front.

In laboratory experiments on collisionless shock waves transverse to the magnetic field at large Mach numbers, the appearance of a discontinuity in the potential and in the density of scale $<100 r_D$ is observed; this is much smaller than the thickness of the front—the so-called "isomagnetic discontinuity."

As a possible mechanism of formation of the discontinuity, the density the dispersion of ion-sound waves was discussed in Ref. 4. Their role is reduced to a limitation of the nonlinear steepening of the density over a scale of the order of tens of Debye lengths. The results of the experiments described above enable us to admit the turbulent ion viscosity as an alternative mechanism of formation of the isomagnetic discontinuity, not only to assure the small size of the discontinuity, but only to explain the energy of the ions in the wave-front observed in this dissipation.

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[1]O. L. Volkov, V. G. Eselevich, G. N. Kichigin and V. L. Papernyĭ, Zh. Eksp. Teor. Fiz. **67**, 1689 (1974) [Sov. Phys.-JETP **40**, 841 (1974)].
[2]O. L. Volkov, V. G. Eselevich, G. N. Kichigin and V. L. Papernyĭ, Preprint Siberian Inst. Terrest. Magnetism, Ionosphere and Radiowave Propagation, No. 5-74, 1974.
[3]V. Formisano and C. Hedgcock, J. Geoph. Res. **78**, 3705 (1973).
[4]A. G. Es'kov, V. G. Eselevich, R. Kh. Kurtmullaev and A. I. Malyutin, Zh. Eksp. Teor. Fiz. **60**, 5 (1971) [*sic*!].
[5]K. B. Kartashev, V. I. Pistunovich, V. V. Platonov, V. D. Ryutov and E. A. Filimonova, Fizika plazmy (**1**, 742 (1975) [Sov. J. Plasma Phys. **1**, 406 (1975)].
[6]F. Chen, translation in: Diagnostika plazmy (Plasma Diagnostics) (R. Huddlestone and S. Leonard, eds.) Mir, 1967, p. 107.
[7]Élektrodinamika plazmy (Electrodynamics of plasma) A. I. Akhiezer, ed., Nauka, 1974.
[8]V. S. Koĭdan, A. G. Ponomarenko, A. I. Rogozin, D. D. Ryutov, in: Diagnostika plasmy (Plasma Diagnostics) No. 3, Atomizdat, 1973.
[9]N. V. Astrakhantsev, V. G. Eselevich, G. N. Kichigin, V. L. Papernyĭ, Zh. Tekh. Fiz. **48**, 297 (1978) [Sov. Phys. Tech. Phys. **23**, 178 (1978)].
[10]B. M. Glukhovskoĭ, V. M. Evtikhneva, I. B. Maksyutov and V. L. Papernyĭ, Élektronnaya tekhnika, ser. 5, No. 5, 17 (1976).
[11]A. A. Ivanov, S. I. Krashennikov, T. K. Soboleva, and P. N. Yushmanov, Fiz. Plazmy **1**, 753 (1975) [Sov. J. Plasma Phys. **1**, 412 (1975)].
[12]N. V. Astrakhantsev, O. L. Volkov, V. G. Eselevich, G. N. Kichigin, V. L. Papernyĭ and I. G. Shukhman, Preprint, Siberian Inst. Terrest. Magnetism, Ionosphere and Radiowave Propagation, No. 11-77, 1977.
[13]B. B. Kadomtsev, in: Voprosy teorii plasmy (Problems of Plasma Theory) No. 4, Atomizdat, 1964, p. 216.
[14]D. Biskamp and D. Pfirsch, Phys. Fluids **12**, 732 (1969).
[15]S. G. Alikhanov, A. I. Alinovskiĭ, G. G. Dolgov-Savel'ev, V. G. Eselevich, R. Kh. Kurtmullaev, V. K. Malinovskiĭ, Yu. E. Nesterikhin, V. I. Pil'skiĭ, R. Z. Sagdeev and V. N. Semenov, III Intern. Conf. on Plasma Physics, Novosibirsk, 1968, paper 24/A.

