 there is present an *NN* pair and there is no vertex of the decay of the initial deuteron into nucleons. In diagram 2c there is likewise no intermediate state that contains only nucleons from the initial deuteron and the initial electron (in the 3—2 state there is present also a gamma quantum), and the block with the vertices 3, 5, and 6 is likewise not expressed in terms of wave functions. It is the presence of such Fock components in the state vectors $|in\rangle$ and $\langle out|$ which leads to the appearance of diagrams which are not expressed in terms of wave functions.

The method of suppressing diagrams that are not expressed in terms of wave functions consists in the following. As already explained, the wave functions depend on the hypersurface on which they are defined. Also dependent on this hypersurface are the values of the components of the state vector $|in\rangle$ (and $\langle out|$) which are "undesirable" for us, and their contributions to the amplitudes of the processes. We investigate below the dependence of these amplitudes on the hypersurface (i.e., on the 4-vector ω), and show how to find values of ω such that the contribution of diagrams that are not expressed in terms of wave functions is minimal. In the example considered above, of *ed* scattering, it becomes possible to obtain a value of ω such that the contribution of the diagrams which are not expressed in terms of wave functions vanishes completely. After ω is fixed by the indicated condition, the amplitude of the *ed* scattering (formula (8)) becomes completely unambiguous.

3. CONNECTION WITH OLD PERTURBATION THEORY IN THE SIM

As already mentioned, the non-single-time dynamic scheme in the ISR retains the favorable features of the Feynman diagram technique (the possibility of carrying out explicitly relativistically invariant calculations in any reference frame) and of the old perturbation theory (separation of the particles from the antiparticles, and consequently the possibility of making meaningful the concept of a composite system and of describing it with the aid of wave functions). To explain in greater detail the non-single-time diagram technique, we shall show in explicit form that the ISR calculation formalism is a relativistically invariant form of the old perturbation theory in the SIM.

We consider the arbitrary intermediate state shown in Fig. 3. The expression for the amplitude of the diagram of Fig. 3 is of the form

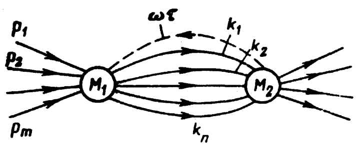

FIG. 3.

$$M=\int M_1M_2\delta^{(4)}\left(\sum_1^m p_j-\sum_1^n k_j+\omega\tau\right)\delta^{(4)}\left(\sum_1^n k_j+\omega\tau-\sum_1^n p_j'\right)$$
$$\times\frac{d\tau}{2\pi(\tau-i0)}\prod_{j=1}^n\theta(\omega k_j)\delta(k_j^2-m_j^2)d^4k_j. \tag{9}$$

Recognizing that the equality $\sum p_j+\omega\tau=\sum k_j$ leads to

$$\tau=\left[\left(\sum k_j\right)^2-\left(\sum p_j\right)^2\right]\Big/2\omega\sum p_j,$$

and integrating with respect to dk_{0j}, we get

$$M=\delta^{(4)}\left(\sum p_j-\sum p_j'\right)$$
$$\times\int\frac{M_1M_2(-2\omega\sum p_j)\delta^{(4)}(\sum p_j-\sum k_j+\omega\tau)d\tau}{2\pi[(\sum p_j)^2-(\sum k_j)^2+i0]}\prod_j\frac{d^3k_j}{2\varepsilon_j}. \tag{10}$$

To rewrite (10) in the form of old perturbation theory in the SIM, we introduce the variables

$$R_j^{ext}=p_j-y_j\sum_i p_i,\quad y_j=\omega p_j\Big/\left(\omega\sum_i p_i\right),$$
$$R_j=k_j-x_j\sum_i k_i,\quad x_j=\omega k_j\Big/\left(\omega\sum_i k_i\right); \tag{11}$$

in which case $\sum y_j=\sum x_j=1$, and the 4-vectors R satisfy the condition $R_j\omega=R_j^{ext}\omega=0$. Introducing the projections $R=(R_0,\mathbf{R}_\perp,\mathbf{R}_\parallel)$, where $\mathbf{R}_\perp\omega=0$, $\mathbf{R}_\parallel\|\omega$, and recognizing that $\mathbf{R}^2=-\mathbf{R}_\perp^2$, we find that in terms of the variables R_j^{ext} and R_j the denominator in (10) takes the form

