selective verification of the quantization conditions or by varying the manner in which the interaction is turned on (if there is no violation of adiabatic invariance, the final result must obviously not depend on the way the interaction is turned on). At the present time it is evidently impossible to calculate the spectra of these systems in the quasiclassical approximation in any other way.

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1)Caustics are hypersurfaces in configuration space with separate regions in which a solution of the Hamilton-Jacobi equation has different multiplicities.

2)In this paper we use the atomic system of units $e=m=\hbar=1$.

3)Commensurability of frequencies cannot play any special role in the adiabatic method, since the Hamiltonian is nonstationary and resonance effects occur at a set of points of measure zero along the time axis.

4)An example of the use of this method in the old Bohr theory is the calculation of the Stark effect for the hydrogen atom.[9]

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Radiative recombination and its application in experiments on electron cooling

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A simple closed expression is derived, in the dipole approximation, for the cross section for radiative recombination of an electron into an arbitrary level of a hydrogenlike atom. The effect of a magnetic field on this process is estimated. The possibility of using it in experiments on electron cooling of heavy charged particles is discussed.

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1. In connection with work now in progress on electron cooling[1] there is increasing interest on the process of recombination of electrons with protons. For example, detection of the resulting hydrogen atoms has been used directly in the NAP-M storage ring to bring the proton and electron beams into coincidence and to obtain a rough estimate of the temperature of the latter beam. As is shown in what follows, for values of the parameters corresponding to the experiment of Ref. 1 recombination is due to radiative transitions. For a formulation of the problem and a survey of the literature on this question, see Ref. 2. In the present paper an expression in closed analytic form for the cross section of radiative recombination to the level n of a hydrogenlike atom is obtained for the first time, by successfully performing the sum over all quantum numbers. An analysis of the experimental situation with electron cooling is also made, and in particular the influence of a magnetic field on the recombination rate is estimated.

Recombination can occur both through the involvement of three particles (ternary recombination) and owing to the emission of a photon (radiative recombination). The total number $\dot{N}$ of recombinations per unit time is given by

$$\dot{N}=\beta n_e N_p, \tag{1.1}$$

where n_e is the density of electrons, N_p is the total number of protons, and β is the recombination coefficient.

In ternary recombination there is a transfer of energy of the order of the mean kinetic energy from one electron to another. The characteristic length for this process is $\rho\sim e^2/T$ (the temperature T of the electron beam is in energy units), and its probability per unit time $\sim v_T\rho^2 n_e$ (v_T is the velocity corresponding to the temperature T). For recombination to occur, the electron that loses energy must be at a distance $\sim\rho$ from a proton. The number of such electrons per proton is

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$\sim\rho^3 n_e$. From this and the definition (1.1) we get the following estimate for the ternary recombination coefficient β_3:

$$\beta_3 \sim v_T \rho^5 n_e \approx 4\cdot 10^{-27} T^{-9/2} n_e \, \mathrm{cm}^3/\mathrm{sec} \tag{1.2}$$

here the temperature T is expressed in electron volts. As will be clear from what follows, a suitable estimate for the radiative recombination coefficient β_R, with $\alpha c/v_T > 1$ ($\alpha = 1/137$ is the fine-structure constant), is

$$\beta_R > 10^{-13} T^{-1/2} \, \mathrm{cm}^3/\mathrm{sec} \tag{1.2a}$$

In the experiment of Ref. 1 the parameters were $n_e = 2.8 \times 10^8$ cm^{-3} and $T > 1/6$. Under these conditions the ratio of the coefficients is $\beta_3/\beta_R < 10^{-2}$, i.e., negligibly small, and only the radiative recombination need be considered. It must be kept in mind that the contribution of ternary recombination increases with increasing electron density (at a given temperature), and in a sufficiently dense plasma this can be the governing process.