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On the constancy of an adiabatic invariant when the nature of the motion changes

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The motion of a charged particle in a spatially periodic field with amplitude increasing with the time is considered. The change of an adiabatic invariant when the particle is captured by a wave is calculated. The expressions obtained can also be used to describe the motion of a pendulum of variable length in a gravitational field as it goes from rotation to vibration about a position of stable equilibrium.

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1. It is well known that the motion of mechanical systems with slowly changing parameters can be characterized by a quantity which is conserved to very high accuracy, an adiabatic invariant (cf., e.g., Ref. 1). This assertion holds both for finite and for infinite motions of the representative point on the phase surface. In many cases, however, it is necessary to trace the transition from one type of notion to the other. For example, charged particles moving in a spatially periodic field whose amplitude increases with time can be captured by a wave. When this happens the trajectories on the phase surface go from the domain of infinite motions to that of finite motions (see, e.g., Ref. 2). This problem has a simple mechanical analog, the motion of a pendulum of variable length in a gravitational field; here vibrations of the pendulum relative to a position of equilibrium correspond to finite motion, and rotations around the point of support, to infinite motion.

Besides these problems there are a number of others whose solution requires an analysis of the transition from one type of motion to another. In particular, there are certain problems of celestial mechanics (see,

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e.g., Ref. 3), and also the problem of the motion of charged particles in open magnetic traps in the presence of electromagnetic oscillations.[4,5]

One of the first papers that dealt with the constancy of an adiabatic invariant when there is a change of the type of motion was one by Best.[2] In this paper (see also Ref. 6) it was concluded on the basis of Liouville's theorem that in zeroth approximation in the parameter $\varepsilon=(\omega\tau)^{-1}\ll 1$ the adiabatic invariant remains unchanged. Here ω is the characteristic frequency of the oscillations of the charged particle in the field of the wave of electric potential, and τ is the characteristic time for the change of amplitude of the wave. At the same time a numerical integration of the equations of motion, which was carried out in Ref. 2 for one particular case, showed that the adiabatic invariant changes by a small quantity of the order of ε. Numerical calculations of the change of the adiabatic invariant when a particle is captured in the field of a wave of increasing amplitude were made by Aamodt and Jaeger.[7] The results are discussed in the text of the paper.

In the present paper, as in Refs. 2, 6, and 7, the question of the constancy of an adiabatic invariant is discussed for the case of motion of a charged particle in a wave of electric potential with variable amplitude. To and including quantities of order ε the particle projectories on the phase surface were found near the separatrix that divides the regions of finite and infinite morions. Knowledge of the trajectories made it possible to obtain an analytic expression for the change of the adiabatic invariant when a trajectory passed across the separatrix.

2. Let us consider the motion of a charged particle in a harmonic wave of electric potential whose amplitude varies with the time. In a coordinate system moving with the wave, the electric potential is of the form

$$\varphi(x,t)=\varphi_0 A(t)(1-\cos kx),$$

where $A(t)$ is a dimensionless amplitude and k is the wave number. If we change in the e uation of motion of a particle with charge e to the dimensionless coordinates $kx\to x$ and $kt(m/e\varphi_0)^{1/2}\to t$, the equation becomes

$$\ddot{x}+A(t)\sin x=0. \qquad (1)$$

We shall assume that the amplitude of the wave changes slowly with the time, $A(t)=1+\varepsilon t$, where $\varepsilon\ll 1$. For short time intervals the solution of Eq. (1) must take the form of a power series in ε. We consider the problem of finding the first two terms of this series. The term proportional to ε^0 describes the solution of the stationary equation (1) for $A(t)=\text{const}$. As is well known, it can be expressed in terms of elliptic functions (cf., e.g., Ref. 2). However, it is very difficult to use these general expressions to find the next approximation. Therefore we shall not give them, and to get a general idea of the character of the motion we shall use the phase plane (see Fig. 1). On it we can delineate the regions of finite and infinite motions. They are separated by trajectories that pass through the hyperbolic points $(\pm\pi, 0)$. It is customary to designate particles with finite trajectories as trapped by the wave, and those with infinite trajectories as untrapped. The energy of the untrapped particles satisfies the condition