$$\left[\sum_1^m\frac{(\mathbf{R}_{j\perp}^{ext})^2+m_j^2}{y_j}\right.$$
$$\left.-\sum_1^n\frac{\mathbf{R}_{j\perp}^2+m_j^2}{x_j}+i0\right], \tag{12}$$

which coincides with the form given by Weinberg's rules.[6] The expression $d^3k_j/2\varepsilon_j$ in terms of these variables goes over into $d^2R_jdx_j/2x_j$, and the limits of the integration with respect to dx_j are zero and unity. The expression

$$\left(\omega\sum p_j\right)\delta^{(4)}\left(\sum p_j-\sum k_j+\omega\tau\right)d\tau$$

goes over into

$$\delta^{(2)}\left(\sum\mathbf{R}_{j\perp}^{ext}-\sum\mathbf{R}_{j\perp}\right)\delta\left(\sum x_j-1\right).$$

The expression for any intermediate state can be transformed in exactly the same manner. Thus, we obtain the same expression for the amplitude as in the old perturbation theory in the SIM, wherein the role of the SIM momentum components perpendicular to the direction of motion is assumed by the vectors $\mathbf{R}_\perp$, while the roles of the ratio of the particle momenta to the infinite momentum (divided by the sum of the fractions of the momenta of the initial particles) is assumed by the variables x_j and y_j.

It is seen also from (12) that the ISR amplitudes depend on the 4-vector ω via the ratios of the scalar products of ω with the 4-momenta of the particles (the variables y_j). This is due in final analysis to the invariance of the theory relative to the substitution $\omega\to\omega'=\alpha\omega$. We

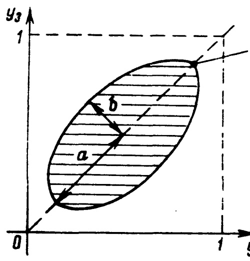

FIG. 4.

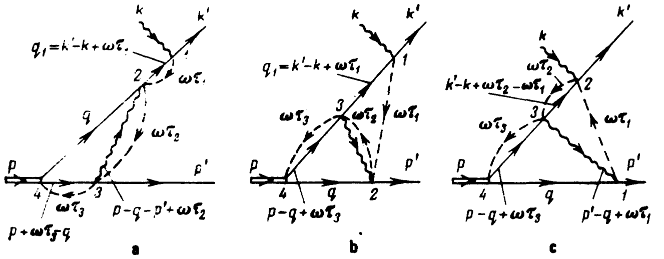

FIG. 5.

note that the number of independent variables y_j is equal to two (there are three independent scalar products of ω with the 4-vectors, and consequently two independent ratios of these scalar products). In the case of the reaction $1+2\to 3+4$, it is convenient to choose the following variables (besides s and t):

$$y_1=\omega p/\omega(p+k),\quad y_2=\omega k/\omega(p+k),$$
$$y_3=\omega p'/\omega(p+k),\quad y_4=\omega k'/\omega(p+k),\tag{13}$$

where p,k and p',k' are the 4-momenta of the initial and final particles ($p^2=p'^2=M^2, k^2=k'^2=\mu^2$). In view of the relations $y_1+y_2=1$ and $y_3+y_4=1$, we choose y_1 and y_2 as the independent variables. Thus, for the two-particle amplitude corresponding to the ISR diagram we obtain $M=M(s,t,y_1,y_3)$. Within the framework of perturbation theory, the sum of the amplitudes of all the ISR diagrams, obtained from a given Feynman diagram by different ordering in time, coincides with the Feynman amplitude, and is independent of y_1 or y_3. The problem of finding a hypersurface on which of the Fock components of the state vectors $|\text{in}\rangle$ and $\langle\text{out}|$ are not expressed in terms of the wave functions of the deuteron will give a minimum contribution to the amplitude reduces now to finding those values of the variables y_1 and y_3 that minimize the relative contributions of the diagrams not expressed in terms of wave functions.