2. Let us calculate the cross section for radiative recombination. We note that for $Z\alpha \ll 1$ (Z is the charge of the nucleus) the dipole approximation can be used We shall confine ourselve to this case. It is convenient to make the calculation in parabolic coordinates (χ, η, φ), in which the radius vector $\mathbf{r}$ has the components $R = [(\chi\eta)^{1/2}\cos\varphi,\ (\chi\eta)^{1/2}\sin\varphi,\ \frac{1}{2}(\chi-\eta)]$. The wave functions of the discrete spectrum corresponding to the level n have the form (in Coulomb units)

$$\psi_D = \frac{2^{1/2}}{n^2} f_{n_1,m}\left(\frac{\chi}{n}\right) f_{n_2,m}\left(\frac{\eta}{n}\right) \frac{e^{im\varphi}}{(2\pi)^{1/2}}. \tag{2.1}$$

Here n_1, n_2 are parabolic quantum numbers, m is the magnetic quantum number, and n is the principal quantum number, $n = n_1 + n_2 + |m| + 1$, and the function $f_{l,m}(x)$ is

$$f_{l,m}(x) = \frac{1}{|m|!}\left(\frac{(l+|m|!)}{l!}\right)^{1/2} \Phi(-l, |m|+1, x)\, e^{-x/2} x^{|m|/2}, \tag{2.2}$$

where Φ is the confluent hypergeometric function. The continuous-spectrum wave function of a particle propagated along the third axis is

$$\psi_c = e^{\pi\xi/2}\,\Gamma(1+i\xi)\, e^{i(\chi-\eta)/2\xi}\, \Phi\left(-i\xi, 1, -\frac{i\chi}{\xi}\right). \tag{2.3}$$

Here $\xi = Z\alpha c/v$, where v is the speed of the particle. In the dipole approximation the matrix element M is expressed in terms of an integral $\mathbf{M}$: $m \propto \mathbf{e}^* \cdot \mathbf{M}$, where $\mathbf{e}$ is the polarization vector of the photon and $\mathbf{M}$ is given by

$$\mathbf{M} = \frac{e^{\pi\xi/2}\,\Gamma(1+i\xi)}{4n^2\pi^{1/2}} \int d\chi\, d\eta\, d\varphi\, (\chi+\eta)\, \mathbf{r}\, e^{i(\chi-\eta)/2\xi}$$
$$\times \Phi\left(-i\xi, 1, -\frac{i\chi}{\xi}\right) e^{-im\varphi} f_{n_1,m}\left(\frac{\chi}{n}\right) f_{n_2,m}\left(\frac{\eta}{n}\right). \tag{2.4}$$

This integral can be done by differentiating the expression (see Ref. 3, p. 875; p. 861 in Engl. transl.)

$$\int_0^\infty dt\, e^{-st} t^{c-1}\, \Phi(a, c, t)\, \Phi(\alpha, c, \lambda t)$$
$$= \Gamma(c)\,(s-1)^{-a}(s-\lambda)^{-\alpha} s^{a+\alpha-c} F\left(a, \alpha; c; \frac{\lambda}{(s-1)(s-\lambda)}\right), \tag{2.5}$$

with respect to a parameter; here F is the hypergeometric function.

Following the standard procedure, i.e., calculating $|M|^2$ and summing over the possible final states, we find the expression for $\sigma^{(n)}$, the cross section for radiative recombination to the level n:

$$\sigma^{(n)} = \frac{16}{3}\alpha\pi^2 \lambda_c^2 \xi^2 \frac{\exp\{-4\xi \operatorname{arctg}(n/\xi)\}}{(1-e^{-2\pi\xi})\,|\lambda_n|^4} S_n,$$
$$S_n = \sum_{n_1=0}^{n-1} \left| \left(1 - \frac{\lambda_n}{n} z \frac{d}{dz}\right) F_{n_1}(z) \right|^2 \tag{2.6}$$
$$+ \sum_{n_1=0}^{n-2} \left| \frac{\lambda_n}{n} (F_{n_1+1}(z) - F_{n_1}(z)) \right|^2 (n_1+1)(n-n_1-1),$$

where

$\lambda_c = \hbar/mc$, $\quad z = -in/\xi\lambda_n^{*2}$, $\quad \lambda_n = {}^1/_2(1+in/\xi)$,
$F_l(z) \equiv F(-l, i\xi; 1; z)$.