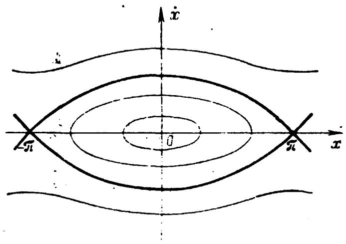

FIG. 1.

$$W={}^1/_2\dot{x}^2+1-\cos x>2,$$

and for trapped particles $W<2$. It is essential for what follows that at the critical energy value $W=2$ the solution of the stationary equation (1) with $\varepsilon=0$ is comparatively simple in form:

$$x_0(t)=4\operatorname{arctg} e^t-\pi. \qquad (2)$$

In the vicinity of the separatrix ($|W-2|\ll 1$) the solution of Eq. (1) differs slightly from the form (2):

$$x(t)\approx x_0(t)+x_{1\alpha}(t)+x_{1\varepsilon}(t),$$

where $x_{1\alpha}(t)$ and $x_{1\varepsilon}(t)$ are the corrections to the solution and are due respectively to the difference $W-2\neq 0$ and to the variation of the amplitude $A(t)$. These corrections can be calculated independently:

$$x_{1\alpha}(t)=\alpha(\operatorname{sh} t+t\operatorname{ch}^{-1}t)+\alpha'\operatorname{ch}^{-1}t, \qquad (3)$$

$$x_{1\varepsilon}(t)=\frac{\varepsilon}{2}(-\operatorname{sh} t\operatorname{th} t+t^2\operatorname{ch}^{-1}t). \qquad (4)$$

In Eq. (3) we have introduced the notation $\alpha=\frac{1}{4}(W-2)$. The last term in this equation, proportional to α', does not contribute to the energy. It describes a shift of the origin from which time is measured, and hereafter we shall disregard it.

The trajectory we have found describes a motion with the energy

$$W(t)=2(1+2\alpha+\varepsilon t-\varepsilon\operatorname{th} t). \qquad (5)$$

In the neighborhood of the hyperbolic points $(\pm\pi, 0)$ the solution (2)–(4) takes the form

$$x_-(t)\approx-\pi+4e^t-{}^1/_2(\alpha+\varepsilon/2)e^{-t}, \qquad (6)$$

$$x_+(t)\approx\pi-4e^{-t}+{}^1/_2(\alpha-\varepsilon/2)e^t. \qquad (7)$$

Here the signs minus and plus refer to the representations that hold in the neighborhoods of the points $(-\pi, 0)$ and $(\pi, 0)$.

It follows from Eqs. (6) and (7) that if $|t|\gg 1$ the terms proportional to α and ε increase exponentially. This can carry us beyond the range of the method of successive approximations. However, if $|t|\gg 1$ the trajectories must pass close to the hyperbolic points, i.e., within a region where Eq. (1) can be linearized in x ($\sin x\approx x\pm\pi$). If at the same time the condition $\varepsilon|t|\ll 1$ holds, the solution of the linearized equation (1) is given by the expression

$$x(t)\pm\pi=C_1e^t+C_2e^{-t},$$

so that it is of the same form as Eqs. (6) and (7).[1)]

Accordingly, the expressions (2)–(4) give solutions of Eq. (1) which are correct for an entire cycle of the particle's motion. By a cycle we mean the motion of an untrapped particle along the segment $(-\pi, \pi)$ or that of a trapped particle from one turning point to the other. We also note that the expressions (2)–(4) can be continued onto another cycle of the motion.