In concluding this section, let us find the physical ranges that the variables connected with ω cover at fixed s and t and at arbitrary variation of ω. We find first the region of variation of the variable $y=\omega p'/\omega p$, on which, in particular, the ed-scattering amplitude depends (see formula (8)). In the system where $\mathbf{p}=0$ we have $y=(\varepsilon(p')-p'\cos\theta)/M(\theta$ is the angle between ω and $\mathbf{p}'$), y varies in the range $y_-\leq y\leq y_+$, where $y_\pm=(\varepsilon(p')\pm p')/M$. In an arbitrary system we get

$$y_\pm=\frac{(pp')\pm[(pp')^2-M^4]^{1/2}}{M^2}\tag{14}$$

The region of variation of y_1 and y_3 (see (13)) is obtained similarly. In the case of elastic scattering, it is an ellipse in the (y_1,y_3) plane with a center having coordinates $(y_1,y_3)=(\varepsilon^*/\sqrt{s},\varepsilon^*/\sqrt{s})$, and with a semiaxis lying on the line $y_1=y_3$ and equal to $a=(2^{1/2}p^*/\sqrt{s})\cos(\theta^*/2)$. The second semiaxis is equal to $b=(2^{1/2}p^*/\sqrt{s})\sin(\theta^*/2)$ (ε^* is the energy of the particle with mass M in the c.m.s., p^* is the momentum of the particles in the c.m.s., and θ^* is the scattering angle in the c.m.s.). This region is shown in Fig. 4.

4. SINGULARITIES OF THE AMPLITUDES

To obtain the invariant variables y_j at which the contribution of diagrams which are not expressed in terms of wave functions is minimal, we need to know the positions of the singularities of the ISR amplitudes. The method of finding the singularities will be explained with the simplest diagrams of Fig. 5 as examples. The internal wavy lines show exchange of a meson with mass μ. The expression for the amplitude of diagram 5a, integrated with respect to $b\tau_j$ and with respect to dq_0, is of the form

$$M_a=F(t,y),\qquad y=\omega p'/\omega p,$$
$$F(t,y)=c\int\frac{\theta(\omega(p-q))}{m^2-(p-q)^2-i0}\,\frac{\theta(\omega(p-p'-q))}{\mu^2-(p-p'-q)^2-i0}\,\frac{d^3q}{2\varepsilon_q}.\tag{15}$$

The problem of finding the singularities of the function $F(t,y)$ differs from the case of the Feynman diagram in that the integration momentum q lies on the mass shell, and the region of integration is limited by the θ function $\theta(\omega(p-p'-q))$ (at $\omega(p-p'-q)>0$ we have $\theta(\omega(p-q))=1$ and the latter imposes no restrictions). The last restriction leads to the appearance of singularities corresponding to the approach of the singularity of the integrand to the limit of the region of variation of q.

Were it not for the indicated restrictions, then, in accordance with the usual method of finding the singularities of Feynman amplitudes (see the paper by Landau[7]), these singularities would be determined from the extremum condition of the function $\varphi_1=\alpha_1(m^2-(p-q)^2)+\alpha_2(\mu^2-(p-p'-q)^2$. The restrictions lead to the problem of a conditioned extremum and can be accounted for by introducing in φ_1 corresponding terms with Lagrange multipliers. We arrive thus to the problem of finding the extremum of the function

$$\varphi=\alpha_1(m^2-(p-q)^2)+\alpha_2(\mu^2-(p-p'-q)^2)+\gamma_1(m^2-q^2)+\gamma_2\omega(p-p'-q)\tag{16}$$

with respect to the variables α_1, α_2, γ_1, γ_2 and q, where γ_1 and γ_2 are Lagrange multipliers. It is also necessary to find the extrema of the functions obtained from φ by successively equating to zero all the coefficients except γ_1.

The case $\gamma_2=0$ returns us to the expression for φ which is considered when the singularities of the Feynman triangular diagram are determined. These singularities are located at the points

$$t=(m+\mu)^2,\tag{17}$$
$$t=m^2+2\mu^2.\tag{18}$$

The position of the singularity (18) was obtained in the approximation $|\varepsilon|=|M-2m|\ll m$ (M is the mass of the particle with momentum p).