The first sum in S_n corresponds to the value $m = 0$ of the magnetic quantum number, and the second to $|m| = 1$. Using the recurrence relations for the hypergeometric functions, we can rewrite the expression for S_n in the form

$$S_n = \sum_{n_1=0}^{n-1} \left\{ [(n_1+1)R_{n_1}(z) + n_1 R_{n_1-1}(z)]^2 + \frac{(n_1+1)(n-n_1-1)}{|\lambda_n|^2} R_{n_1}^2(z) \right\}, \tag{2.7}$$

where

$$R_l(z) \equiv (\lambda_n^*/\lambda_n)^l F(-l, 1+i\xi; 2; z).$$

We note that according to the relation

$$F(\alpha, \beta; \gamma; z) = (1-z)^{-\alpha} F(\alpha, \gamma-\beta; \gamma; z/(z-1)) \tag{2.8}$$

the quantity $R_1(z)$ is real. Now, by means of the formula

$$(2\alpha-\gamma-\alpha z+\beta z)F(\alpha, \beta; \gamma; z) + (\gamma-\alpha)F(\alpha-1, \beta; \gamma; z)$$
$$+ \alpha(z-1)F(\alpha+1, \beta; \gamma; z) = 0$$

we can put S_n in the following form:

$$S_n = \sum_{n_1=0}^{n-1} \{n_1^2 R_{n_1-1}^2 - (n_1+1)^2 R_{n_1}^2$$
$$+ n_1(n_1+1) R_{n_1} R_{n_1-1} - (n_1+1)(n_1+2) R_{n_1+1} R_{n_1}\}. \tag{2.9}$$

The summation over n_1 is now trivial, and we finally get

$$S_n = -nR_{n-1}(z)\,[nR_{n-1}(z) + (n+1)R_n(z)]. \tag{2.10}$$

The quantity S_n depends on $x_n \equiv \lambda_n^2$. We list the forms of $S_n(x)$ for a few values of n:

$$S_1 = 1, \quad S_2 = 2 + \frac{3}{x_2} + \frac{1}{x_2^2},$$
$$S_3 = 3 + \frac{14}{x_3} + \frac{19}{x_3^2} + \frac{8}{x_3^3} + \frac{1}{x_3^4}, \tag{2.11}$$
$$S_4 = 4 + \frac{38}{x_4} + \frac{346}{3x_4^2} + \frac{409}{3x_4^3} + \frac{622}{9x_4^4} + \frac{43}{3x_4^5} + \frac{1}{x_4^6}.$$

In the limiting case $\xi \gg 1$, $n \ll \xi$ the quantity S_n can be expressed in terms of Laguerre polynomials:

$$S_n \approx {}^1/_8[L_n^2(4n) - L_{n-1}^2(4n)]. \tag{2.12}$$

For $n \gg \xi$ we have $S_n \approx n$. A particular case of this limit is the Born approximation, for which $\xi \ll 1$.

Using the expression for the generating function (see Ref. 4, p. 93)

$$(1-s)^{a-c}(1-s+sz)^{-a} = \sum_{m=0}^{\infty} \frac{s^m (c)_m}{m!} F(-m, a; c; z), \tag{2.13}$$

we can derive a convenient integral expression for $R_n(z)$:

$$nR_{n-1}(z)=\frac{\exp\{2\xi\,\mathrm{arctg}(n/\xi)\}(1-e^{-2\pi\xi})(-1)^n|\lambda_n|^2}{\pi}$$
$$\times\int_{-\xi/n}^{\xi/n}\frac{dx}{1+x^2}\exp\left\{2in\left(\mathrm{arctg}\,x-\frac{\xi}{n}\,\mathrm{Arth}\frac{nx}{\xi}\right)\right\}, \qquad (2.14)$$

by means of which we can rewrite the expression (2.6) for the cross section for radiative recombination:

$$\sigma^{(n)}=\frac{32\pi\alpha\lambda_c^2\xi^2(1-e^{-2\pi\xi})}{3^{3/2}}U(n,\xi). \qquad (2.15)$$