3. In the nonstationary case the condition for capture of particles by the wave is $W(t) < 2A(t)$. It follows from Eq. (5) that particles with $|\alpha| < \varepsilon/2$, which are "untrapped" at $x = -\pi$ are captured by the wave before they can get to the line $x = \pi$ on the phase surface (for definiteness we are considering particles with positive velocity). Let us examine how the adiabatic invariant changes in such a transition. As is well known, an adiabatic invariant characterizes a quasiperiodic motion whose period T changes by an amount $\Delta T \ll T$ during a time equal to T. Froms Eqs. (2)–(7) it is not hard to find that this condition is satisfied if $|\alpha| \gg \varepsilon$, i.e., for motions along trajectories sufficiently far from the separatrix. In this region the adiabatic invariant changes exponentially slowly: $\dot{I} \sim \exp(-|\alpha|/\varepsilon)$. We emphasize that there exists an expression, an infinite series in powers of $\varepsilon \ll 1$, which has this property (cf., e.g., Ref. 8). In practice one can find only one or two terms of the series. The derivatives of the truncated expression for I are of the respective orders of $\varepsilon/|\alpha|$ and (ε/α).[2]

The adiabatic invariant for Eq. (1) has been calculated to terms of order ε in Ref. 2. We present these expressions;

$$I_0(W, A(t)) = \begin{cases} 4(2W)^{1/2}\mathbf{E}(1/k), & k>1 \\ 8(A(t))^{1/2}(\mathbf{E}(k)+(k^2-1)\mathbf{K}(k)), & k<1 \end{cases}, \tag{8}$$

$$I_1(x, W, A(t)) = 4\,\mathrm{sign}(x\dot{x})\frac{\dot{A}}{A}\begin{cases} \mathbf{K}(1/k)E(|x|/2, 1/k) - \mathbf{E}(1/k)K(|x|/2, 1/k), & k>1 \\ \mathbf{K}(k)E(|y|, k) - \mathbf{E}(k)K(|y|, k), & k<1 \end{cases} \tag{9}$$

Here $k = (W/2A)^{1/2}$, $y = \arcsin(k^{-1}\sin x)$, and K and E are elliptic integrals of the first and second kinds. Their values when the first argument is equal to $\pi/2$ are denoted by $\mathbf{K}$, $\mathbf{E}$. We note that I_1 goes to zero at $x = \pm\pi$ for untrapped particles and at the turning points ($x = \pm \arcsin k$) for trapped particles.

In the neighborhood of the separatrix ($|k-1| \ll 1$) the expression (8) takes the simple form

$$I_0 \approx 4\left[2 + (k-1)\ln\frac{8}{|k-1|} + (k-1) + \varepsilon t\right]. \tag{10}$$

We calculate how much this quantity changes when the particle goes from untrapped to trapped. It follows from Eq. (6) that the particle crosses the line $x = -\pi$ at the time

$$t_- \approx -\frac{1}{2}\ln\left(\frac{8}{\alpha+\varepsilon/2}\right).$$

Using Eq. (5), we find that at this time the quantity k has a value $k_- \approx 1 + \alpha + \varepsilon/2$.

From Eq. (7) we find the time when the particle is reflected from the potential hump:

$$t_+ \approx \frac{1}{2}\ln\left(\frac{8}{-\alpha+\varepsilon/2}\right),$$

and here k is $k_+ \approx 1 + \alpha - \varepsilon/2$. Since I_1 vanishes at $t = t_\pm$, the change of I during the interval of time ($t_- < t < t_+$) can be obtained, up to terms of order ε inclusive, by means of Eq. (10):

$$\Delta I^{(1)} = 4\left(\alpha\ln\left(\frac{\alpha+\varepsilon/2}{-\alpha+\varepsilon/2}\right) - \varepsilon\right). \tag{11}$$

This expression has been derived on the assumption $|\alpha| < \varepsilon/2$, since only in this case will the particle be untrapped ($k_- > 1$) at $t = t_-$ and become trapped ($k_+ < 1$) at $t \to t_+$. The further evolution leads to the descent of the particle to the bottom of the potential well; meanwhile, so long as $|k-1|$ is comparable with ε the motion does not satisfy the adiabaticity condition. In fact, during one passage between the turning (reflection) points the quantity k changes by ε, and the period of the motion, determined formally for a constant value of k, is given by

$$T = \partial I_0/\partial W \approx \ln(8/|k-1|).$$

There is an analogous band of nonadiabatic behavior in the region of infinite trajectories with $k - 1 > 0$.