In the case $\gamma_2\neq 0$, differentiating φ with respect to q,

α_1, γ_1, and γ_2 we obtain the equation

$$2\alpha_1(p-q)-2\gamma_1 q-\gamma_2\omega=0 \tag{19}$$

under the conditions

$$(p-q)^2=m^2,\quad q^2=m^2,\quad \omega(p-p'-q)=0.$$

Multiplying (19) in succession by q, p, and ω and equating to zero the determinant of the obtained system of equations, we arrive at a quadratic equation in $y=\omega p'/\omega p$, which gives the position of the singularities with respect to y at the points

$$y=\frac{1}{2}\pm\frac{i}{2}\left(\frac{|\varepsilon|}{m}\right)^{1/2}. \tag{20}$$

Thus, besides the singularities of the Feynman diagram, the amplitude of the diagram 5a has singularities in the variable y.

We can obtain analogously the singularities of the diagram 5b and 5c. The amplitude of the diagram 5b has a singularity in y, defined by formula (20), and singularities in t, which depend on the variable y and are located at the points

$$t=m^2+\frac{1-y}{y}\mu(\mu+2m), \tag{21}$$

$$t=m^2+\frac{1-y}{y}2\mu^2. \tag{22}$$

The amplitude of the diagram 5c has singularities determined by formulas (18), (20), (21), and (22). We note that the sum of the diagrams 5a, 5b, and 5c leads to the Feynman amplitude. Therefore the singularities in y and the singularities in t, which depend on y, cannot be encountered in only one amplitude, since they should cancel out in the sum.

5. DETERMINATION OF THE VALUES OF ADDITIONAL VARIABLES IN THE AMPLITUDE

The question of finding the values of the additional variables in the scattering amplitudes will be considered by using as an example double-scattering diagrams (see Fig. 6). The amplitude of diagram 6a is given by

$$M=\frac{1}{2\pi}\int\psi_4^*\frac{\theta(\omega(p-q_1))}{1-\omega q_1/\omega p}\,\frac{\theta(\omega(p+k-q_1-q_2))}{\mu^2-(p+k-q_1-q_2)^2-i0}\times M_2M_3\frac{\theta(\omega(p'-q_2))}{1-\omega q_2/\omega p'}\psi_1\frac{d^3q_1}{2\varepsilon_1}\,\frac{d^3q_2}{2\varepsilon_2}, \tag{23}$$

where the wave functions ψ_1 and ψ_4 depend on the momenta at the vertices 1 and 4.

Changing over to the variables

$$R_1=q_1-\frac{\omega q_1}{\omega p}p,\quad R_2=q_2-\frac{\omega q_2}{\omega p'}p',$$

we obtain

$$M=\frac{1}{2\pi}\int\frac{\psi^*(\mathbf{R}_{1\perp}+x_1\mathbf{Q}_{1\perp},x_1)\,\psi(\mathbf{R}_{2\perp}+x_2\mathbf{Q}_{2\perp},x_2)M_2M_3}{\mu^2+(\mathbf{R}_{1\perp}+\mathbf{R}_{2\perp})^2+(1-x_1y_1-x_2y_3)\zeta-i0}\times\theta(1-x_1)\theta(1-x_2)\theta(1-x_1y_1-x_2y_3)\frac{d^2R_{1\perp}dx_1}{2x_1(1-x_1)}\,\frac{d^2R_{2\perp}dx_2}{2x_2(1-x_2)}, \tag{24}$$

where

$$\zeta=\frac{m^2+\mathbf{R}_{1\perp}^2}{x_1y_1}+\frac{m^2+\mathbf{R}_{2\perp}^2}{x_2y_3}-s,$$

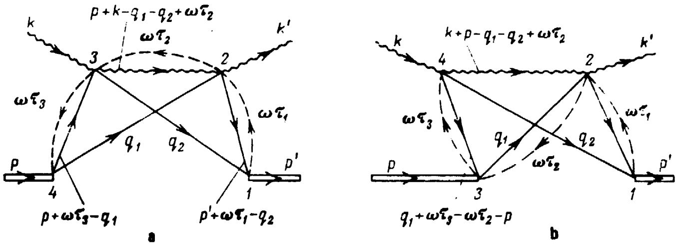

FIG. 6.

$\mathbf{Q}_{1\perp}$ and $\mathbf{Q}_{2\perp}$ are projections, orthogonal to $\boldsymbol{\omega}$, of the spatial parts of the 4-vectors $Q_1=p-y_1(p+k)$ and $Q_2=p'-y_3(p+k)$, while y_1 and y_3 are defined in (13).