Here

$$U(n,\xi)=3^{3/2}U_1(n,\xi)U_2(n,\xi)/2\pi,$$
$$U_1(n,\xi)=\int_{-\xi/n}^{\xi/n}\frac{dx}{1+x^2}\exp\left\{2in\,\mathrm{arctg}\,x-2i\xi\,\mathrm{Arth}\frac{nx}{\xi}\right\}, \qquad (2.16)$$
$$U_2(n,\xi)=\int_{-\xi/n}^{\xi/n}dx\left(\frac{2ix}{1+x^2}+\frac{n}{n^2+\xi^2}\right)\exp\left\{2in\,\mathrm{arctg}\,x-2i\xi\,\mathrm{Arth}\frac{nx}{\xi}\right\}.$$

This form is convenient for deriving asymptotic expressions. For example, let us find the asymptotic form of $U(n,\xi)$ and thus also for $\sigma^{(n)}$ at $n\gg 1$ and at a fixed value of ξ. Under these conditions the main contribution to the integrals in Eq. (2.16) comes from small values $x\sim N^{-1/3}$, where $N=n(1+n^2/\xi^2)$; using this fact, we find[1)]

$$U(n,\xi)=\frac{1}{N}[(1-A)(1-B)+AB], \qquad (2.17)$$

where

$$A=\frac{a}{N^{2/3}}\left(1-\frac{n^2}{\xi^2}\right),\quad B=\frac{1}{350aN^{4/3}}\left(3\left(1-\frac{n^2}{\xi^2}\right)^2+10\frac{n^2}{\xi^2}\right)$$
$$a=\Gamma(^1/_3)/5\Gamma(^2/_3)\,(12)^{1/3}\approx 0.1728\ldots.$$

where $\Gamma(x)$ is the gamma function. We note that although Eq. (2.17) has been derived on the assumption $n\gg 1$, a comparison with the exact formula (2.16) shows that in any case for $\xi>1$ the difference is smaller than 1 percent, even for $n=1$, and decreases as n increases.

The total recombination cross section is the sum of the partial cross sections

$$\sigma=\sum_n\sigma^{(n)}.$$

It does not seem possible to perform this sum in analytic form. In the case $\xi\gg 1$ we can obtain an asymptotic formula for σ, starting from the expression (2.17):

$$\sigma\approx\frac{32\pi\alpha\lambda_c^2\xi^2}{3^{3/2}}\left[\ln\xi+b_0+\frac{b_1}{\xi^{2/3}}+\frac{b_2}{\xi^{4/3}}+\frac{1}{\xi^2}(b_3-b_4\ln\xi)\right]. \qquad (2.18)$$

Here

$$b_0=C-a\zeta(^5/_3)-\frac{3}{350a}\zeta(^7/_3)+\frac{3\zeta(3)}{175},$$

where C is Euler's constant, $\zeta(z)$ is the Riemann zeta function, and

$$b_0\approx 0.161,\quad b_1=3a\approx 0.518,\quad b_2=\frac{9}{700a}\approx 0{,}074,$$
$$b_3=\frac{1}{12}-\frac{8C-2}{175}\approx 0.068,\quad b_4=\frac{8}{175}\approx 0.046.$$

Here the values $n\lesssim\xi$ make the main contribution to the sum.

3. Since the experiment on electric cooling of protons is done without a magnetic field, it seems useful to consider the effect of such a field on the radiative recombination process.