Proceeding as in the derivation of Eq. (11), we find that the change of the adiabatic invariant during one cycle of the motion is given, both for untrapped ($k - 1 > \varepsilon/2$) and for trapped ($k - 1 < -\varepsilon/2$) particles by the same expression:

$$\Delta I^{(2)} = 4\left(\alpha\ln\left(\frac{\alpha+\varepsilon/2}{\alpha-\varepsilon/2}\right) - \varepsilon\right). \tag{12}$$

In the expressions (11) and (12) the quantity α is equal to the difference $k - 1$ evaluated at the time of crossing the line $x = 0$. During one cycle of the motion it decreases by ε. Summing over all cycles, we get

$$\Delta I = 2\varepsilon\left\{\beta\ln\left(\frac{1+\beta}{1-\beta}\right) - 2 - \sum_{n=1}^{\infty}\left[\beta\ln\left(\frac{4n^2-(1+\beta)^2}{4n^2-(1-\beta)^2}\right) + 2n\ln\left(\frac{(2n+1)^2-\beta^2}{(2n-1)^2-\beta^2}\right) - 4\right]\right\}. \tag{13}$$

Here we have introduced the notation $\beta = 2\alpha_0/\varepsilon$, where the quantity α_0 is the value for the cycle in which the particle changes directly from untrapped to trapped, and is connected with the energy at the time of passage across the line $x = 0$ by the relation $W = 2(1 + 2\alpha_0)$. By means of Eqs. (2) and (5) we can express α_0 in terms of the coordinate x_s of the point where the trajectory of the particle intersects the separatrix:

$$\alpha_0 = \frac{\varepsilon}{2}\sin\left(\frac{x_s}{2}\right).$$

Using a summation formula (see ref. 9) and some simple transformations, we can put Eq. (13) in the form

$$\Delta I = 4\varepsilon\left\{-1 + \lim_{n\to\infty}\ln\frac{((n+1/2)^2-\beta^2/4)^n\pi e^{2n}}{\Gamma(n+1/2+\beta/2)\Gamma(n+1/2-\beta/2)\cos(\pi\beta/2)}\right\}. \tag{14}$$

By means of asymptotic representations for the Γ functions we finally get

$$\Delta I = -4\varepsilon\ln(2\cos(\pi\alpha_0/\varepsilon)). \tag{15}$$

Our discussion has been for the case of a wave of increasing amplitude, in which charged particles are trapped by the wave. It is not hard to show that the expression (15) is valid also for the opposite process, in which particles change from trapped to untrapped as the amplitude of the wave decreases, with $\varepsilon = dA/dt < 0$.

We note that the values of ΔI for particles crossing the separatrix at the same point but moving in opposite directions are equal. This follows from the spatial symmetry of the potential in which the particle moves and from the symmetry of the expression (15) in the sign of α_0, and also follows from the connection between α_0 and $x_s[\alpha_0 = \frac{1}{2}\varepsilon\sin(x_s/2)]$.

The expression (15) diverges logarithmically as $|\alpha_0| \to \varepsilon/2$. This singularity can be very simply explained. If $|\alpha_0| \approx \varepsilon/2$ the trajectory intersects the separatrix close to the hyperbolic points, in whose neighborhood the particles linger for considerable times:

$$\Delta t \approx \frac{1}{2}\ln\left(\frac{8}{\alpha_0+\varepsilon/2}\right) \quad \text{at} \quad \alpha_0 \approx -\frac{\varepsilon}{2},$$

$$\Delta t \approx \frac{1}{2}\ln\left(\frac{8}{-\alpha_0+\varepsilon/2}\right) \quad \text{at} \quad \alpha_0 \approx \frac{\varepsilon}{2}$$

(see the foregoing arguments). During this time the area of the region bounded by the separatrices increases by $8\varepsilon\Delta t$ [see Eq. (8)]. The adiabatic invariant, which is approximately equal to half of this area, also increases with this area.