The amplitude of the diagram 6b (and the amplitudes of other similar diagrams) cannot be expressed in terms of wave functions, since the vertex 3 is not connected here with wave functions.

We choose the variables y_1 and y_3 such as to suppress the contribution of the diagrams of type 6b. We note for this purpose that the amplitude of the diagram 6b contains the product of the following θ functions:

$$\theta(\omega(q_1-p))\theta(\omega(k+p-q_1-q_2))\cdot\theta(\omega q_1)\theta(\omega q_2),$$

corresponding to lines 43, 42, 32, and 41. Changing over to the variables $z_{1,2}=\omega q_{1,2}/\omega(p+k)$ and y_1, we obtain

$$\theta(z_1-y_1)\theta(1-z_1-z_2)\theta(z_1)\theta(z_2).$$

Therefore the amplitude of diagram 6b can be represented in the form

$$M_b=\int_{y_1}^{1}F_b(s,t,z_1,y_1)\,dz_1. \tag{25}$$

Allowance for the diagram obtained from Fig. 6a by changing the order of the vertices 1 and 2 leads to an integral with limits from y_3 to unity. If the function F_b in (25) does not tend to infinity as $y_1\to1$ and $z_1\to1$, then the amplitude M_b decreases as $y_1\to1$. However, as shown by the examples, F_b can have singularities in z_1, which tend to unity as $y_1\to1$. These singularities turn out to be weak and do not prevent the amplitude M_b from decreasing as $y_1\to1$. To determine the character of the singularities of the function F_b, we use the Landau formula[7]:

$$F_b\sim\varphi^{m/2-n},$$

where φ is the value, near the singularity, of the diameter of the function in integral that defines F_b, m is the number of integrations, and n is the degree of φ in the denominator. In diagram 6b there are three intermediate states for M_b, therefore after changing over to the Feynman parametrization we obtain $n=3$. There are two contours of integration with respect to $d^3q_{1,2}$, and integration with respect to the parameters α_i ($i=1,2,3$). Taking into account the condition $\sum\alpha_i=1$ and the fact that F_b contains one less integration than M_b, we get $m=7$, thus obtaining a square-root singularity. Allowance for the restrictions connected with the θ functions, and consideration of examples of more complicated diagrams, does not lead to pole singularities or stronger singularities.

Thus, to suppress diagrams of the type 6b it is necessary to choose the variables y_1 and y_3 in formula (24) to be closest to unity. The point closest to unity in the physical region of the variables y_1 and y_3 (see Fig. 4) is the point $y_1 = y_3 = y_{max}$, where

$$y_{max} = [\varepsilon(p^*) + p^* \cos(\theta^*/2)]/\sqrt{s}, \qquad (26)$$

while $\varepsilon(p^*)$ and p^* are the energy and momentum of the deuteron in the c.m.s., and θ^* is the scattering angle. The condition $|y_{1,3} - 1| \ll 1$, at which the suppression of the amplitude takes place, is satisfied at high energy and at a small scattering angle.

The physical cause of the suppression of the diagram 6b is the decrease of the phase space in which the intermediate particles can be located. A "reaction" allowed by the energy and momentum conservation law (including the spurion momentum) takes place in each vertex of diagram 6b. Since the direction of the spurion momentum remains unchanged, this imposes limitations on the particle momenta at which such reactions are kinematically allowed. The condition $|y_{1,3} - 1| \ll 1$ denotes the choice of a direction of ω such that the allowed region of the particle momenta is minimal. Bearing in mind that intermediate states correspond to denominators contained in the variables $\mathbf{R}_\perp$ and x of the expression $\sum_i (\mathbf{R}_{\perp i}^2 + m^2)/x$, we can state that under the conditions $|y_{1,3} - 1| \ll 1$ the intermediate states of the diagrams expressed in terms of the wave functions are farthest from the physical region, and the intermediate states of diagrams which are expressed in terms of the wave functions are closest to the physical region.