Let us compare the forces exerted on an electron which is on the level $n=1$ by a magnetic field, $f_{\mathrm{mag}}\sim eH(Z\alpha)$, and by the Coulomb force center, $f_{\mathrm{Coul}}\sim eH_0(Z\alpha)^3$, where $H_0=m^2c^3e\hbar\approx 4.4\times 10^{13}$ Oe. For fields that satisfy the condition

$$H\ll H_0(Z\alpha)^2, \qquad (3.1)$$

which certainly holds in all actual installations, the effect of the magnetic field on an electron in any state, with $n\sim 1$ is negligibly small compared with the Coulomb force.

As the principal quantum number increases the Coulomb force decreases, but since for $\xi\lesssim 1$ the recombination goes to the lower levels (in this case the contribution of states with $n\gg 1$ falls off as $1/n^3$), if fields satisfying the condition (3.1) can have any effect on the recombination it must be for $\xi\gg 1$. In that case, as we have already noted, values $n\lesssim\xi$ give the main contribution.

Accordingly, it is sufficient to solve this problem for ξ, $n\gg 1$, and the problem can be treated essentially in the framework of classical electrodynamics. In the absence of a magnetic field we can directly use the formula for the spectral density of the radiation from an electron passing by with initial impact parameter ρ (see Ref. 5, Sec. 70); dividing this expression by the energy $\hbar\omega$ of the emitted photon, we have for the probability of emission at the given impact parameter

$$dW_\omega=\frac{2\pi\alpha\xi^2\lambda_c^2\omega\,d\omega}{3v^2}\left\{[H_{i\nu}^{(1)\prime}(i\nu\varepsilon)]^2-\frac{\varepsilon^2-1}{\varepsilon^2}[H_{i\nu}^{(1)}(i\nu\varepsilon)]^2\right\}, \qquad (3.2)$$

where

$$\nu=\xi\frac{\hbar\omega}{mv^2},\quad \varepsilon^2-1=\frac{l^2}{\xi^2},\quad l=\frac{mv\rho}{\hbar}.$$

In our case $\nu\gg 1$ and we can replace the Hankel functions with their asymptotic expressions

$$H_{i\nu}^{(1)}(i\nu\varepsilon)=\frac{1}{i\pi}\int_{-\infty}^{\infty}\exp\{i\nu[\eta-\varepsilon\,\mathrm{sh}\,\eta]\}\,d\eta$$
$$\approx\frac{1}{i\pi}\int_{-\infty}^{\infty}\exp\left\{-i\nu\left[(\varepsilon-1)\eta+\varepsilon\frac{\eta^3}{6}\right]\right\}d\eta \qquad (3.3)$$
$$=\frac{2}{i\pi}\left(\frac{2(\varepsilon-1)}{3\varepsilon}\right)^{1/2}K_{1/3}\left(\frac{\nu[2(\varepsilon-1)]^{3/2}}{(3\varepsilon)^{1/2}}\right),$$
$$H_{i\nu}^{(1)\prime}(i\nu\varepsilon)\approx\frac{1}{\pi}\frac{4}{3^{1/2}}\frac{\varepsilon-1}{\varepsilon}K_{2/3}\left(\frac{\nu[2(\varepsilon-1)]^{3/2}}{(3\varepsilon)^{1/2}}\right).$$

From Eqs. (3.2) and (3.3) it follows that the main contribution to dW_ω comes from nearly parabolic trajectories ($\varepsilon-1\sim\nu^{-2/3}\ll 1$) corresponding to large orbital angular momenta l.