It must be pointed out that the expression (15) becomes inaccurate if $|\alpha_0|$ is very nearly equal to $\varepsilon/2$. In fact, we have been assuming that the time spent in one cycle of the motion is small in comparison with ε^{-1}. This assumption is violated if $1 - |2\alpha_0/\varepsilon| < e^{-1/\varepsilon}$; here, according to (15), ΔI will be of order of magnitude unity. However, if we assume that particles are distributed uniformly over the phase plane, then the fraction of them for which our argument is not justified will be exponentially small, $\sim e^{-1/\varepsilon}$.

This problem has also been analyzed in Ref. 7 by numerical methods. For $|\alpha_0| \lesssim \varepsilon/3$ the two approaches give practically the same result.[2)] In the greater part of the range $\varepsilon/3 \lesssim |\alpha_0| \lesssim \varepsilon/2$ the quantitative results are also very nearly the same. However, they differ qualitatively in two respects. First, the values of ΔI calculated in Ref. 7 are finite even at $|\alpha_0| = \varepsilon/2$. Second, the numerical calculations give different values of ΔI for particles crossing the separatrix with different signs of the velocity. The first of these differences is due to the very nature of numerical calculations. Indeed, numerical methods permit analysis of the behavior of a system only over finite time intervals. Therefore it was assumed in Ref. 7 that the amplitude of the wave changes during some finite time Δt. But during this time particles with a sufficiently small value of the difference $\varepsilon/2 - |\alpha_0|$ can fail to get through the band where the behavior is nonadiabatic. The result is that the calculated values of ΔI are too low. We point out that the boundaries of the band in question cannot be defined exactly. Consequently, even for particles that intersect the separatrix far from the hyperbolic points, the results of a calculation must depend on the way the time dependence of the wave's amplitude is specified, and are not of universal significance.

The difference between values of ΔI for particles crossing the separatrix with different signs of the velocity is possibly due to an substantial simplification used in Ref. 7. Namely, in that paper the adiabatic invariant was identified with the zeroth term of the expansion of the exact expression in powers of ε [cf. Eqs. (8), (9)]. Since the change of the adiabatic invariant is itself of the order of magnitude of ε, the calculations made in Ref. 7 must, strictly speaking, be regarded as incorrectly formulated. It cannot be excluded that this may be the very reason for the splitting of the dependence of ΔI on ε. In fact, the term of order ε is antisymmetric in the velocity [cf. Eq. (9), and also the work of Best[2]]. Including it might eliminate the splitting.

1) Calculations show that to take effects of the nonlinear terms in Eq. (1) as $|x| \to \pi$ into account one must go to higher orders in ε.

2) In comparing our results with those of Ref. 7 it must be noted that the quantities I and ε introduced in that paper differ from those used here: $I \to a_s I$, $\varepsilon \to 0.141 a_s^{-4}\varepsilon$, where $a_s = 2.65$.

1. L. D. Landau and E. M. Lifshitz, Mekhanika, (Mechanics) Nauka, 1973. Engl. transl. Pergamon Press, 1976.
2. R. W. B. Best, Physica 40, 182 (1968).
3. V. V. Beletskiĭ, Ocherk o dvizhenii kosmicheskikh tel (Essay on the Motion of Cosmic Bodies), Nauka, 1977.
4. R. E. Asmodt, Phys. Rev. Lett. 27, 135 (1971).
5. A. V. Timofeev, Nucl. Fusion, 14, 165 (1974).
6. D. D. Ryutov, II Internat. Cong. on Waves and Instabilities in Plasmas, Innsbruck, Austria, 1975.
7. R. E. Aamodt and E. F. Jaeger, Phys. Fluids 17, 1386 (1974).
8. M. Kruskal, Adiabatic Invariants, (Russ. transl.), IIL, 1962.
9. I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov, i proiavedeniĭ (Tables of Integrals, Sums, Series, and Products), Fizmatgiz, 1963. Engl. transl. Academic Press, 1965.

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