It seems at first glance that the uncertainties connected with the diagrams of the type 6b will vanish completely if we put $y_1 = y_3 = 1$. The amplitude M_b (see (25)) at $y_1 = y_3 = 1$ does indeed vanish, but when y_1 and y_3 tend to the value 1, which is beyond the limits of the physical region of Fig. 4, the singularities of the integrand in (24) cross the real axis in the course of their motion, hook the integration contours, and move the latter to the complex region. The extrapolation of the functions under the integral sign in (24) to the complex region is quite ambiguous.

The prescription for finding the values of the variables y_i can be easily formulated for an arbitrary diagram if it is noted that the only reason why diagram 6b is suppressed when y_1 is close to unity ($y_2 = 1 - y_1 \ll 1$) is that this diagram contains in the vertex 4 only outgoing internal lines. Since the sum of the variables $z_i (z_i > 0)$ corresponding to these lines is equal to y_2, it follows that as $y_2 \to 0$ the region of integration with respect to the variables z_i tends to zero. We consider an arbitrary diagram with an external line that enters into a vertex that contains only outgoing internal lines, or with an external line that emerges from a vertex that contains only incoming internal lines. It is precisely diagrams with such vertices which are expressed in terms of wave functions. To enhance such a diagram it is necessary to choose the variables y_i corresponding to these external lines to be maximal in the physical region. On the contrary, to suppress diagrams with vertices of this type (for example vertex 1 in diagram 5c), the corresponding variables y_i must be chosen to be minimal.

We note that these results can apparently be obtained also by direct calculation of the asymptotic form of the considered amplitudes in a region such that the minimal or maximal values of the variables y_i tend to zero or unity.

By way of illustration, we turn to the expression for the ed-scattering amplitude (formula (8)). The external line with respect to the triangular diagram of Fig. 2. is the photon line. To suppress diagrams of the type 2b, the variable $y_{1a} = \omega q/\omega(p+k)$ ($q = p' - p + \omega\tau_3$ is the 4-momentum of the photon) must be chosen equal to zero, and this leads to the condition $\omega(p - p') = 0$ or $y = \omega p'/\omega p = 1$. Since $t = (p - p')^2 < 0$, the condition $y = 1$ is satisfied in the physical region of the variable y, and the diagram 2b makes no contribution. It can be shown that at $y = 1$ the diagram of Fig. 2c also vanishes. Expression (8) for the only diagram of Fig. 2a which does not vanish at $y = 1$ takes the form

$$M = -F(t)/t,$$

where the form factor of the scalar deuteron $F(t)$, after changing over to the variables $\mathbf{R}_\perp$ and x, takes the form

$$F(t) = \frac{1}{2\pi}\int \psi(\mathbf{R}_\perp^2, x)\,\psi((\mathbf{R}_\perp + x\mathbf{Q}_\perp)^2, x)\frac{d^2R_\perp\,dx}{2x(1-x)^2}. \qquad (27)$$

This expression agrees with the formula obtained for the form factor from Weinberg's rules and from the SIM moving in a direction orthogonal to the spatial part of the momentum transfer (see, e.g., Ref. 8). We note that if other components become significant in the Fock column (for example $\Delta\Delta$ in the deuteron), then they make an additional contribution to the form factor, and this contribution is not eliminated by a suitable choice of ω.

6. CONCLUSION

The prescriptions formulated above make it possible to express uniquely the amplitudes of the processes in terms of the wave functions. This was accomplished because the diagrams with production of particles from vacuum vanished after going over to the light front, and the diagrams not expressed in terms of wave functions became minimal by suitably locating the surface of the light wave front. Within the framework of the old perturbation theory in the SIM, such diagrams are minimal when the infinite momentum is directed along the vector ω. The direction of ω relative to the particle momenta can be easily determined from the obtained values of the variables y_i.

Thus, the formalism developed here, when account is taken of the spin, makes possible a consistent approach to a theoretical description of relativistic nuclear reactions with large momentum transfers, and to an investigation of the high-momentum components of the nuclear wave functions.

One of the primary problems at the present time is the identification of the mechanisms of the reactions with large momentum transfers. This calls for an in-

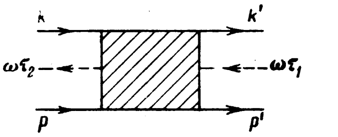

FIG. 7.

dication of the distinguishing "attributes" of the particular mechanism, by which this mechanism can be reliably identified, and this would make it possible in turn to extract from the experimental data information on the relativistic wave functions and, in particular, on the character of their dependence on the variable n. The present status of the problem of determining the mechanisms of a number of reactions with large momentum transfers was discussed in an earlier review.[9] It is also of importance to investigate the general features of the dependence of the wave functions on their arguments in the relativistic region, and their asymptotic behavior.