For n, $l\gg 1$ we can use the approximate relations

$$\int d^2\rho\to 2\pi\sum_l\left(\frac{\hbar}{mv}\right)^2 l,\quad d\omega_n\approx\frac{I_n-I_{n-1}}{\hbar}\approx\frac{(Z\alpha)^2mc^2}{\hbar n^3}. \qquad (3.4)$$

We have taken the conservation of energy in the recombination into account: $\hbar\omega_n=\frac{1}{2}mv^2+I_n$, where $I_n=(Z\alpha)^2mc^2/2n^2$ is the ionization energy of the n-th level. Substituting Eqs. (3.3) into Eq. (3.2) and expanding in powers of $(\varepsilon - 1)$, on using (3.4) we get an expression for the partial cross section $\sigma_l^{(n)}$ for transition into a state with definite (large!) values of n and l

$$\sigma_l^{(n)} = \frac{8}{9}\alpha\xi^2\lambda_c^2\left(1+\frac{n^2}{\xi^2}\right)\left(\frac{l}{n}\right)^5$$
$$\times\left[K_{1/3}^2\left(\frac{l^3}{6n^2}\left(1+\frac{n^2}{\xi^2}\right)\right)+K_{2/3}^2\left(\frac{l^3}{6n^2}\left(1+\frac{n^2}{\xi^2}\right)\right)\right]. \tag{3.5}$$

It follows from Eq. (3.5) that the values of l that contribute to the cross section satisfy the inequality $l \ll n^{2/3}$. Summing the $\sigma_l^{(n)}$ given by Eq. (3.5) over l (this operation can be replaced by an integration) we find

$$\sigma^{(n)} = \frac{32\pi\alpha\xi^2\lambda_c^2}{3^{3/2}}\frac{1}{n(1+n^2/\xi^2)}. \tag{3.6}$$

This expression, obtained by Kramers,[6] corresponds to the first term of the expansion (2.17) of the quantity $U(n, \xi)$; if we set $A = B = 0$ in this expression, we get the expression (3.6) for $\sigma^{(n)}$.

We now take account of the effect of a magnetic field H, assuming that the corrections due to H are small. Then the Fourier components of the velocity, in terms of which the probability of emission of radiation is expressed, will be of the form

$$|\mathbf{v}_\omega| = \left|\mathbf{v}_{0\omega}+\frac{i\omega_H}{\omega}[\mathbf{v}_{0\omega}\times\mathbf{h}]\right|, \tag{3.7}$$

where $\mathbf{v}_0$ is the corresponding Fourier component of the velocity for $H = 0$ and $\omega_H = eH/mc$;

$$\frac{\omega_H}{\omega_n} = \frac{2n^3H}{(Z\alpha)^3H_0(1+n^2/\xi^2)},$$

and $\mathbf{h}$ gives the direction of the magnetic field. In the case when the magnetic field is perpendicular to the plane in which the electron moves, we find, proceeding as in the derivation of Eq. (3.6),

$$\sigma_H^{(n)} = \frac{32\pi\alpha\xi^2\lambda_c^2}{3^{3/2}}\frac{1}{n(1+n^2/\xi^2)}\left[1+\left(\frac{\omega_H}{\omega_n}\right)^2\right]. \tag{3.8}$$

We note that the terms linear in H have gone to zero in the integration over the azimuthal angle.

Accordingly, the effect of a magnetic field on the process of radiative recombination becomes important at field strengths

$$H \sim H_0\left(\frac{Z\alpha}{\xi}\right)^3 = H_0\left(\frac{v}{c}\right)^3 \approx 4H_0 10^{-9}\, T\ [\mathrm{eV}].$$

In particular, in the NAP-M accelerator $H \sim 10^3$ Oe, $T \sim (1/6)$ eV, and the effect of the magnetic field on the process can be neglected.

4. The recombination coefficient β_R is expressed in terms of the known cross section σ of the process and the velocity distribution of the electrons (in the rest system of the proton) in the following way:

$$\beta_R = \int \sigma(v)vf(\mathbf{v})d^3v. \tag{4.1}$$

To find β_R in the general case one must carry out a numerical calculation. This is especially simple if for $U(n, \xi)$ we use the approximate expression (2.17), which is very accurate. If also the quantity $\xi_T = Z\alpha c/v_t$ ($v_T/c = 2\times 10^{-3}T^{1/2}$ [eV]) is large, we can use the expression (2.18) for σ. In this case, we can find β_R in analytic form.