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APPENDIX: PARAMETRIZATION OF AMPLITUDE OFF THE ENERGY SHELL

We consider the parametrization of the two-particle amplitudes shown in Fig. 7 off the energy shell. Let initially $\tau_2 = 0$. The the amplitude, just as any five-point diagram, depends on five invariant variables. Addition of an external spurion line with 4-momentum $\omega\tau_2$ leads to the appearance of only one additional variable, since the 4-momenta of the spurions are "parallel." Thus, the amplitude of the reaction $1+2 \to 3+4$ off the energy shell depends, besides on $s=(k+p)^2$ and $t=(k-k')^2$ on four additional variables, which we choose in the following manner:

$$s_1=(p+\omega\tau_2)^2,\quad s_2=(p'+\omega\tau_1)^2,\quad s_3=(p'+\omega\tau_2)^2,\quad s_4=(k+p-\omega\tau_2)^2=(k'+p'-\omega\tau_1)^2. \tag{28}$$

An arbitrary n-point diagram with two external spurion lines also depends on four additional variables. We note that in this formalism, there exists in principle amplitudes with arbitrary numbers of spurion lines, and the addition of each spurion line, starting with the second, leads to the appearance of one additional variable.

We consider now the parametrization of the form factor of a particle off the energy shell. This form factor enters, for example, in the amplitude of the diagram of Fig. 2. The amplitude of the scattering of an electron by a particle off the energy shell is shown graphically in Fig. 8a. The expression for the amplitude (integrated

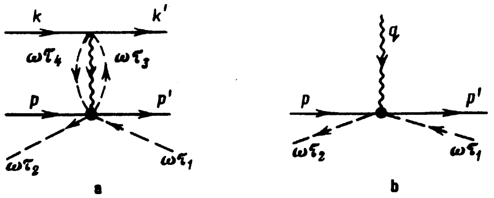

FIG. 8.

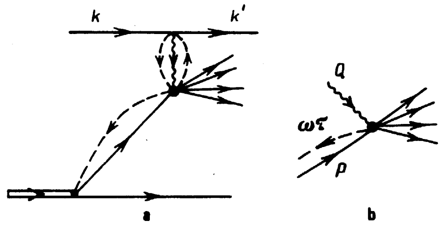

FIG. 9.

with respect to $d\tau_3$) is of the form

$$M=\frac{e^2}{(2\pi)^{5/2}}\int\frac{d\tau_4}{(2\omega q\tau_4-q^2-i0)\,(\tau_4-i0)}F_1(\tau_4), \tag{29}$$

where $q=k-k'$. Since the 4-momenta k and k' enter in the amplitude of the one-photon exchange (29) in the form of the difference $k-k'$, the amplitude depends also on $t=(k-k')^2$ and on the variables s_1, s_2, and s_3, but does not depend on s or s_4. Let us examine this amplitude at $\omega q=0$, the condition under which the form factor of the deuteron was calculated (see Sec. 5). At $\omega q=0$ the amplitude M takes the form

$$M=\frac{e^2}{(2\pi)^{1/2}}\frac{1}{-t}F, \tag{30}$$

where

$$F=\int\frac{d\tau_4}{(2\pi)^2(\tau_4-i0)}F_1(\tau_4)\,d\tau_4.$$

The condition $\omega q=0$ is equivalent to the condition $s_1=s_3$, and consequently the form factor depends, besides on t, on the two variables s_1 and s_2 (see formulas (28): $F=F(t,s_1,s_2)$ and can be shown on Fig. 8b with a virtual γ photon.