For example, for the isotropic velocity distribution

$$f_1(\mathbf{v}) = \frac{1}{(\pi v_T^2)^{3/2}}\exp\left\{-\frac{\mathbf{v}^2}{v_T^2}\right\} \tag{4.2}$$

we find

$$\beta_R^{(1)} = \frac{64}{3}\left(\frac{\pi}{3}\right)^{1/2}Zr_e^2c\xi_T\left[\ln\xi_T+b_0+\frac{1}{2}C+b_1\Gamma\left(\frac{4}{3}\right)\xi_T^{-2/3}\right.$$
$$\left.+b_2\Gamma\left(\frac{5}{3}\right)\xi_T^{-4/3}+\left(b_3+\frac{1}{2}b_4(1-C)-b_4\ln\xi_T\right)\xi_T^{-2}\right]; \tag{4.3}$$

here $r_e = e^2/mc^2$ is the classical electron radius.

For electron cooling in the NAP-M machine the spread of longitudinal (relative to the motion of the protons) velocities is much (two or three orders of magnitude) smaller than that of transverse velocities, so that

$$f_2(\mathbf{v}) = \frac{1}{\pi v_T^2}\exp\left\{-\frac{\mathbf{v}_\perp^2}{v_T^2}\right\}\delta(v_\parallel). \tag{4.4}$$

With Eq. (4.4) we find

$$\beta_R^{(2)} \approx A\left[\ln\xi_T+b_0+\frac{1}{2}C+\ln 2+b_1\Gamma\left(\frac{5}{6}\right)\Big/\xi_T^{2/3}\pi^{1/2}\right.$$
$$\left.+b_2\Gamma\left(\frac{7}{6}\right)\Big/\xi_T^{4/3}\pi^{1/2}+\frac{1}{2\xi_T^2}\left(b_3-b_4\ln\xi_T+b_4\left(1-\ln 2-\frac{1}{2}C\right)\right)\right], \tag{4.5}$$

$$A = 32(1/3\pi)^{1/2}Zr_e^2c\xi_T.$$

In the experimental study of the effect of the relative longitudinal velocities of electrons and protons on the cooling effects a modulation of the longitudinal velocity (energy) of the electron beam was produced. This means that in Eq. (4.4) we make the replacement

$$\delta(v_\parallel) \to \delta(v_\parallel - v_0\cos\psi).$$

The resulting value of the recombination coefficient must be averaged over ψ. We give the result for the case in which the quantity $\gamma = v_0/v_T$ is small ($\gamma \ll 1$):

$$\beta_R^{(3)} = \left(1+\frac{\gamma^2}{2}\right)\beta_R^{(2)}-\frac{4A}{\pi^{1/2}}\gamma\left[\ln\xi_0+b_0+2-\ln 2\right.$$
$$+b_1\pi^{1/2}\Gamma\left(\frac{4}{3}\right)\xi_0^{-2/3}\Big/4\Gamma\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)^2$$
$$\left.+b_2\pi^{1/2}\Gamma\left(\frac{5}{3}\right)\xi_0^{-4/3}\Big/4\Gamma\left(\frac{7}{6}\right)\left(\frac{7}{6}\right)^2+\ldots\right], \quad \xi_0 = \frac{\xi_T}{\gamma} = \xi(v_0). \tag{4.6}$$