As the last example we consider the structure function of deep-inelastic eN scattering off the energy shell. This function is connected with the square of the $eN\to eX$ amplitude off the energy shell, summed over the final states of X. It appears in calculations of deep-inelastic ed scattering in the impulse approximation (see Fig. 9a). Recognizing that in the calculation of the diagram na it is also necessary to impose the condition $\omega Q=0$ (by virtue of which, in analogy with the form factor, we have represented the amplitude on Fig. 9b with a virtual photon with 4-momentum Q, not "entangled" by a spurion line), we see that this structure function depends, besides on the arguments Q^2 and pQ (which enter in the structure function on the energy shell in the scaling region in the combination $x=-Q^2/2pQ$), also on the variable $s=(p+\omega\tau)^2$ (since there are no other independent scalar products). Therefore, generally speaking, it does not coincide with the structure function measured in experiments on nucleons. These conclusions remain in force also for calculations within the framework of the old perturbation theory in the SIM. The character of the dependence of the form factors and of the amplitudes off the energy shell on the additional variables is determined by the dynamics.

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The self-screening of classical Yang-Mills fields

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It is proved that any static system of Yang-Mills fields produced by charges without currents has Coulomb solutions. However, as a consequence of the lack of a uniqueness theorem, for given charges and asymptotic behavior at infinity there exists a multiplicity of solutions containing a "magnetic" field. Fields produced by an infinite uniformly charged plane are considered. Solutions containing a "magnetic" field and decaying at a distance of the order $\ell_0 = (gh/\sigma c)^{-1/2}$ have minimal energy.

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1. The intrinsic nonlinearity of Yang-Mills (YM) fields endows them with remarkable properties, leading to the hope that there is a possibility that they exist in reality. However, a serious difficulty is the macroscopic inobservability of the Yang-Mills fields if the theory contains long-range Coulomb solutions. In the present paper we shall show that owing to the nonlinearity of the YM equations, for the same charge distribution, in addition to the Coulomb fields there may exist a series of solutions containing a "magnetic" field (there is no uniqueness theorem). In the example we consider (a charged plane), the solution of minimal energy is found among the latter; this allows one to assume that something similar occurs also in more complicated cases: the fields become localized near the charges.

2. Static YM fields have been investigated by Khriplovich,[1] who considered nonlinear properties of the static fields produced by charges. Since in that paper it was assumed that the cause of the appearance of nonlinear solutions was the isotopic nonparallelism of charges producing the fields, Khriplovich has chosen as his object of investigation a relatively complicated two-particle problem, in which it was difficult to find solutions.

In fact, the nonparallelism of the isopins of the charges has no importance, since the relative orientations of isopin spaces at different points is arbitrary in a YM theory: it is meaningless to speak of parallelism or nonparallelism of sources situated at different points.

In order to illustrate this point we consider a static system of YM fields defined by the matrix equations

$$\operatorname{div} \mathbf{E}+[A_iE_i]=4\pi\rho,$$
$$(\operatorname{rot} \mathbf{H})_i+e_{ijk}[A_jH_k]+[A_0E_i]=0. \quad (1)$$

In the absence of currents the source of the "magnetic" field is the commutator $[A_0E_i]$. We shall assume that both the vector and scalar potentials are expanded in terms of the generators of the gauge group:

$$A_0=\Phi^a(\mathbf{r})I^a, \quad A_i=a_i^a(\mathbf{r})I^a, \quad i=1,2,3.$$

In A_0 we separate the factor $\Phi(\mathbf{r})$—the absolute value of the potential and the unit vector $I(\mathbf{r})$ of the direction in isospin space

$$A_0(\mathbf{r})=\Phi(\mathbf{r})I(\mathbf{r}), \quad I(\mathbf{r})=u^{-1}(\mathbf{r})I_0u(\mathbf{r}),$$
$$\langle I_0^2\rangle=1.$$

Then

$$\partial_iI=[IB_i],$$

where

$$B_i=u^{-1}\partial_iu.$$

The corresponding field strength E_i has the form

$$E_i=-\partial_iA_0-[A_iA_0]=-I\partial_i\Phi-\Phi[(A_i-B_i)I].$$

If the vector potential is chosen as "longitudinal":

$$A_i=B_i=u^{-1}\partial_iu \quad (H=0), \quad (2)$$

we obtain

$$E_i=-(\partial_i\Phi)I$$

and the source of the "magnetic" field in the second equation of (1) is absent: the equation becomes an identity. The first of the equations (1) takes the form

$$\Delta\Phi I=-4\pi\rho.$$

At each point the quantities A_0 and ρ are isotopically

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