The experiments on electron cooling are carried out in the presence of a magnetic field, usually directed along the proton beam. The electrons move along helical lines with the axis along the direction of the field. To investigate (see Ref. 1) the influence of the transverse velocities of the electrons on the cooling effects, one either produced a change of the velocity of revolution of the electrons, or else, by slightly turning the magnetic field, imparted to the electrons a systematic transverse velocity. In both cases one obtains the corresponding distribution function from Eq. (4.4) by the replacement $\mathbf{v}_\perp \to \mathbf{v}_\perp - \mathbf{v}_0$, except that in the first case $\mathbf{v}_0$ is a rotating vector and in the second case it is constant. However, after intergration over the angles of the vector $\mathbf{v}_\perp$ the only remaining dependence is that on $|\mathbf{v}_0| = v_0$, and the recombination coefficients in the two cases are identical[2)]:

$$\beta_R^{(4)}=A\left\{\left[\ln\xi_T+b_0+\frac{1}{2}C+\ln 2+\frac{b_4}{2\xi_T^2}\right]\Phi\left(\frac{1}{2},1,-\gamma^2\right)\right.$$
$$+\frac{b_1\Gamma(5/6)}{\pi^{1/2}\xi_T^{2/3}}\Phi\left(\frac{1}{6},1,-\gamma^2\right)+\frac{b_2\Gamma(7/6)}{\pi^{1/2}\xi_T^{4/3}}\Phi\left(-\frac{1}{6},1,-\gamma^2\right)$$
$$\left.+\frac{1}{2\xi_T^2}\left[b_3-b_4\ln\xi_T-b_4\left(\frac{1}{2}C+\ln 2\right)\right]\Phi\left(-\frac{1}{2},1,-\gamma^2\right)-\varphi(\gamma^2)\right\},$$
$$\varphi(\gamma^2)=e^{-\gamma^2}\sum_{n=1}^{\infty}\left(\frac{\gamma^2}{2}\right)^n\frac{(2n-1)!!}{(n!)^2}\left(\sum_{k=0}^{n-1}\frac{1}{2k+1}\right)\left(1-\frac{b_4}{2\xi_T^2}\right). \tag{4.7}$$

For $\gamma\ll 1$ we find from Eq. (4.7)

$$\beta_R^{(4)}=\beta_R^{(2)}-\frac{A}{2}\gamma^2\left\{\ln\xi_T+b_0+\frac{1}{2}C+\ln 2+1\right.$$
$$+b_1\Gamma\left(\frac{5}{6}\right)\Big/3\pi^{1/2}\xi_T^{2/3}-b_2\Gamma\left(\frac{7}{6}\right)\Big/3\pi^{1/2}\xi_T^{4/3}$$
$$\left.+\frac{1}{2\xi_T^2}\left(b_4\ln\xi_T-b_3+b_4\left(\frac{1}{2}C+\ln 2\right)\right)\right\}. \tag{4.8}$$

For $\gamma\gg 1$ the main contribution to the integral (4.1) comes from values $|\mathbf{v}_\perp|\approx v_0$, and in this case the coefficient β_R differs from $v_0\sigma(v_0)$ by a quantity $\sim 1/\gamma^2$:

$$\beta_R^{(4)}=v_0\sigma(v_0)\left(1+\frac{1}{4\gamma^2}\right)$$
$$+\frac{16\pi}{3^{3/2}}\frac{Zr_e^2c\xi_0}{\gamma^2}\left(1-\frac{4}{9}\frac{b_1}{\xi_0^{2/3}}-\frac{4}{9}\frac{b_2}{\xi_0^{4/3}}+\frac{b_4}{\xi_0^2}\right). \tag{4.9}$$

At present the experimental results do not have the complete clarity necessary for a detailed comparison with the theory. Nevertheless, the results obtained so far show that the process of radiative recombination can be used effectively for monitoring in various situations which occur in experiments on electron cooling. In particular, the effective temperature of the electron beam can be determined with high accuracy. If, as was the case in the experiment of Ref. 1, the size of the proton beam is much smaller than that of the electron beam, then by shifting the proton beam across the section of the electron beam one can in principle study the structure of the latter beam.

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[1)]The first two terms of this expansion were previously known (see Ref. 2, p. 226).

[2)]If the electron beam is radially inhomogeneous, for example, with v_T depending on r, these coefficients will be different, since in the second case the proton beam moves at an angle to the electron beam and the expression (4.7) has to be averaged over r.

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